Shirmayil G. BAGIROV

ON ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NONLINEAR ELLIPTIC EQUATION

Abstract

In the paper, asymptotic behavior of solutions to second order semi-linear elliptic equations in the divergent form with Neumann condition on lateral surface is investigated. It is proved that in the vicinity of infinity, the constant sign solutions decrease as a power function, the alternating solutions decrease exponentially.

Denote $\Pi_{a,b} = G \times (a,b)$, $\Pi_{a,\infty} = \Pi_a$, $\Gamma_{a,b} = \partial G \times (a,b)$, $\Gamma_{a,\infty} = \Gamma_a$, where G is a bounded domain in R^n with Lipschits boundary.

Let L be an operator of the form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i},$$

where $x = (x_1, ..., x_n) \in R^n$, the coefficients $a_{ij}(x)$, $a_i(x)$ are bounded, measurable functions, $a_{ij} = a_{ji}$, and the following condition be fulfilled

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geq \nu_1 |\xi|^2, \quad x \in G, \quad \nu_1 = const > 0, \quad |\xi|^2 = \sum_{i=1}^{n}\xi_i^2, \ \xi \in R^n.$$

As $t \to +\infty$ we'll investigate the behavior of the solution of the equation:

$$u_{tt} + Lu - |u|^{\sigma} = 0 \quad in \quad \Pi_0, \tag{1}$$

satisfying the condition

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(x_i, n) \quad in \quad \Gamma_0, \tag{2}$$

where $\sigma > 1$, n is a unit vector of the external normal to ∂G.

Notice that the similar problem with nonlinearity of the form $|u|^{\sigma-1} \cdot u$ was investigated in the papers [1], [2], [3].

The generalized solution is understood as the solution of problem (1), (2). The function $u(x,t)$ is called the generalized solution to equation, (1) satisfying condition (2) if $u(x,t) \in W_2^1(\Pi_{a,b}) \cap L_\infty(\Pi_{a,b})$ for any $0 < a$, $b < \infty$, and it holds the equality:

$$\int\limits_{\Pi_{a,b}} u_t \cdot \varphi_t dxdt + \sum_{i,j=1}^{n} \int\limits_{\Pi_{a,b}} a_{ij}(x) \cdot u_{x_j} \cdot \varphi_{x_i} dxdt -$$

$$- \sum_{i=1}^{n} \int\limits_{\Pi_{a,b}} a_i(x) \cdot u_{x_j} \cdot \varphi_{x_i} dxdt + \int\limits_{\Pi_{a,b}} |u|^{\sigma} \cdot \varphi dxdt = 0$$

for any function $\varphi(x,t) \in W_2^1(\Pi_{a,b})$ such that $\varphi(x,a) = \varphi(x,b) = 0$.

Prove some auxiliary facts. Consider the following linear problem:

$$\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\left(a_{ji}(x)\frac{\partial u}{\partial x_j}\right)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial u}{\partial x_j}\right)=0, \tag{3}$$

$$\frac{\partial u}{\partial \nu^*}=\sum_{i,j=1}^{n}a_{ji}\frac{\partial u}{\partial x_j}\cos(n,x_i)-\sum_{i=1}^{n}a_i\frac{\partial u}{\partial x_i}\cos(n,x_i)=0. \tag{4}$$

Problem (3), (4) has a solution $k(x)$ such that $0<m_1\le k(x)\le m_2$. Such a solution exists since problem (3), (4) is conjugated to the Neumann problem for the equation:

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{n}a_i(x)\frac{\partial u}{\partial x_j}=0.$$

This equation has a solution $u(x)\equiv 1$, satisfying the Neumann boundary condition on ∂G. Consequently, by the Fredholm theorem, problem (3), (4) has a nontrivial solution $k(x)$ that is continuous in G, and is positive.

Lemma 1. *For any $\sigma>1$ problem (1), (2) has no negative solutions.*

Proof. In definition of the solution, as the test function take $\varphi(x,t)=t\cdot\psi(t)\cdot k(x)$, where $\psi(t)\in C_0^\infty(R)$, $\psi(t)=1$ for $0\le t\le R$, $\psi(t)=0$ for $t\ge 2R$, and $k(x)$ is the solution of problem (3), (4).

Then we have

$$\int_{\Pi_{0,2R}}|u|^\sigma\cdot t\cdot\psi\cdot k\,dxdt=-\int_{\Pi_{0,2R}}u_t\cdot\left(t\psi'+\psi\right)\cdot k\,dxdt-$$

$$-\sum_{i,j=1}^{n}\int_{\Pi_{0,2R}}a_{ij}(x)\cdot u_{x_j}\cdot k_{x_i}\cdot t\cdot\psi\,dxdt+\sum_{i=1}^{n}\int_{\Pi_{0,2R}}a_i(x)\cdot u_{x_i}\cdot k\cdot t\cdot\psi\,dxdt=$$

$$=-\int_{\Pi_{0,2R}}u_t\left(t\psi'+\psi\right)k\,dxdt=\int_{\Pi_{0,2R}}uk\left(t\psi''+2\psi'\right)dxdt+\int_G ku(x,0)\,dx\le$$

$$\le m_2\left(\int_{\Pi_{0,2R}}|u|^\sigma\cdot t\cdot\psi\,dxdt\right)^{\frac{1}{\sigma}}\cdot\left(\int_{\Pi_{0,2R}}\frac{\left|t\cdot\psi'+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dxdt\right)^{\frac{1}{q}}+$$

$$+m_2\int_G u(x,0)\,dx\le\frac{\varepsilon m_2}{\sigma}\int_{\Pi_{0,2R}}|u|^\sigma t\cdot\psi\,dxdt+$$

$$+\frac{m_2}{\varepsilon^{q-1}\cdot q}\int_{\Pi_{0,2R}}\frac{\left|t\cdot\psi''+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dxdt+m_2\int_G u(x,0)\,dx,$$

where $\frac{1}{\sigma}+\frac{1}{q}=1$.

Hence we get

$$\left(\frac{m_1}{m_2}-\frac{\varepsilon}{\sigma}\right)\cdot\int_{\Pi_{0,2R}}|u|^\sigma t\cdot\psi\,dxdt\le$$

$$\leq \frac{m_2}{\varepsilon^{q-1}\cdot q}\int\limits_{\Pi_{0,2R}}\frac{\left|t\psi''+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dxdt+\int\limits_G u\left(x,0\right)dx. \qquad (5)$$

Make the substitution $\tau = t/R$. Take $\psi\left(t\right)$ in the form $\psi\left(t\right)=\psi\left(\tau R\right)=\left(\varphi_0\left(\tau\right)\right)^{\mu}=\theta\left(\tau\right)$, where $\varphi_0\left(\tau\right)=1$ for $\tau\leq 1$, $\varphi_0\left(\tau\right)=0$ for $\tau\geq 2$, $\varphi_0\left(\tau\right)\in C_0^{\infty}$, μ is a number rather large in modulus.

Estimate the first integral in the right hand side of inequality (5):

$$\int\limits_{\Pi_{0,2R}}\frac{\left|t\psi''+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dxdt=\int\limits_G\int\limits_{1\leq\tau\leq 2}\frac{\left|\tau R^{-1}\theta''+2R^{-1}\theta'\right|}{R^{q-1}\tau^{q-1}\theta^{q-1}}Rdxd\tau=$$

$$=R^{2(1-q)}\cdot mesG\int\limits_{1\leq\tau\leq 2}\frac{\left|\tau\theta''+2\theta'\right|}{\tau^{q-1}\theta^{q-1}}d\tau=R^{2(1-q)}\cdot mesG\times$$

$$\times\int\limits_{1\leq\tau\leq 2}\frac{\left|\tau\mu\varphi_0^{\mu-1}\varphi_0''+\tau\mu\left(\mu-1\right)\varphi_0^{\mu-2}\varphi_0'^2+2\mu\varphi_0^{\mu-1}\varphi_0'\right|^q}{\tau^{q-1}\varphi_0^{\mu(q-1)}}d\tau=$$

$$=R^{2(1-q)}\cdot A\left(\varphi_0\right),$$

where

$$A\left(\varphi_0\right)=mesG\int\limits_{1\leq\tau\leq 2}\frac{\left|\tau\mu\varphi_0^{\mu-1}\varphi_0''+\tau\mu\left(\mu-1\right)\varphi_0^{\mu-2}\cdot\varphi_0'^2+2\mu\varphi_0^{\mu-1}\varphi_0'\right|^q}{\tau^{q-1}\mu^{\mu(q-1)}}d\tau,$$

we can choose μ,φ_0 so that $A\left(\varphi_0<\infty\right)$.

It we take into account all these facts in (5), then:

$$\left(\frac{m_1}{m_2}-\frac{\varepsilon}{\sigma}\right)\cdot\int\limits_{\Pi_{0,2R}}\left|u\right|^{\sigma}tdxdt\leq\left(\frac{m_1}{m_2}-\frac{\varepsilon}{\sigma}\right)\cdot\int\limits_{\Pi_{0,2R}}\left|u\right|^{\sigma}t\cdot\psi\left(t\right)dxdt\leq$$

$$\leq\frac{1}{q\varepsilon^{q-1}}\cdot R^{2(1-q)}\cdot A\left(\varphi_0\right)+\int\limits_G u\left(x,0\right)dx. \qquad (6)$$

Since $q=\frac{\sigma}{\sigma-1}>1$, if $\int\limits_G u\left(x,0\right)dx\leq 0$, then as $R\to\infty$, from (6) we get $\int\limits_{\Pi_0}\left|u\right|^{\sigma}tdxdt=0$.

Hence we have that $u\equiv 0$ in Π_0,if $\int\limits_G u\left(x,0\right)dx\leq 0$. This proves lemma 1.

We get that if $u\left(x,t\right)$ is a non-trivial solution of problem (1), (2), then $\int\limits_G u\left(x,0\right)dx>0$. If in place of the test function we take $\varphi\left(x,t\right)=\left(t-t_0\right)\cdot\psi\left(t\right)\cdot k\left(x\right)$ for $t\geq t_0,\varphi\left(x,t\right)=0$ for $t<t_0$, then for any non-trivial solution $u\left(x,t\right)$

$$\int\limits_G u\left(x,t_0\right)dx>0.$$

Lemma 2. *If $u(x,t)$ is the solution of problem (1), (2), then*

$$\lim_{t\to\infty} u(x,t) = 0.$$

Proof. At first prove that any solution of problem (1), (2) is bounded. If $u(x,t)$ is the solution equation (1), then $u(x,t)$ is the subsolution of the equation:

$$u_{tt} + Lu - |u|^{\sigma-1} u = 0. \tag{7}$$

Really:

$$u_{tt} + Lu - |u|^{\sigma-1} u \geq u_{tt} + Lu - |u|^{\sigma} = 0.$$

Equation (7) has a strong positive solution $\omega(t)$ satisfying the relations $\omega(t_0) = 1$, $\omega'(t_0) = 0$ in the form of a parabola with asymptotes at the points $t_0 \pm T$ (where T is independent of t_0). Then for rather large t from the maximum principle, the subsolution is less than the solution, i.e. $u(x,t) \leq \omega(t)$ at Π_{t_0-T,t_0+T}. Thus, $u(x,t)$ is upper bounded, since for large t it is less than the value at the vertex of the parabola.

The function $v(x,t) = u(x,t) - C_0 \cdot t^{-\frac{2}{\sigma-1}}$, where $C_0 = \left[\frac{2(\sigma+1}{(\sigma-1)^2}\right]^{\frac{1}{\sigma-1}}$, is also an upper bounded subsolution of equation (7). Then:

$$u_{tt} + Lv - a(x,t) v \geq 0, \tag{8}$$

where $a(x,t) \geq 0$.

Consider the function $v - \varepsilon t$. This function also satisfies equality (8) and is negative for $t = 0$. There exists $T_0(\varepsilon)$ such that for $T \geq T_0(\varepsilon)$, $v - \varepsilon T \leq 0$. Then it follows from the maximum principle that $v - \varepsilon T \leq 0$ for $t \geq 0$. Tending ε to zero, we get $v \leq 0$.

So,

$$u(x,t)^{+} \leq C_0 \cdot t^{-\frac{2}{\sigma-1}}. \tag{9}$$

Having made in (1) the substitution $v = -u$, consider the equation:

$$u_{tt} + Lv + |v|^{\sigma} = 0.$$

Since

$$|v| = v^{+} - v^{-}, \quad v = v^{+} + v^{-},$$

then

$$\int_G |v|\, dx \leq -\int_G v^{-} dx \leq 2\int_G C_0 \cdot t^{-\frac{2}{\sigma-1}} dx = C_1 \cdot t^{-\frac{2}{\sigma-1}}.$$

If $\sigma < 3$, then:

$$\int_1^{\infty}\int_G |v|\, dxdt \leq C_1 \cdot \int_1^{\infty} t^{-\frac{2}{\sigma-1}} = -C_2 \cdot t^{-\frac{\sigma-3}{\sigma-1}}\Big|_1^{\infty} = C_2.$$

If $\sigma > 3$, then for large T

$$\int_{\Pi_{T-2,T+2}} |v|\, dxdt \leq C_3,$$

where C_3 is independent of T. Really:

$$\int\limits_{\Pi_{T-2,T+2}} |v|\, dxdt \le C_1 \int\limits_{T-2}^{T+2} t^{-\frac{2}{\sigma-1}} dt = C_1 \frac{\sigma-1}{\sigma-3}\left((T+2)^{\frac{\sigma-3}{\sigma-1}} - (T-2)^{\frac{\sigma-3}{\sigma-1}}\right) =$$

$$= 4C_1 \left(T-2+\xi\cdot 4\right)^{-\frac{2}{\sigma-1}} = 4C_1 \frac{1}{(T-2+\xi\cdot 4)^{\frac{2}{\sigma-1}}} \le 4C_1,$$

if $T>3$. Here $0<\xi<1$.

For $\sigma=3$ we similarly get:

$$\int\limits_{\Pi_{T-2,T+2}} |v|\, dxdt \le C_1 \int\limits_{T-2}^{T+2} t^{-1} dt = C_1 \ln T \Big|_{T-2}^{T+2} = C_1 \frac{4}{(T-2+4\xi)^{\frac{2}{\sigma-1}}} \le 4C_1,$$

if $T>3$.

From the theory of linear differential equations we know that (see [4])

$$\max_{\Pi_{T-1,T+1}} |u| \le C \int\limits_{\Pi_{T-2,T+2}} |u|\, dxdt \le C_3, \quad for \quad T>3.$$

So, everywhere $|u|<C$.

From (6) we get:

$$\int\limits_{\Pi_{1,\infty}} |u|^q\, dxdt \le C_4. \tag{10}$$

Then for each T_ε, there exists such a point $(x_\varepsilon, t_\varepsilon) \in \Pi_{T_\varepsilon-1,T_\varepsilon+1}$ and such a C that

$$|u(x_\varepsilon, t_\varepsilon)| \le \frac{C}{mes^{1/q} G} \left(\int\limits_{\Pi_{T_\varepsilon-1,T_\varepsilon+1}} |u|^q\, dxdt \right)^{1/q}. \tag{11}$$

This is easily proved by contradiction. Hence $u(x_\varepsilon, t_\varepsilon) \to 0$ as $T_\varepsilon \to 0$. Using this, prove that $u(x,t) \to 0$ as $t \to +\infty$.

Having again made the substitution $v=-u$, write equation (1) in the form

$$u_{tt} + Lv + |v|^{\sigma-1} signv \cdot v = 0.$$

Denote $q(x,t) = |v|^{\sigma-1} sign\ v$. Since, $|v| = |u| < C$, then $|q(x,t)| < C_1$.

Consider the function:

$$W(x,t) = v(x,t) + C_0 t^{-\frac{2}{\sigma-1}}. \tag{12}$$

From (9) it follows that $W(x,t) \ge 0$.

$W(x,t)$ satisfies the equation:

$$W_{tt} + LW + q(x,t) W = -C_0 \frac{2(\sigma+1)}{(\sigma-1)^2} t^{-\frac{2\sigma}{\sigma-1}} - q \cdot C_0 \cdot t^{-\frac{2}{\sigma-1}}.$$

Then by the Harnack inequality (see [5]) we have:

$$\max_{\Pi_{T-1,T+1}} W(x,t) \le C_1 \inf_{\Pi_{T-1,T+1}} W(x,t) +$$

$$+C_2 \|f\|_{L_{q/2}(\Pi_{T-2,T+2})} \le C_1 \cdot \inf_{\Pi_{T-1,T+1}} W(x,t) +$$

$$+C_2 \left(\int\limits_{\Pi_{T-2,T+2}} t^{-\frac{q}{\sigma-1}} \left[-C_0 \frac{2(\sigma+1)}{(\sigma-1)^2} t^{-2} - qC_0 \right]^{\frac{q}{2}} dxdt \right)^{\frac{2}{q}} \le$$

$$\le C_1 \cdot \inf_{\Pi_{T-1,T+1}} W(x,t) + C_2 \cdot C_3 \left(\int\limits_{\Pi_{T-2,T+2}} t^{-\frac{q}{\sigma-1}} dt \right)^{2/q} \to 0, \quad as \quad T \to \infty,$$

by (11) and (12).

Hence it follows that $u = -v = C_0 \cdot t^{-\frac{2}{\sigma-1}} - W \to 0$ as $t \to +\infty$. This proves lemma 2.

Now, prove that $u(x,t) = O\left(t^{-\frac{2}{\sigma-1}}\right)$. It $u(x,t)$ is a non-negative solution, this is obvious.

Make the substitution $v = -u$. Then

$$u_{tt} + Lu + |v|^q = 0. \tag{13}$$

Since $u \le Ct^{-\frac{2}{\sigma-1}}$, then $v \ge -Ct^{-\frac{2}{\sigma-1}}$. Denote $h(t) = -Ct^{-\frac{2}{\sigma-1}}$, $z = v - h(t)$. Then $z \ge 0$ and

$$z_{tt} + Lz + |z+h|^\sigma = h_{tt}.$$

Write this as follows:

$$z_{tt} + Lz + \frac{|h+z|^\sigma - |h|^\sigma}{z} z = C_1 \cdot t^{-\frac{2\sigma}{\sigma-1}} + C_2 \cdot t^{-\frac{2\sigma}{\sigma-1}} = O\left(t^{-\frac{2\sigma}{\sigma-1}}\right).$$

Hence

$$z_{tt} + Lz + B(x,t) z = C \cdot t^{-\frac{2\sigma}{\sigma-1}}, \tag{14}$$

where $B(x,t) = \frac{|h+z|^\sigma - |h|^\sigma}{z} z = C \cdot t^{-2} + \frac{o(z)}{z}$ tends to zero as $z \to 0$.

Since $z \ge 0$, then applying the Harnack inequality to (15), we get:

$$\max_{\Pi_{T-1,T+1}} |z(x,t)| \le C_1 \min_{\Pi_{T-1,T+1}} |z(x,t)| + C_2 \cdot \|f\|_{L_{q/2}(\Pi_{T-2,T+2})}, q > n+1.$$

Let at first $T = t_\varepsilon$. Then by the Harnack inequality we have:

$$\max_{\Pi_{T-1,T+1}} |z(x,t)| \le C_1 t_\varepsilon^{-\frac{2}{\sigma-1}} + C_2 \cdot t^{-\frac{2\sigma}{\sigma-1}}.$$

$$t_\varepsilon^{-\frac{2}{\sigma-1}} = C_3 (t_\varepsilon + 1 + t_\varepsilon - 1)^{-\frac{2}{\sigma-1}} = C_3 (t_\varepsilon + 1)^{-\frac{2}{\sigma-1}} \left(1 + \frac{t_\varepsilon - 1}{t_\varepsilon + 1}\right)^{-\frac{2}{\sigma-1}} =$$

$$= C_3 (t_\varepsilon + 1)^{-\frac{2}{\sigma-1}} \left(1 + \frac{1 - \frac{1}{t_\varepsilon}}{1 + \frac{1}{t_\varepsilon}}\right)^{-\frac{2}{\sigma-1}} \le C_4 (t_\varepsilon + 1)^{-\frac{2}{\sigma-1}} \le$$

$$\leq C_4 (T+1)^{-\frac{2}{\sigma-1}} \leq C_4 t^{-\frac{2}{\sigma-1}},$$

if $T-1 \leq t \leq T+1$. So,

$$|z(x,t)| \leq C_4 \cdot t^{-\frac{2}{\sigma-1}}, \quad if \quad T-1 \leq t \leq T+1.$$

Having taken successively $T = t_\varepsilon + 1$, $t_\varepsilon + 2$ and etc , we get:

$$|z(x,t)| \leq C \cdot t^{-\frac{2}{\sigma-1}}, \quad for \quad t \geq T_0.$$

Then

$$|v| = |z+h| \leq |z| + |h| \leq C \cdot t^{-\frac{2}{\sigma-1}},$$

$$|u| = |v| = O\left(t^{-\frac{2}{\sigma-1}}\right), \quad |u|^{\sigma-1} = O\left(t^{-2}\right).$$

The following theorem is the main result.

Theorem 1.

I. For any $\sigma > 1$ equation (1) has no solution satisfying condition (2) negative in Π_a, $a > 0$.

II. Let $u(x,t) > 0$ be a solution of equation (1) satisfying condition (2). Then $u(x,t) = O\left(t^{-\frac{2}{\sigma-1}}\right)$.

III. Let $u(x,t)$ be a solution of equation (1) satisfying condition (2), that changes sign at each domain Π_a, $a > 0$. Then $u(x,t) = O\left(e^{-ht}\right)$, where h is independent of $u(x,t)$.

Proof. Above we proved I and II. Prove III.

Write equation (1) in the form

$$u_{tt} + Lu - q(x,t)\,u = 0, \qquad (15),$$

where $q(x,t) = |u|^{\sigma-1} \cdot sign\ u$.

Since $\lim\limits_{t\to\infty} |u(x,t)| = 0$, then there exists a t_o such that for any $t \geq t_0$

$$|u(x,t)|^{\sigma-1} < \varepsilon.$$

Take $\theta(t) \in C^\infty$ such that for $\theta(t) = 1$ for $t > t_0 + 1$, $\theta(t) = 0$ for $t \leq t_0$ and $0 \leq \theta(t) \leq 1$.

Assume

$$v(x,t) = \theta(t) \cdot u(x,t).$$

The function $v(x,t)$ satisfies the equation:

$$v_{tt} + Lv - q(x,t)\,v = F(x,t) \qquad (16)$$

and the boundary conditions

$$\frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \qquad (17)$$

where

$$q(x,t) = \begin{cases} |u|^{\sigma-1} sign\ u & for \quad t \geq t_0 + 1, \\ 0 & for \quad t \leq t_0, \end{cases}$$

$$F(x,t) = (\theta_t \cdot u)_t + \theta_t \cdot u_t.$$

Obviously, the function $F(x,t)$ has a compact support.

Show that $|v(x,t)| \leq C \cdot \exp\{-ht\}$, $C = const$. From the theory of linear equations [see 6,7] it follows that problem (16), (17) has a solution $v_1(x,t)$ such that

$$v_1(x,t) = \begin{cases} O\left(e^{-ht}\right) & as \quad t \to +\infty, \\ at + b + O\left(e^{ht}\right) & as \quad t \to +\infty. \end{cases} \tag{18}$$

The function $\omega(x,t) = v_1(x,t) - v(x,t)$ satisfies the equation:

$$\omega_{tt} + L\omega - q(x,t)\,\omega = 0 \tag{19}$$

and then boundary condition:

$$\frac{\partial \omega}{\partial \nu} = 0 \quad on \quad \Gamma,$$

$\omega(x,t) \to 0$ as $t \to +\infty$ and $\omega = at + b + O\left(e^{ht}\right)$ as $t \to -\infty$.

If we prove $\omega \equiv 0$, the statement of the theorem will follow. Show that $a = 0$, $b = 0$. Suppose $a > 0$. So, $\omega(x,t) < 0$ for $t < -T_1$, where $T_1$is a rather large positive number. Prove that $\omega < 0$ for $t > -T_1$. Since $q(x,t) = |u|^{\sigma-1}\, sign\ u$ as $t \geq t_0 + 1$, then $q(x,t) = O\left(t^{-2}\right)$ as $t \to +\infty$.

Denote $l = \max\limits_{t=T} \omega(x,t)$, and $W(x,t) = (\omega - l)^+$, where T is a rather large positive number. Obviously, $W(x,t) = 0$ for $t = -T_1$, $t = T$ and

$$W(x,t) \in \overset{\circ}{W}{}_2^1\left(Q_{T_1,T}\right).$$

In the definition of the solution, as a test function take $\varphi(x,t) = W(x,t) \cdot k(x)$, where $k(x)$ is a positive solution of problem (3), (4) such that $0 < m_1 \leq k(x) \leq m_2$.

Then from the definition of the solution we have:

$$\int\limits_{A_l^+} |\omega_t|^2\, kdxdt + \nu_1 \int\limits_{A_l^+} |\nabla \omega|^2\, kdxdt \leq -\int\limits_{A_l^+} q(x,t) \cdot k \cdot \omega \cdot (\omega - l)^+\, dxdt, \tag{20}$$

where $A_l^+ = \{(x,t), W > 0\}$. Here we used

$$-\sum_{i,j=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_{ij} \frac{\partial \omega}{\partial x_j} \frac{\partial k}{\partial x_i} W dxdt + \sum_{i=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_i \frac{\partial \omega}{\partial x_i} \cdot W \cdot kdxdt =$$

$$= -\frac{1}{2} \sum_{i,j=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_{ij} \frac{\partial \omega^2}{\partial x_j} \frac{\partial k}{\partial x_i} dxdt + \frac{1}{2} \sum_{i=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_i \frac{\partial \omega^2}{\partial x_i} kdxdt = 0.$$

Estimate the first part of (20) using the inequality [see. 4]

$$\|u\|_{\frac{2n}{n-2}} \leq C \, \|\nabla u\|_{2,\Omega}, \tag{21}$$

where C is a constant dependent on dimension of n.

Then

$$-\int\limits_{A_l^+} q(x,t)\, \omega\, (\omega - l)^+\, kdxdt \leq \int\limits_{A_l^+} |q(x,t)|\, (\omega - l + l)\, (\omega - l)\, kdxdt =$$

$$= \int\limits_{A_l^+} |q(x,t)| \cdot |\omega - l|^2 \, k dx dt + l \cdot \int\limits_{A_l^+} |q(x,t)| \cdot |\omega - l| \, k dx dt \le$$

$$\le m_2 \int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - l|^2 \, dx dt + l \cdot m_2 \int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - l| \, dx dt. \qquad (22)$$

At first estimate the first summand:

$$F_1 = \int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - l|^2 \, dx dt \le$$

$$\le \left(\int\limits_{\substack{A_l^+ \\ t>t_0}} |\omega - l|^{\frac{2(n+1)}{n-1}} \, dx dt \right)^{\frac{n-1}{n+1}} \left(\int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t|^{\frac{n+1}{2}} \, dx dt \right)^{\frac{2}{n+1}} \le$$

$$\le \left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\omega - l|^{\frac{2(n+1)}{n-1}} \, dx dt \right)^{\frac{n-1}{n+1}} \left(\int\limits_{A_l^+ \cap \{t>t_0\}} |q(x,t|^{\frac{n+1}{2}} \, dx dt \right)^{\frac{2}{n+1}} \le$$

$$\le \left[\left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\omega - k|^{\frac{2(n+1)}{n-1}} \, dx dt \right)^{\frac{n-1}{n+1}} \right] \left(\int\limits_{A_l^+ \cap \{t>t_0\}} |q(x,t|^{\frac{n+1}{2}} \, dx dt \right)^{\frac{2}{n+1}} \le$$

$$\le C \cdot \left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\nabla (\omega - l)|^2 \, dx dt \right) \cdot I_2, \qquad (23)$$

where

$$I_2 = \left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |q(x,t)|^{\frac{n+1}{2}} \, dx dt \right)^{\frac{2}{n+1}}.$$

Then estimate I_2.

$$I_2 = \left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |q(x,t)|^{\frac{n+1}{2}} \, dx dt \right)^{\frac{2}{n+1}} \le I_2 = C_1 \cdot \left(\int\limits_{A_l^+ \cap \{t>t_0\}} t^{-(n+1)} dx dt \right)^{\frac{2}{n+1}} \le$$

$$\le C_1 \cdot \left(\int_{t_0}^{T} t^{-(n+1)} dx dt \right)^{\frac{2}{n+1}} \le C_2 \cdot \left(\left. \frac{t^{-n}}{-n} \right|_{t_0}^{T} \right)^{\frac{2}{n+1}} =$$

$$= C_2 \cdot \left(\frac{T^{-n}}{-n} + \frac{t_0^{-n}}{n} \right)^{\frac{2}{n+1}} = C_3 \cdot \left(t_0^{-n} - T^{-n} \right)^{\frac{2}{n+1}}.$$

take t_0 so that $|u(x,t)| < \varepsilon$ and $C_3 \cdot t_0^{-\frac{2n}{n+1}} < \frac{m_1 \nu_1}{4Cm_2}$. Then we get

$$I_2 \leq \frac{m_1 \nu_1}{4C \cdot m_2}.$$

From (23) it follows that

$$F_1 \leq \frac{\nu_1 m_1}{4m_2} \cdot \int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\nabla (\omega - l)|^2 \, dxdt. \tag{24}$$

Estimate the second summand of the right hand side of (22).

$$F_2 = l \cdot \int\limits_{A_l^+} |q(x,t)| \cdot |\omega - l| \, dxdt \leq$$

$$\leq l \cdot \left(\int\limits_{A_l^+} |q(x,t)|^{p_1} \, dxdt \right)^{\frac{1}{p_1}} \left(\int\limits_{A_l^+} |\omega - l|^{\frac{2(n+1)}{n-1}} \, dxdt \right)^{\frac{n-1}{2(n+1)}} \leq$$

$$\leq l \cdot C_1 \left(\int\limits_{\substack{t > t_0 \\ A_l^+}} t^{-2p_1} dt \right)^{\frac{1}{p_1}} \left(\int\limits_{A_l^+} |\nabla (\omega - l)|^2 \, dxdt \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\nu_1 m_1}{4m_2} \int\limits_{A_l^+} |\nabla (\omega - l)|^2 \, dxdt + l^2 \cdot C_2 \left(\int\limits_{\substack{t > t_0 \\ A_l^+}} t^{-2p_1} dt \right)^{\frac{1}{p_1}}, \tag{25}$$

here $\frac{1}{p_1} + \frac{n-1}{2(n+1)} = 1$.

Hence $p_1 = 1 + \frac{n-1}{n+3}$.

Combining (24) and (26), we get

$$m_1 \int\limits_{A_l^+} |\omega_t|^2 \, dxdt + m_1 \nu_1 \int\limits_{A_l^+} |\nabla \omega|^2 \, dxdt = \frac{m_1}{2} \int\limits_{A_l^+} |\omega_t|^2 \, dxdt +$$

$$+ \frac{m_1 \nu_1}{2} \int\limits_{A_l^+} |\nabla \omega|^2 \, dxdt + l^2 \cdot C_2 \left(\int_{t_0}^{T} t^{-2p_1} dt \right)^{2/p_1}.$$

As a result, for $n > 1$ we have:

$$\frac{m_1}{2}\int\limits_{A_l^+} |\omega_t|^2\, dxdt + \frac{m_1\nu_1}{2}\cdot \int\limits_{A_l^+} |\nabla\omega|^2\, dxdt \le l^2 \cdot C\left(\int\limits_{t_0}^{T} t^{-2p_1} dt\right)^{2/p_1}. \quad (26)$$

From the fact that $l(T) \to 0$ as $T \to 0$ and from the convergence of the integral $\int\limits_{t_0}^{T} t^{-2p_1} dt$ we get $A_l^+ = 0$.

So, $\omega - l \le 0$. Since l tends to zero, then $\omega < 0$.

In the similar way we can prove that if $a < 0$ then $\omega(x,t) > 0$.

Show that $a = b = 0$. Assume $a > 0$. So $\omega(x,t) < 0$ for $t > t_1$. The function $\omega_1 = -t^\beta$ will be a supersolution of equation (19) for negative β rather large in modulus.

Really,

$$\omega_{1tt} + L\omega_1 - q(x,t)\,\omega_1 = -\beta(\beta-1)\,t^{\beta-2} + q(x,t)\,t^\beta = -t^{\beta-2}\left(\beta(\beta-1) - qt^{-2}\right) < 0.$$

Let t_0 be rather great. Choose A such a small positive number that $-At_2^\beta \ge \omega(x,t_2)$. Then from $W = \omega(x,t_2) + At_2^\beta \le 0$, $\omega(x,t) + At^\beta \to 0$ as $t \to +\infty$ and

$$W_{tt} + LW - q(x,t)\,W \ge 0,$$

as above we can prove that

$$\omega(x,t) + At^\beta \le 0 \quad for \quad t \ge t_2.$$

Consider the point set where $v = u < 0$, for them we have

$$-A\cdot t^\beta \ge \omega(x,t) \ge v_1 \ge -C_1 e^{-ht}.$$

This contradiction proves that a may not be positive. In the similar way we can show that a may not be negative, and that $b = 0$. Since $\omega \to 0$ as $t \to \pm\infty$, consequently, $\omega \equiv 0$.

References

[1]. Kondratiev V.A., Oleinik O. A. *Some results for nonlinear elliptic equation in cylindrical domains* // Operator Theory: Advances and Applications. 1992, vol. 57, pp. 185-195.

[2]. Kondratiev V.A *On positive solutions to weakly linear elliptic equations of second order in cylindrical domains.* // Trudy mat. Institut im. V.A. Steklova, 2005, vol. 250, pp. 183-191 (Russian).

[3]. Khachlaiv T. S. *Asymptotics of solutions of a semi-linear elliptic equation satisfying the Neumann condition on lateral surface of cylindrical domain.* // Trudy seminara im. I.G. Petrovskogo, 2004, issue 24, pp. 304-324 (Russian).

[4]. Ladyzhenskaya O.A., Uraltseva N.N. *Linear and quasilinear equations of elliptic type.* M.: Nauka, 1973.

[5]. Gilbarg D., Trudinger N. *Elliptic partial differential equations of second order.* M.: Nauka, 1989 (Russian).

[6]. Agmon S., Nirenberg L. *Properties of solutions of ordinary differential equations in BAnach space* // Comm. Pure Appl. Math. 1963. 16, pp. 121-239.

[7]. Pazy A. *Archive Rat'e Mech.* And Anal. 1967, vol. 24, pp.193-218.

Shirmayil G. Bagirov
Baku State University,
23, Z.I. Khalilov str., AZ 1148, Baku, Azerbaijan
Tel.: (99412) 539 47 20 (off).

Received October 04, 2010; Revised December 16, 2010