Transactions of NAS of Azerbaijan, 2011, vol. XXXI, No 1, pp. 3-6. 3

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ASYMPTOTIC BEHAVIOR OF EIGEN VALUES OF A BOUNDARY VALUE PROBLEM FOR LAPLACE EQUATION

Abstract

Asymptotic behavior of eigen values of a boundary value problem for the Laplace equation in the square is studied in the paper.

In the given paper we investigate the asymptotic behavior of eigen values of the following boundary value problems in the square $\Omega = [0, 2\pi] \times [0, 2\pi]$

$$-\frac{\partial u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} = \lambda u(x,y), \tag{1}$$

$$\frac{\partial u(2\pi,y)}{\partial x} + i\frac{\partial u(0,y)}{\partial y} = 0,$$

$$\frac{\partial u(0,y)}{\partial x} = 0, \quad y \in [0, 2\pi] \tag{2}$$

$$u(x,0) = u(x,2\pi), \qquad \frac{\partial u(x,0)}{\partial y} = \frac{\partial u(x,2\pi)}{\partial y}, \quad x \in [0,2\pi] \tag{3}$$

The asymptotic formula for eigen values of problem (1)-(3) is found.

Note that if instead of condition (2) the condition

$$\frac{\partial u(2\pi,y)}{\partial x} + i\frac{\partial u(2\pi,y)}{\partial y} = 0, \quad u(0,y) = 0, \quad y \in [0,2\pi], \tag{2'}$$

is considered, then the behavior of eigen values of boundary value problems (1), (2′), (3) was investigated in the paper of S. Ya. Yakubov [1]. In [1], it is proved that fore problem (1), (2′), (3), there exists a sequence of eigen values λ_k converging to zero.

In the papers of V. A. Il'in, A.F. Filippov [2] and M.M. Hechtman [3] the form of boundary conditions are found for the Laplace equation in the square when the classic situation (for example, asymptotic behavior of eigen values) violates. In these papers, the boundary conditions contain non-differential operators as well.

Theorem. *The following asymptotic formula holds for the eigen values of problem (1)-(3)* $\lambda_{n,k} \sim k^2 + \left(n + \frac{1}{2}\right)^2$.

Proof. In the Hilbert space $L_2(0,2\pi)$ consider the operators A and B that are defined by the following equalities

$$D(A) = W_2^2\left((0,2\pi);\; u(0) = u(2\pi),\; u'(0) = u'(2\pi)\right), \quad Au = -\frac{d^2u}{dy^2};$$

$$D(B) = W_2^1\left((0,2\pi);\; u(0) = u(2\pi)\right), \quad Bu = i\frac{du}{dy}.$$

As it was noted in [1], the eigen values of the operator A are the numbers $\mu_k(A) = k^2$, $k = 0, 1, 2, ...$, for $k > 0$. The pair of eigen functions

$$u_{k1}(y) = \frac{1}{\sqrt{2\pi}} e^{iky}, \quad u_{k2}(y) = \frac{1}{\sqrt{2\pi}} e^{-iky}, \quad k = 1, 2, ...$$

correspond to these numbers.

The eigen functions $u_{k1}(y)=\frac{1}{\sqrt{2\pi}}e^{iky}$, $k=1,2,...$, correspond to the eigen values $\mu_k(B)=-k$, $k=0,1,2,...$, the eigen functions $u_{k2}(y)=\frac{1}{\sqrt{2\pi}}e^{-iky}$, $k=1,2,...$ correspond to the eigen values $\mu_{-k}(B)=k$, $k=1,2,....$

Then, problem (1)-(3) in $L_2(0,2\pi)$ is reduced to the boundary value problem for a second order elliptic differential-operator equation with the operator boundary conditions

$$-u''(x)+Au(x)=\lambda u(x), \quad x\in(0,2\pi) \tag{4}$$

$$\begin{gathered} u'(2\pi)+Bu(0)=0, \\ u'(0)=0, \end{gathered} \tag{5}$$

where $u(x)$ is a vector-functions with the values in $L_2(0,2\pi)$.

It is known that the system of functions $\left\{\frac{1}{\sqrt{2\pi}}e^{iky}\right\}$, $k=0,\pm1,\pm2,...$ forms an orthonormed base in $L_2(0,2\pi)$, i.e. any function $u(x)\in L_2(0,2\pi)$ expands in series

$$u(x)=\sum_{k=1}^{\infty}\left[(u(x),u_{k1})u_{k1}+(u(x),u_{k2})u_{k2}\right],$$

where $\widetilde{u}_{k1}(x)=(u,u_{k1})$ and $\widetilde{u}_{k2}(x)=(u,u_{k2})$ are the Fourier coefficients.

Then the following spectral expansions hold for the operators A and B

$$Au=\sum_{k=1}^{\infty}k^2\left[(u(x),u_{k1})u_{k1}+(u(x),u_{k2})u_{k2}\right], \quad u\in D \tag{A}$$

$$Bu(x)=\sum_{k=1}^{\infty}\left[-k(u(x),u_{k1})u_{k1}+k(u(x),u_{k2})u_{k2}\right], \quad u\in D \qquad (B).$$

Considering the above mentioned spectral expansions, we can rewrite problem (4) – (5) in the form

$$\begin{cases} -\left[\sum\limits_{k=1}^{\infty}(u'',u_{k1})u_{k1}+(u'',u_{k2})u_{k2}\right]+\sum\limits_{k=1}^{\infty}k^2\left[(u,u_{k1})u_{k1}+(u,u_{k2})u_{k2}\right]= \\ \qquad =\lambda\sum\limits_{k=1}^{\infty}\left[(u,u_{k1})u_{k1}+(u,u_{k2})u_{k2}\right], \\ \sum\limits_{k=1}^{\infty}\left[(u'(2\pi),u_{k1})u_{k1}+(u'(2\pi),u_{k2})u_{k2}\right]+ \\ \qquad +\sum\limits_{k=1}^{\infty}\left[-k(u(0),u_{k1})u_{k1}+k(u(0),u_{k2})u_{k2}\right]=0, \\ \sum\limits_{k=1}^{\infty}\left[(u'(0),u_{k1})u_{k1}+(u'(0),u_{k2})u_{k2}\right]=0 \end{cases}$$

For the coefficients $\widetilde{u}_{k1}(x)=(u(x),u_{k1})$ we get the following boundary value problem

$$-\widetilde{u}''_{k1}+k^2\widetilde{u}_{k1}(x)=\lambda\widetilde{u}_{k1}(x), \quad x\in(0,2\pi) \tag{6}$$

$$\begin{gathered} \widetilde{u}'_{k1}(2\pi)-k\widetilde{u}_{k1}(0)=0, \\ \widetilde{u}'_{k1}(0)=0, \end{gathered} \tag{7}$$

for the coefficients $\widetilde{u}_{k2}(x)=(u(x),u_{k2})$ the following boundary value problem

$$-\widetilde{u}''_{k2}+k^2\widetilde{u}_{k2}(x)=\lambda\widetilde{u}_{k2}(x), \quad x\in(0,2\pi) \tag{8}$$

$$\begin{aligned}\widetilde{u}'_{k2}(2\pi)+k\widetilde{u}_{k2}(0)=0,\\ \widetilde{u}'_{k2}(0)=0\end{aligned} \tag{9}$$

Thus, the finding of the eigen values of problems (4)-(5) is reduced to the finding of the eigen values of boundary value problems (6)-(7) and (8)-(9).

At first find the eigen values of boundary value problems (6)-(7). The general solution of ordinary differential equation (6) is of the form

$$\widetilde{u}_{k1}(x)=c_1e^{-x\sqrt{k^2-\lambda}}+c_2e^{-(2\pi-x)\sqrt{k^2-\lambda}} \tag{10}$$

where $c_i\,(i=1,2)$ are arbitrary constants.

Substituting (10) in (7), we get the system with respect to $c_i\,(i=1,2)$ whose determinant is of the form

$$K(\lambda)=-\sqrt{k^2-\lambda}e^{-2\pi\sqrt{k^2-\lambda}}\left(\sqrt{k^2-\lambda}e^{-2\pi\sqrt{k^2-\lambda}}+k\right)+$$

$$+\sqrt{k^2-\lambda}\left(\sqrt{k^2-\lambda}+ke^{-2\pi\sqrt{k^2-\lambda}}\right).$$

The eigen values of boundary value problem (6)-(7) consists of those real $\lambda\neq k^2$ that even if for one k^2 satisfy the equation

$$K(\lambda)=0. \tag{11}$$

Equation (11) is equivalent to the equation

$$e^{4\pi\sqrt{k^2-\lambda}}-1=0 \tag{12}$$

that for $\lambda<k^2$ has no solutions, i.e. boundary value problem (6)-(7) has no eigen values for $\lambda<k^2$. Let $\lambda>k^2$. Assume $\sqrt{\lambda-k^2}=y$. Then, equation (12) is represented in the form

$$e^{4\pi iy}=e^{2\pi ni},\quad y\in(0,+\infty).$$

Hence, $y=\frac{n}{2}$, $n=1,2,...$. Then for the values of (6)-(7) we get the formula

$$\lambda^{(1)}_{n,k}=k^2+\left(\frac{n}{2}\right)^2,\quad k,n=1,2,...$$

The eigen values of boundary value problems (8)-(9) are found it the same way.

The eigen values of boundary value problem (8)-(9) consists of those $\lambda\neq k^2$ that even if for one k^2 satisfy the equation

$$2k+\sqrt{k^2-\lambda}\left(e^{2\pi\sqrt{k^2-\lambda}}-e^{-2\pi\sqrt{k^2-\lambda}}\right)=0. \tag{13}$$

Let $\lambda<k^2$. Denote $\sqrt{k^2-\lambda}=y,\ \ 0<y<k$. Then, equation (13) takes the form

$$k+y\ sh\ 2\pi y=0,\quad y\in(0,k). \tag{14}$$

Obviously, for each fixed $k=1,2,...$ the function $\varphi_k(y)=ky\ sh\ 2\pi y$ in the interval $(0,k)$ is continuous and positive. Therefore, equation (14) has no solutions on $(0,k)$ for any k, and by the same token boundary value problem (8)-(9) has no eigen values satisfying the condition $\lambda<k^2$.

Now let $\lambda > k^2$. Assume $\sqrt{\lambda - k^2} = z$, $0 < z < +\infty$. Then, equation (13) takes the form

$$2k + ize^{2\pi iz} - e^{-2\pi iz} = 0. \tag{15}$$

Rewrite equation (15) in the form

$$\sin 2\pi z = \frac{k}{z}, \quad z \in (0, +\infty), \quad k = 1, 2, ... \tag{16}$$

Obviously, the abscissa of the intersection points of the curves $f(z) = \sin 2\pi z$, $z \in (0, +\infty)$ and $\psi_k(z) = \frac{k}{z}$ will be the desired roots of equation (16). The function $2\pi z$ vanishes at the points $z_n = \frac{n}{2}\,(n = 1, 2, ...)$ and is positive in the intervals $\left(n, n + \frac{1}{2}\right)$. On the other hand, for each $k = 1, 2, ...$ the function $\psi_k(z) = \frac{k}{z}$ is continuous and decreases on $(0, +\infty)$, and the positive semi-axis of the abscissa axis is the horizontal asymptote of the function $\psi_k(z)$. Therefore, for the roots of equation (16) we can approximately take the abscissa of the intersection points of the curves $f(z) = \sin 2\pi z$ and $\psi_k(z) = \frac{k}{z}$.

Denote these points by $z_{n,k} : n < z_{n,k} < n + \frac{1}{2}\,(n = 1, 2, ...)$. Obviously, for each fixed n, while increasing k, the z_{nk} will approach to the right end of the interval $\left(n, n + \frac{1}{2}\right)$, i.e. $z_{n,k} \sim n + \frac{1}{2}$. Hence, for the eigen values we get the asymptotic formula

$$\lambda_{n,k}^{(2)} \sim k^2 + \left(n + \frac{1}{2}\right)^2, \quad k = 1, 2, ... \quad n = N_0, N_0 + 1, ...,$$

where N_0 is some rather large natural number.

Obviously, $\lambda_{n,k}^{(2)}$ is a subsequence of $\lambda_{n,k}^{(1)}$, therefore, we can affirm that the following asymptotic formula

$$\lambda_{n,k} \sim k^2 + \left(n + \frac{1}{2}\right)^2, \quad k = 1, 2, ... \quad n = N_0, \quad N_0 + 1, ... \,.$$

holds for the eigen values of problem (1)-(3).

The theorem is proved.

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan- Grant No EIF - 2011-1(3)-82/17/1.

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Received October 04, 2010; Revised December 12, 2010.