Mahammad-Rza I. ARAZM, Vafa A. MAMEDOVA, Zeynab S. GAFAROVA

ON REMOVABLE SETS OF SOLUTIONS OF NEUMANN PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS OF DIVERGENT FORM

Abstract

In this paper we consider a nondivergent elliptic equation of second order whose leading coefficients are from some weight space. The sufficient condition of removability of a compact with respect to this equation in the weight space of Hölder functions was found.

Let D be a bounded domain situated in n-dimensional Euclidean space E_n of the points $x=(x_1,...,x_n)$, $n\geq 3$, ∂D be its boundary. Consider in D the following elliptic equation

$$\begin{cases} Lu=\sum\limits_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum\limits_{i=1}^{n}b_i(x)u_{x_i}+c(x)u+b(x,u,\nabla u)=0 \\ u|_{\partial D\backslash E}=0 \end{cases} \tag{1}$$

in supposition that $\|a_{ij}(x)\|$ is a real symmetric matrix, moreover

$$\gamma|\xi|^2\omega(x)\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq\gamma^{-1}\omega(x)|\xi|^2;\ \xi\in E_n,\ x\in D, \tag{2}$$

$$a_{ij}(x)\in C^1_\omega\left(\overline{D}\right);\ i,j,1,...,n, \tag{3}$$

$$|b_i(x)|\leq b_0;\ -b_0\leq c(x)\leq 0;\ i=1,...,n;\ x\in D. \tag{4}$$

$$|b(x,u,\nabla u)|\leq g(u)\omega(x)|\nabla u|,\ \int\limits_0^a g(u)\,du<\infty,\ a<\infty. \tag{5}$$

Here $g(x)$ is non-negative function from u, $u_i=\dfrac{\partial u}{\partial x_i}$, $u_{ij}=\dfrac{\partial^2 u}{\partial x_i\partial x_j}$; $i,j=1,...,n$; $\gamma\in(0,1]$ and $b_0\geq 0$ are constants. Besides we'll assume that the minor coefficients of the operator L are measurable in D. Let $\lambda\in(0,1)$ be some number.

The compact $E\subset\overline{D}$ is called removable with respect to the equation (1) in the space $C^\lambda_\omega(D)$ if from

$$Lu=0,\ x\in D\backslash E;\ u|_{\partial D\backslash E}=0;\ u(x)\in C^\lambda_\omega(D) \tag{6}$$

it follows that $u(x)\equiv 0$ in D.

The aim of the given paper is finding sufficient condition of removability of a compact with respect to the equation (1) in the space $C^\lambda_\omega(D)$. This problem have been investigated by many researchers. For the Laplace equation the corresponding result was found by L. Carleson [1]. Concerning the second order elliptic equations of divergent structure, we show in this direction the papers [2], [3]. For a class of non-divergent elliptic equations of the second order with discontinuous coefficients the removability condition for a compact in the space $C^{\lambda}(D)$ was found in [4]. Mention also papers [5-7] in which the conditions of removability for a compact in the space of continuous functions have been obtained.The removable sets of solutions of the second order elliptic and parabolic equations in nondivergent form were considered in [10]-[12]. In [13], T. Kilpelainen and X. Zhong have studied the divergent quasilinear equation without minor members, proved the removability of a compact. Removable sets for pointwise solutions of elliptic partial differential equations were found by J. Diederich [14]. Removable singularities of solutions of linear partial differential equations were considered in R. Harvey, J. Polking paper [15]. Removable sets at the boundary for subharmonic functions have been investigated by B. Dahlberg [16]. Also we mentioned the papers of A.V.Pokrovskii [17], [18].

Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x : |x-z| < R\}$ and the sphere $\{x : |x-z| = R\}$ of radius R with the center at the point $z \in E_n$ respectively. We'll need the following generalization of mean value theorem belonging to E.M. Landis and M.L. Gerver [8] in weight case.

Lemma. *Let the domain D be situated between the spheres $S_R(0)$ and $S_{2R}(0)$, moreover the intersection $\partial D \cap \{x : R < |x| < 2R\}$ be a smooth surface. Further, let in $\overline{D}$ the uniformly positive definite matrix $\|a_{ij}(x)\|$; $i,j = 1,...,n$ and the function $u(x) \in C^2(D) \cap C^1_{\omega}(\overline{D})$ be given. Then there exists the piece-wise smooth surface Σ dividing in G the spheres $S_R(0)$ and $S_{2R}(0)$ such that*

$$\int_{\Sigma} \omega \left|\frac{\partial u}{\partial \nu}\right| ds \leq K \underset{D}{osc}\, u \cdot \frac{\omega(D)}{R^2}.$$

Here $K > 0$ is a constant depending only on the matrix $\|a_{ij}(x)\|$ and n, $\frac{\partial u}{\partial \nu}$ is a derivative by a conormal determined by the equality

$$\frac{\partial u(x)}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \cos(\bar{n}, x_j)^{\frac{1}{2}},$$

where $\cos(\bar{n}, x_j)$; $j = 1,...,n$ are direction cosines of a unit external normal vector to Σ.

Theorem 1. *Let D be a bounded domain in $\mathbb{E}_n$, $E \subset \overline{D}$ be a compact. If with respect to the coefficients of the operator L the conditions (2)-(5) are fulfilled, then for removability of the compact E with respect to the equation (1) in the space $C^{\lambda}_{\omega}(D)$ it sufficies that*

$$m_H^{n-2+\lambda}(E) = 0. \quad (7)$$

Proof. At first we show that without loss of generality we can suppose the condition $\partial D \in C^1$ is fulfilled. Suppose, that the condition (7) provides the removability of the compact E for the domains, whose boundary is the surface of the class C^1, but $\partial D \in C^1$ and by fulfilling (7) the compact E is not removable. Then the problem (6) has non-trivial solution $u(x)$, moreover $u|_E = f(x)$ and $f(x) \neq 0$. We always can suppose the lowest coefficients of the operator L are infinitely differentiable in D. Moreover, without loss of generality, we'll suppose that the coefficients of the operator L are extended to a ball $B \supset \overline{D}$ with saving the conditions (2)-(5). Let $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \min\{f(x), 0\}$, and $u^{\pm}(x)$ be generalized by Wiener (see [8]) solutions of the boundary value problems

$$Lu^{\pm} = 0,\ x \in D\backslash E;\ u^{\pm}\big|_{\partial D\backslash E} = 0;\ u^{\pm}\big|_E = f^{\pm}.$$

Evidently, by $u(x) = u^+(x) + u^-(x)$. Further, let D' be such a domain, that $\partial D' \in C^1$, $\overline{D} \subset D'$, $\overline{D'} \subset B$, and $\vartheta^{\pm}(x)$ be solutions of the problems

$$L\vartheta^{\pm} = 0,\ x \in D'\backslash E;\ \vartheta^{\pm}\big|_{\partial D'} = 0;\ \vartheta^{\pm}\big|_E = f^{\pm};\ \vartheta^{\pm}(x) \in C_{\omega}^{\lambda}(D').$$

By the maximum principle for $x \in D$

$$0 \le u^+(x) \le \vartheta^+(x),\ \vartheta^-(x) \le u^-(x) \le 0.$$

But according to our supposition $\vartheta^+(x) \equiv \vartheta^-(x) \equiv 0$. Hence, it follows, that $u(x) \equiv 0$. So, we'll suppose that $\partial D \in C^1$. Now, let $u(x)$ be a solution of the problem (6), and the condition (7) be fulfilled. Give an arbitrary $\varepsilon > 0$. Then there exists a sufficiently small positive number δ and a system of the balls $\{B_{r_k}(x^k)\}$, $k = 1, 2, ...,$ such that $r_k < \delta$, $E \subset \bigcup\limits_{k=1}^{\infty} B_{r_k}(x^k)$ and

$$\sum_{k=1}^{\infty} r_k^{n-2+\lambda} < \varepsilon. \tag{8}$$

Consider a system of the spheres $\{B_{2r_k}(x^k)\}$, and let $D_k = D \cap B_{2r_k}(x^k)$, $k = 1, 2, ...,$. Without loss of generality we can suppose that the cover $\{B_{2r_k}(x^k)\}$ has a finite multiplicity $a_0(n)$. By lemma for every k there exists a piece-wise smooth surface Σ_k dividing in D_k the spheres $S_{r_k}(x^k)$ and $S_{2r_k}(x^k)$, such that

$$\int_{\Sigma_k} \omega \left|\frac{\partial u}{\partial \nu}\right| ds \le K \underset{D_k}{osc} u \frac{\omega(D_k)}{r_k^2}. \tag{9}$$

Since $u(x) \in C_{\omega}^{\lambda}(D)$, there exists a constant $H_1 > 0$ depending only on the function $u(x)$ such that

$$\underset{D_k}{osc\omega} u \le H_1 (2r_k)^{\lambda}. \tag{10}$$

Besides,

$$\omega(D_k) \le mes_n B_{2r_k}\left(x^k\right) = \Omega_n 2^n r_k^n;\ k = 1, 2, ..., \tag{11}$$

where $\Omega_n = mes_n B_1(0)$. Using (10)-(11) in (9), we get

$$\int_{\Sigma_k} \omega \left|\frac{\partial u}{\partial \nu}\right| ds \le C_1 r_k^{n-2+\lambda};\ k = 1, 2, ..., \tag{12}$$

where $C_1 = KH_1 2^{n+\lambda}$.

Let D_{Σ} be an open set situated in $D\backslash E$ whose boundary consists of unification of Σ and Γ, where $\Sigma = \bigcup\limits_{k=1}^{\infty} \Sigma_k$, $\Gamma = \partial D \backslash \bigcup\limits_{k=1}^{\infty} D_k^+$, D_k^+ is a part of D_k remaining after the removing of points situated between Σ and $S_{2r_k}\left(x^k\right);\ k=1,2,....$ Denote by D'_Σ the arbitrary connected component D_Σ, and by $\mathcal{M}$ we denote the elliptic operator of divergent structure

$$\mathcal{M}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\left(x\right)\frac{\partial}{\partial x_j}\right).$$

According to Green formula for any functions $z\left(x\right)$ and $W\left(x\right)$ belonging to the intersection $C^2\left(D'_\Sigma\right)\cap C^1\left(\overline{D}'_\Sigma\right)$, we have

$$\int\limits_{D'_\Sigma}\left(z\mathcal{M}\beta-\beta\mathcal{M}z\right)dx=\int\limits_{\partial D'_\Sigma}\left(z\frac{\partial\beta}{\partial\nu}-\beta\frac{\partial z}{\partial\nu}\right)ds. \tag{13}$$

Since $\partial D\in C^1$, then $\mathrm{u}\left(x\right)\in C^1\left(D'_\Sigma\right)\cap C^1\left(\overline{D'_\Sigma}\right)\left(x\right)\in C^1\left(\overline{D}_{\Sigma'}\right)$ (see [9]). From (13) choosing the functions $z=1,\ \beta=\omega u^2$ we have

$$\int\limits_{D'_\Sigma}\mathcal{M}\left(\omega u^2\right)dx=2\int\limits_{\partial D'_\Sigma}\omega u\frac{\partial u}{\partial\nu}ds+\int\limits_{\partial D_\Sigma}\omega_{x_i}u^2ds.$$

But $\left|u\left(x\right)\right|\leq M<\infty$ for $x\in\overline{D}$. Let's assume that the condition

$$\omega_{x_i}<c\omega. \tag{*}$$

is fulfilled. By virtue of condition (*) and $\int\limits_{\partial D_\Sigma}\omega u^2ds<C_3M\varepsilon$, subject to (12) and (8) we conclude

$$\int\limits_{D'_\Sigma}\mathcal{M}\left(\omega u^2\right)dx\leq 2Ma_0\sum_{k=1}^{\infty}\int\limits_{\Sigma_k}\omega\left|\frac{\partial u}{\partial\nu}\right|ds+\int\limits_{D'_\Sigma}\omega u^2dx\leq$$

$$\leq 2Ma_0C_1\sum_{k=1}^{\infty}r_k^{n-2+\alpha}+\varepsilon MC_2<C_3\varepsilon, \tag{14}$$

where $C_3=2Ma_0C_1$.

On the other hand

$$\mathcal{M}\left(\omega u^2\right)=6u\omega\mathcal{M}\left(u\right)+2\sum_{i,j=1}^{n}\omega a_{ij}u_iu_j+\left(2u+1\right)\sum_{i,j=1}^{n}a_{ij}u_{x_j}\omega_{x_i}+$$

$$+\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_i}u\omega_{x_j}+\sum_{i,j=1}^{n}a_{ij}u\omega_{x_ix_j}$$

and besides,

$$\mathcal{M}u=\mathcal{L}u-\sum_{i=1}^{n}d_i\left(x\right)u_i+c\left(x\right)u-b\left(x,u,\nabla u\right),$$

where

$$d_i(x) = \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} - b_i(x), \ i = 1, ..., n.$$

It is evident that by virtue of conditions (3)-(4) $|d_i(x)| \le d_0 < \infty; \ \ i = 1, ...n$. Thus, from (13) we obtain

$$6\int_{D'_\Sigma} u\omega \sum_{i=1}^{n} d_i(x) u_i dx - 6\int_{D'_\Sigma} u^2 c(x) dx + 2\int_{D'_\Sigma} \sum_{i,j=1}^{n} \omega(x) a_{ij} u_i u_j dx +$$

$$+ (2u+1)\int_{D'_\Sigma} \sum_{i,j=1}^{n} a_{ij} u_j \omega_{x_i} dx + \int_{D'_\Sigma} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} u\omega_{x_i} dx + |\nabla u|^2 dx +$$

$$+ \int_{D'_\Sigma} \sum_{i,j=1}^{n} a_{ij} u \omega_{x_i x_j} dx + b(x, u, \nabla u) < C_3 \varepsilon.$$

Let's estimate the nonlinear member on the right part applying the inequality

$$\int_{D'_\Sigma} b(x, u, \nabla u) dx \le \int_{D'_\Sigma} g(x) \omega(x) |\nabla u| dx \le \frac{1}{2\alpha}\int_{D'_\Sigma} g^2(u) dx + \int_{D'_\Sigma} \omega^2(x) |\nabla u|^2 dx.$$

Hence, for any $\alpha > 0$ applying Cauchy inequality we have

$$2\gamma\int_{D'_\Sigma} \omega |\nabla u|^2 dx < 6d_0\int_{D'_\Sigma} \omega |u| |u_i| dx + 6\int_{D'_\Sigma} u^2 \omega(x) + (2u+1)\int_{D'_\Sigma} a_{ij} u_j \omega_{x_i} dx +$$

$$+ d_0\int_{D'_\Sigma} u\omega_{x_i}^2 dx + \int_{D'_\Sigma} a_{ij} u \omega_{x_i x_j} + C_3\varepsilon \le 6\frac{d_0}{\varepsilon}\int_{D'_\Sigma} |u|^2 dx + 6\frac{d_0\varepsilon}{2}\int_{D'_\Sigma} \omega^2 |\nabla u|^2 dx +$$

$$+ (2n+1)\int_{D'_\Sigma} u_j \omega dx + d_0\int_{D'_\Sigma} u\omega dx + \gamma C_4 \varepsilon \le 6\frac{d_0}{\varepsilon} M mes_n D +$$

$$+ \frac{(2M+1)\gamma}{\varepsilon} mes_n D + d_0 M\omega(D) + \gamma C_4 M \omega(D) + C_3\varepsilon. \qquad (15)$$

If we'll take into account that

$$\left|\omega_{x_i x_j}\right| < C_4 \omega(x),$$

then from here we have that

$$\int_{D'_\Sigma} \omega^2 |\nabla u|^2 dx \le C_5,$$

where $C_5 = (6d_0 + (2M+1)) M mes_n D + (d_0 M + \gamma C_4 M) \omega(D) + \dfrac{C_3}{\gamma}$. Without loss of generality we assume that $\varepsilon \le 1$. Hence we have

$$\int_D \omega^2 |\nabla u|^2 dx \le C_6.$$

Thus $u(x) \in W^1_{2,\omega}(D)$. From the boundary condition and $mes_{n-1}(\partial D \cap E) = 0$ we get $u(x) \in \overset{\circ}{W}{}^1_{2,\omega}(D)$. Now, let $\sigma \geq 2$ be a number which will be chosen later, $D^+_\Sigma = \{x : x \in D'_\Sigma, u(x) > 0\}$. Without loss of generality, we suppose that the set D^+_Σ isn't empty. Supposing in (13) $z = 1,\ \beta = \omega u^\sigma$, we get

$$\int\limits_{D^+_\Sigma} \mathcal{M}(\omega u^\sigma)\, dx = \sigma \int\limits_{\partial D^+_\Sigma} \left(\omega_\nu u^\sigma + \sigma u^{\sigma-1}\frac{\partial u}{\partial \nu}\right) ds \leq$$

$$\leq M^\sigma \int\limits_{\partial D^+_\Sigma} \omega ds + \sigma M^{\sigma-1} \int\limits_{\partial D^+_\Sigma} \left|\frac{\partial u}{\partial \nu}\right| ds \leq C_5(a_0, M, \sigma, C_1)\,\varepsilon.$$

But, on the other hand

$$\mathcal{M}(u^\sigma) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \omega u^\sigma}{\partial x_j}\right) + b(x,u,\nabla u) =$$

$$= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\omega\left(\sigma u^{\sigma-1}\frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\omega_{x_i}\frac{\partial u^\sigma}{\partial x_j}\right)\right) + b(x,u,\nabla u) =$$

$$= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\omega\sigma u^{\sigma-1}\frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\sigma u^{\sigma-1}\omega x\frac{\partial u}{\partial x_j}\right) + b(x,u,\nabla u) =$$

$$= \sigma\omega u^{\sigma-1}\mathcal{M}(u) + \sigma\omega\frac{\partial}{\partial x_i}\left(a_{ij}u^{\sigma-1}\frac{\partial u}{\partial x_j}\right) + \sigma u^{\sigma-1}\frac{\partial}{\partial x_i}\left(a_{ij}\omega\frac{\partial u}{\partial x_j}\right) + b(x,u,\nabla u) + \beta =$$

$$= \sigma\omega u^{\sigma-1}\mathcal{M}(u) + \sigma\omega u^{\sigma-1}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sigma\omega a_{ij}u_{x_j}(\sigma-1)u^{\sigma-2}u_{x_i} +$$

$$+ \sigma u^{\sigma-1}\omega_{x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sigma u^{\sigma-1}\omega\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \beta + b(x,u,\nabla u) =$$

$$= 3\sigma\omega u^{\sigma-1}M(u) + \sigma(\sigma-1)a_{ij}ux_iu_{x_j}u^{\sigma-2}\omega + \sigma u^{\sigma-1}\omega_{x_i}a_{ij}u_{x_j} + \beta + b(x,u,\nabla u) =$$

$$= \sigma\int\limits_{D^+_\Sigma} d_i(x)\,u_{x_i}u\omega dx - \sigma(\sigma-1)\int\limits_{D^+_\Sigma} u^\sigma\omega(x)\,c(x)\,dx +$$

$$+ \sigma(\sigma-1)\int\limits_{D^+_\Sigma}\sum_{i,j=1}^n u^{\sigma-2}\omega(x)\,a_{ij}u_iu_jdx + (2u+1)\int\limits_{D^+_\Sigma}\sum_{i,j=1}^n a_{ij}u_j\omega_{x_j}u^{\sigma-1} + b(x,u,\nabla u).$$

Hence, we conclude

$$\sigma(\sigma-1)\int\limits_{D^+_\Sigma}\omega^2u^{\sigma-2}|\nabla u|^2 dx \leq d_0\int\limits_{D^+_\Sigma}u^{\sigma-1}\omega u_idx + b(x,u,\nabla u) \leq$$

$$\leq d_0\int\limits_{D^+_\Sigma}u^{\sigma-1}\omega u_idx + b(x,u,\nabla u) \leq \frac{d_0\varepsilon}{2}\int\limits_{D^+_\Sigma}u^\sigma dx + b(x,u,\nabla u). \qquad (16)$$

Let $D^{+}=\{x:x\in D,u\left(x\right)>0\}$, D_1^{+} be an arbitrary connected component of D^{+}. Subject to the arbitrariness of ε from (16) we get

$$(\sigma-1)\gamma\int\limits_{D_1^+}\omega u^{\sigma-2}\left|\nabla u\right|^2dx\le d_0\int\limits_{D_1^+}\omega u^{\sigma-1}\sum_{i=1}^{n}\left|u_i\right|dx.$$

Thus, for any $\mu>0$

$$(\sigma-1)\gamma\int\limits_{D_1^+}\omega u^{\sigma-2}\left|\nabla u\right|^2dx\le\frac{d_0\mu}{2}\int\limits_{D_1^+}\omega u^{\sigma-2}\left(\sum_{i=1}^{n}\left|u_i\right|\right)^2dx+\frac{d_0}{2\mu}\int\limits_{D_1^+}\omega u^{\sigma}dx\le$$

$$\le\frac{d_0\mu n}{2}\int\limits_{D_1^+}\omega u^{\sigma-2}\left|\nabla u\right|^2dx+\frac{d_0}{2\mu}\int\limits_{D_1^+}\omega u^{\sigma}dx. \qquad (17)$$

But, on the other hand

$$I=-\sigma\sum_{i=1}^{n}\int\limits_{D_1^+}x_i\omega u^{\sigma-1}u_idx=-\sum_{i=1}^{n}\int\limits_{D_1^+}x_i\omega\left(u^{\sigma}\right)_idx=n\int\limits_{D_1^+}\omega u^{\sigma}dx,$$

and besides, for any $\beta>0$

$$I=\frac{\sigma\beta}{2}\int\limits_{D_1^+}r^2\omega u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^+}u^{\sigma-2}\omega^2\left|\nabla u\right|^2dx$$

Then

$$I\le\frac{\sigma\beta}{2}\int\limits_{D_1^+}r^2\omega u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^+}\omega^2\left|\nabla u\right|^2u^{\sigma-2}dx,$$

where $r=|x|$. Denote by $k(D)$ the quantity $\sup\limits_{x\in D}|x|$. Without loss of generality we'll suppose, that $k(D)=1$. Then

$$I\le\frac{\sigma}{2\beta}\int\limits_{D_1^+}\omega u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^+}\omega^2u^{\sigma-2}\left|\nabla u\right|^2dx.$$

Thus,

$$\left(n-\frac{\sigma\beta}{2}\right)\int\limits_{D_1^+}\omega u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^+}\omega^2u^{\sigma-2}\left|\nabla u\right|^2dx.$$

Now, choosing $\beta=\dfrac{n}{\sigma}$, we finally obtain

$$\int\limits_{D_1^+}\omega u^{\sigma}dx\le\frac{\sigma^2}{n^2}\int\limits_{D_1^+}\omega^2u^{\sigma-2}\left|\nabla u\right|^2dx. \qquad (18)$$

Subject to (18) in (17) , we conclude

$$(\sigma-1)\gamma\int\limits_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2dx\le\left(\frac{d_0\varepsilon n}{2}+\frac{d_0\sigma^2}{2\varepsilon n^2}\right)\int\limits_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2dx. \quad (19)$$

Now choose μ such that

$$(\sigma-1)\gamma>\frac{d_0\mu n}{2}+\frac{d_0\sigma^2}{2\mu n^2}. \quad (20)$$

Then from (18)-(20) it will follow that $u(x)\equiv 0$ in D_1^+, and thus $u(x)\equiv 0$ in D. Suppose that $\mu=\dfrac{(\sigma-1)\gamma}{d_0 n}$. Then (20) is equivalent to the condition

$$n>\left(\frac{\sigma}{\sigma-1}\right)^2\left(\frac{d_0}{\gamma}\right)^2. \quad (21)$$

At first, suppose that

$$n>\left(\frac{d_0}{\gamma}\right)^2. \quad (22)$$

Let's choose and fix such a big $\sigma\ge 2$ that by fulfilling (22) the inequality (21) was true. Thus, the theorem is proved, if with respect to n the condition (22) is fulfilled. Show that it is true for any $n\ge 3$. For that, at first, note that if $k(D)\ne 1$, then condition (22) will take the form

$$n>\left(\frac{d_0k(D)}{\gamma}\right)^2.$$

Now, let the condition (22) be not fulfilled. Denote by k the least natural number for which

$$n+k>\left(\frac{d_0}{\gamma}\right)^2. \quad (23)$$

Consider $(n+k)$-dimensional semi-cylinder

$$D'=D\times(-\delta_0,\delta_0)\times...\times,\times(-\delta_0,\delta),$$

where the number $\delta_0>0$ will be chosen later. Since $\omega(D)=1$, then $\omega(D')\le 1+\delta_0\sqrt{k}$. Let's choose and fix δ_0 so small that along with the condition (23) the condition

$$n+k>\left(\frac{d_0\omega(D')}{\gamma}\right)^2 \quad (24)$$

was fulfilled too.

Let

$$y=(x_1,...,x_n,x_{n+1},...,x_{n+k}),E'=\underbrace{E\times[-\delta_0,\delta_0]\times...\times[-\delta_0,\delta_0]}_{k\ \text{times}}$$

Consider on the domain D' the equation

$$\mathcal{L}'_{\vartheta}=\sum_{i,j=1}^{n}a_{ij}\left(x\right)\vartheta_{ij}+\sum_{i=1}^{k}\frac{\partial^{2}\vartheta}{\partial x_{n+i}^{2}}+\sum_{i=1}^{n}b_{i}\left(x\right)\vartheta_{i}+c\left(x\right)\vartheta=0. \tag{25}$$

It is easy to see that the function $\vartheta\left(y\right)=u\left(x\right)$ is a solution of the equation (25) in $D'\backslash E'$. Besides, $m_H^{n+k-2+\lambda}\left(E'\right)=\left(2\delta_0\right)^k m_H^{n-2+\lambda}\left(E\right)=0$, the function $\vartheta\left(y\right)$ vanishes on $\left(\partial D\times\underbrace{\left[-\delta_0,\delta_0\right]\times..\times\left[-\delta_0,\delta_0\right]}_{k\ \text{times}}\right)\backslash E'$ and $\dfrac{\partial\vartheta}{\partial\nu'}=0$ at $x_{n+i}=\pm\delta_0,\ i=1,...,k$, where $\dfrac{\partial}{\partial\nu'}$ is a derivative by the conormal generated by the operator $\mathcal{L}'$. Noting that $\gamma\left(\mathcal{L}'\right)=\gamma\left(\mathcal{L}\right),d_0\left(\mathcal{L}'\right)=d_0\left(\mathcal{L}\right)$ and subject to the condition (24), from the proved above we conclude that $\vartheta\left(y\right)\equiv 0$, i.e. D'. The theorem is proved.

References

[1]. Carleson L. *Selected problems on exceptional sets.* D. Van. Nostrand company, Toronto-London-Melbourne, 1967, 126p.

[2]. Moiseev E.I. *On Neumann problems in piecewise smooth domains.* Diff. Uravneniya, 1971, vol. VII, No9, pp.1655-1656. (Russian)

[3]. Moiseev E.I. *On existence and non-existence boundary sets of Neumann problem.* Diff. Uravneniya, 1973, vol. IX, No5, pp.901-911. (Russian)

[4]. Landis E.M. To question on uniqueness of solution of the first boundary value problem for elliptic and parabolic equations of the second order. Uspekhi mat. nauk, 1978, vol.33, No3, 151 p. (Russian)

[5]. Kondratyev V.A.. Landis E.M. *Qualitative theory of linear partial differential equations of second order.* Itogi nauki i tekhniki. Ser. "Modern problems of mathematics". Fundamental directions. Partial differential equations, vol.3, M., VINITI, 1988, pp.99-212. (Russian)

[6]. Mamedov I.T. *On exceptional sets of solutions of Dirichlet problem for elliptic equations of second order with discontinuous coefficients.* Trudy IMM AN Azerb., 1998, vol.III (XVI), pp.137-149. (Russian)

[7]. Gerver M.L., Landis E.M. *One generalization of a theorem on mean value for multivariable functions.* DAN SSSR, 1962, vol. 146, No 4, pp.761-764. (Russian)

[8].Landis E.M. Second order equations of elliptic and parabolic types. M., "Nauka", 1971, 288 p. (Russian)

[9]. Gilbarg D., Trudinger N.S. *Elliptic partial Differential Equations of Second Order.* Berlin-Heidelberg-New York, Springer-Verlag, 1977, 401p.

[10]. Kayzer V., Muller B. *Removable sets for heat conduction.* Vestnik Moscow University. 1973, No 5, pp.26-32. (Russian)

[11].Mamedova V.A. *On removable sets of solutions of boundary value problems for elliptic equations of second order.* Trans. of NAS of Azerb., 2005, vol. XXV, No 1, pp.101-106.

[12]. Gadjiev T.S., Mamedova V.A. *On removable sets of solutions of second order elliptic and parabolic equations in nondivergent form. Ukr. Math. Journ.*, 2009, vol. 61, No 11, pp.1485-1496.

[13]. Kilpelainen T., Zhong X., *Removable sets for continuous solutions of quasilinear elliptic equations.* Proc. Amer. Math. Soc. 2002, 130, No 6, pp. 1681-1688 (electronic).

[14]. Diederich J. *Removable sets for pointwise solutions of elliptic partial differential equations.* Trans. Amer. Math. Soc., 1972, 165, pp.333-352.

[15]. Harvey R., Polking J. *Removable singularities of solutions of linear partial differential equations.* Acta Math., 1970, 125, pp.39-56.

[16]. Dahlberg B.E.J. O*n exceptional sets at the boundary for subharmonic functions.* Arkiv mat., 1977, vol.15, No 2, pp.305-312.

[17]. A. V. Pokrovskii. *Removable singularities of solutions of linear uniformly elliptic second order equations.* Funktsionalnyj Analiz i Prilozhenija, 2008, 42, 2, pp.44-55 (Russian).

[18]. A. V. Pokrovskii. *Removable singularities for solutions of second-order linear uniformly elliptic equations in non-divergence form.* Mat. Sbornik, 2008, 199, 6, pp.137-160 (Russian).

Mahammad-Rza I. Arazm, Vafa A. Mamedova,
Zeynab S.Gafarova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 539 47 20 (off.).

Received October 11, 2010; Revised December 27, 2010.