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ASYPMTOTICS OF THE SOLUTION OF A MIXED PROBLEM FOR A QUASILINEAR HYPERBOLIC EQUATION DEGENERATING INTO PARABOLIC EQUATION

Abstract

The mixed problem for a second order quasilinear hyperbolic eqaution containing a small derivative for higher derivatives and degenerating into a parabolic equation is considered in a rectangle. The asymptotic expansion of the generalized solution of the considered problem is constructed with any accuracy, and the remainder term is estimated.

Far less papers have been devoted to the investigation of singularly perturbed nonlinear hyperbolic equations compared with the papers concerning elliptic and parabolic equations. Here, the papers [1]-[4] should be noted. In these and other known papers concerning singularly perturbed hyperbolic equations, the derivatives of the desired function with respect to t linearly enter the equation.

In the present paper, in $D = \{(t,x)\,|0 \le t \le T,\ 0 \le x \le 1\}$ we consider the following mixed problem

$$L_\varepsilon U \equiv \varepsilon \frac{\partial^2 U}{\partial t^2} + \varepsilon^{2k} \left(\frac{\partial U}{\partial t}\right)^{2k+1} + \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + aU - f(t,x) = 0, \qquad (1)$$

$$U\,|_{t=0} = \frac{\partial U}{\partial t}\,|_{t=0} = 0,\ (0 \le x \le 1);\ U\,|_{x=0} = U\,|_{x=1} = 0,\ \ (0 \le t \le T), \qquad (2)$$

where $\varepsilon > 0$ is a small parameter, k is an arbitrary natural number, $a > 0$ is a constant, $f(t,x)$ is a given function.

Our goal is to construct the complete asymptotic expansion in small parameter of the generalized solution of problem (1) (2).

In the first iterative process the approximate solution of equation (1) is sought in the form $W = \sum_{i=0}^{n} \varepsilon^i W_i(t,x)$, and the function W_i will be chosen so that

$$L_\varepsilon W = 0(\varepsilon^{n+1}). \qquad (3)$$

Substituting the expressions for W into (3), for determining $W_0,\ W_1, ..., W_n$ we get the following equations

$$L_0 W_0 \equiv \frac{\partial W_0}{\partial t} - \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t,x); \quad L_0 W_i = f_i(t,x), \qquad (4)$$

where $f_i(t,x)$ are the known fucntions, moreover $f_i(t,x) = -\dfrac{\partial^2 W_{i-1}}{\partial t^2}$ for $i = 1, 2, ..., 2k-1$ and for $i = 2k,\ 2k+1, ..., n$ the functions f_i are expressed polynomially by the first and second order derivatives $W_0,\ W_1, ..., W_{i-1}$. Find such solutions of equation (4) that satisfy the boundary conditions:

$$W_i\left|_{x=0}\right. = W_i\left|_{x=1}\right. = 0;\ i = 0, 1, ..., n. \tag{5}$$

The first initial condition from (2) will be used for equations (4). Then the second initial condition will be lost. For compensating the lost initial condition, the boundary layer type function should be constructed near the boundary $S_0 = \{(t, x) \in D\,|t = 0\}$.

A new decomposition of the operator L_ε in powers of ε near the boundary S_0 for the auxiliary function $r = \sum\limits_{j=0}^{n+1} \varepsilon^j r_j(\tau, x)$ is of the form

$$\begin{gathered} L_{\varepsilon,1} r \equiv \varepsilon^{-1}\left\{\frac{\partial^2 r_0}{\partial\tau^2} + \left(\frac{\partial r_0}{\partial\tau}\right)^{2k+1} + \frac{\partial r_0}{\partial\tau} + \sum_{j=1}^{n+1}\varepsilon^j\left[\frac{\partial^2 r_j}{\partial\tau^2} + (2k+1)\times\right.\right. \\ \left.\left.\times\left(\frac{\partial r_0}{\partial\tau}\right)^{2k}\frac{\partial r_j}{\partial\tau} + \frac{\partial r_j}{\partial\tau} + h_j(r_0, r_1, ..., r_{j-1})\right] + 0(\varepsilon^{n+2})\right\}, \end{gathered} \tag{6}$$

where $\tau = t/\varepsilon,\ h_j(r_0, r_1, ..., r_{j-1})$ are the known functions dependent on $r_0, r_1, ..., r_{j-1}$ and their first and second derivatives, $r_j(\tau, x)$ are some smooth functions.

We'll look for a boundary layer type function near the boundary S_0 in the form $V = \sum\limits_{i=0}^{n} \varepsilon^{1+i} V_i(\tau, x)$ as the solution of the equation

$$L_{\varepsilon,1}(W + V) - L_{\varepsilon,1}W = 0(\varepsilon^{n+1}). \tag{7}$$

Suppose that the functions $W_i(t, x);\ i = 0, 1, ..., n$ have been already constructed. Then new expansions of the functions W and $W+V$ in powers of ε in the coordinates (τ, x) will be of the form

$$W = \sum_{j=0}^{n+1} \varepsilon^j \omega_j(\tau, x) + 0(\varepsilon^{n+2}), \tag{8}$$

$$W + V = \omega_0 + \varepsilon(\omega_1 + V_0) + ... + \varepsilon^{n+1}(\omega_{n+1} + V_n) + 0(\varepsilon^{n+2}), \tag{9}$$

where $\omega_0 = W_0(0, x)$ is independent of τ, and the remaining functions ω_j are determined from the formula

$$\omega_j = \sum_{s+r=j} \frac{1}{s!}\frac{\partial^s W_r(0, x)}{\partial t^s} t^s; \quad j = 1, 2, ..., n+1.$$

For defining $V_0, V_1, ..., V_n$, from (6)-(9) we get the following equations

$$\frac{\partial^2 V_0}{\partial\tau^2} + \frac{\partial V_0}{\partial\tau} = 0;\ \ \frac{\partial^2 V_j}{\partial\tau^2} + \frac{\partial V_j}{\partial\tau} = \frac{\partial^2 V_{j-1}}{\partial x^2} - aV_{j-1};\ j = 1, 2, ..., n, \tag{10}$$

Now write intial conditions for equations (4). Having substituted the expressions of W and V in the equality $(W+V)\left|_{t=0}\right.=0$ and comparing the terms under the same powers of ε whose powers are less than $n+1$, we have

$$W_0\left|_{t=0}\right.=0;\ W_i\left|_{t=0}\right.=-V_{i-1}\left|_{\tau=0}\right.;\ i=1,2,...,n. \tag{11}$$

Obviously, if the functions $W_i;\ i=0,1,...,n$ will satisfy conditions (11), the sum $W+V$ will satisfy the first initial condition from (2) with ε^{n+1}-th accuracy, i.e.

$$(W+V)\left|_{t=0}\right.=\varepsilon^{n+1}V_n\left|_{\tau=0}\right.. \tag{12}$$

The boundary conditions for equations (10) are found from the requirement

$$\frac{\partial}{\partial t}(W+V)\left|_{t=0}\right.=0 \tag{13}$$

and are of the form

$$\frac{\partial V_j}{\partial \tau}\left|_{\tau=0}\right.=-\frac{\partial W_j}{\partial t}\left|_{t=0}\right.;\ j=0,1,...,n. \tag{14}$$

It should be noted that usually the first iterative process is carried out. Then the second one that serves to construct boundary layer functions, is carried out. From (11) and (14) we get that the functions $W_i,V_i;i=0,1,...,n$ should be constructed in turn, one after another: $W_0,V_0,W_1,V_1,...,W_n,V_n$. In other words, the iterative processes are imbedded one into another.

Introduce the denotation

$$C^{\alpha,\beta}(D)=\left\{U(t,x)\left|\frac{\partial^i U(t,x)}{\partial t^{i_1}\partial x^{i_2}}\in C(D);\right.\right.$$

$$\left. i=i_1+i_2;\ i_1=0,1,...,\alpha;\ i_2=0,1,...,\beta\right\}.$$

The following lemma is valid.

Lemma 1. *Let $f(t,x)\in C^{p-1,2p+2}(D)$, and the following condition be fulfilled*

$$\frac{\partial^{2r}f(t,0)}{\partial x^{2r}}=\frac{\partial^{2r}f(t,1)}{\partial x^{2r}}=0;\ r=0,1,...,p, \tag{15}$$

where p is an arbitrary natural number. Then the function W_0 being a solution of the first equation from (4) and satisfying boundary conditions (5) for $i=0$ and the homogeneous initial condition from (11) is contained in the space $C^{p,2p}(D)$ and satisfies the relation

$$\frac{\partial^{i_1+2i_2}W_0(t,0)}{\partial t^{i_1}\partial x^{2i_2}}=\frac{\partial^{i_1+2i_2}W_0(t,1)}{\partial t^{i_1}\partial x^{2i_2}}=0;\ i_1+i_2\le p, \tag{16}$$

where i_2,i_2 are non-negative integers.

Proof. The function $W_0(t,x)$ may be represented by the formula

$$W_0(t,x)=\sum_{k=1}^{\infty}W_{0k}(t)\sin k\pi x, \tag{17}$$

where $W_{0k}(t)$ is of the form

$$W_{0k}(t)=\int_0^t e^{-(a+k^2\pi^2)(t-\tau)}f_k(\tau)d\tau. \qquad (18)$$

Here $f_k(t)$ denotes the function $f_k(t)=2\int_0^1 f(t,\xi)\sin k\pi\xi d\xi$.

Using condition (15), we can get the estimation

$$\left|f_k^{(i)}(t)\right|\leq\frac{2M_{i,2p+2}}{k^{2p+2}\pi^{2p+2}};\quad i=0,1,...,p-1;\ t\in[0,T], \qquad (19)$$

where

$$M_{i,2p+2}=\max_{(t,x)\in D}\left|\frac{\partial^{i+2p+2}f(t,x)}{\partial t^i\partial x^{2p+2}}\right|.$$

Taking into account (19), from (18) we have

$$\left|\frac{d^{i_1}W_{0k}(t)}{dt^{i_1}}\right|\leq\frac{C_{i1}}{k^{2p+2-2i_1}\pi^{2p+2-2i_1}};t\in[0,T];\ C_{i_1}>0. \qquad (20)$$

Denoting $\overline{W}_{ok}(t,x)=W_{ok}(t)\sin k\pi x$ and following (20), we get

$$\left|\frac{\partial^i\overline{W}_{ok}(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|\leq\frac{C_{i_1}}{k^{2p+2-2i_1-i_2}\pi^{2p+2-2i_1-i_2}};(t,x)\in D;\ i=i_1+i_2. \qquad (21)$$

It follows from (21) that the number series

$$\frac{C_{i_1}}{\pi^r}\sum_{k=1}^{\infty}\frac{1}{k^r};\ r=2p+2-2_{i_1}-i_2 \qquad (22)$$

is majorant for the functional series

$$\sum_{k=1}^{\infty}\frac{\partial^i\overline{W}_{ok}(t,x)}{\partial t^{i_1}\partial x^{i_2}};\quad (t,x)\in D, \qquad (23)$$

that is obtained term by term differentiation of (17). Since the number series (22) converges for $r\geq 2$, functional series (23) converges uniformly in D for $2i_1+i_2\leq$ $\leq 2p$. Hence the validity of the statement of Lemma 1 follows.

Lemma 1 is proved.

Note that the number p in the condition of lemma 1 will be chosen below.

Since the function $W_0(t,x)$ is already known, we can determing the function V_0 as a solution type of the first boundary layer equation from (10) satisfying condition (14) for $j=0$. Obviously, V_0 is determined from the formula

$$V_0(\tau,x)=\frac{\partial W_0(0,x)}{\partial t}e^{-\tau}. \qquad (24)$$

By lemma 1, from (24) it follows that the function V_0 and all its even order derivatives with respect to x up to the $2p$-th order inclusively vanish for $x=0$ and $x=1$.

From (4), (5) (11) for $i = 1$ we get that the function W_1 is a solution of the problem

$$\frac{\partial W_1}{\partial t} - \frac{\partial^2 W_1}{\partial x^2} + aW_1 = -\frac{\partial^2 W_0}{\partial t^2}, \qquad W_1 \left|_{t=0} = \theta_1(x), \; W_1 \right|_{x=0} = W_1 \left|_{x=1} = 0, \right. \tag{25}$$

where $\theta_1(x)$ is a known function: $\theta_1(x) = \dfrac{\partial W_0(0,x)}{\partial t} = f(0,x)$.

We can look for the solution of problem (25) in the form $W_1 = W_1^{(1)} + W_1^{(2)}$, where $W_1^{(1)}$ is a solution of the equation $L_0 W_1^{(1)} = 0$ with the initial condition $W_1^{(1)} |_{t=0} = f(0,x)$, and $W_1^{(2)}$ is a solution of the equation $L_0 W_1^{(2)} = -\dfrac{\partial^2 W_0}{\partial t^2}$ with the initial condition $W_1^{(2)} |_{t=0} = 0$. It is easy to see that the function $W_1^{(1)}(t,x)$ is determined by the formula

$$W_1^{(1)}(t,x) = -\sum_{k=1}^{\infty} \left[f_k(0) e^{-(a+k^2\pi^2)t} \right] \sin k\pi x. \tag{26}$$

Since the problem for $W_1^{(2)}$ is the same type as the problem for W_0, lemma 1 is applicable to it, consequently, $W_1^{(2)}(t,x) \in C^{p-1,2p-2}(D)$ and (16) is fulfilled for $W_1^{(2)}$. Hence and from (26) it follows that the function W_1 and all its even order derivatives with respect to x up to the $(2p-2)$-th order inclusively vanish for $x = 0$ and $x = 1$.

The process shows that at each step, while passing from W_{i-1} to W_i, the smoothness with respect to x decreases by a unit, the smoothness with respect to t by 2 units. In this connection, the natural number p that is contained in the condition of lemma 1 should be chosen so that the smoothness of the function W_0, and conditions (16) allow to construct the remaining function $W_1, W_2, ..., W_n$, and that in future the operator L_ε could act on the function $W = \sum\limits_{i=0}^{n} \varepsilon^i W_i$. For that it suffices to take the number p in such a way: $p = n + 2$.

From (10), (14) for $j = 1$ and (24)we have that V_1 is a boundary layer type solution of the following problem:

$$\frac{\partial^2 V_1}{\partial t^2} + \frac{\partial V_1}{\partial \tau} = \varphi_1(x) e^{-\tau}; \; \frac{\partial V_1}{\partial \tau} \left|_{\tau=0} = \psi_1(x), \right. \tag{27}$$

where by $\varphi_1(x)$, $\psi_1(x)$ denote

$$\varphi_1(x) = \frac{\partial^2 W_0(0,x)}{\partial t \partial x^2} - a\frac{\partial W_0(0,x)}{\partial t}; \; \psi_1(x) = -\frac{\partial W_1(0,x)}{\partial t}. \tag{28}$$

Obviously, the function determined by the formula

$$V_1^{(1)} = -\varphi_1(x) \tau e^{-\tau} \tag{29}$$

is a special solution to the equation for V_1 in (27). Having sought V_1 in the form $V_1 = V_1^{(1)} + V_1^{(2)}$, we have that $V_1^{(2)}$ is a boundary layer type solution of the problem

$$\frac{\partial^2 V_1^{(2)}}{\partial \tau^2} + \frac{\partial V_1^{(2)}}{\partial \tau} = 0, \quad \frac{\partial V_1^{(2)}}{\partial \tau} \left|_{\tau=0} = \varphi_1(x) + \psi_1(x). \right.$$

The boundary layer type solution of the last problem is of the form

$$\partial V_1^{(2)} = -\left[\varphi_1(x) + \psi_1(x)\right] e^{-\tau}. \tag{30}$$

From (28)-(30) we get that the function V_1 being the sum of $V_1^{(1)}$ and $V_1^{(2)}$ is determined by the formula

$$V_1 = \left[b_{10}(x) + b_{11}(x)\tau\right] e^{-\tau}, \tag{31}$$

where $b_{10}(x),\ b_{11}(x)$ denote

$$\begin{gathered} b_{10}(x) = a\frac{\partial W_0(0,x)}{\partial t} - \frac{\partial^3 W_0(0,x)}{\partial t \partial x^2} + \frac{\partial W_1(0,x)}{\partial t}, \\ b_{11}(x) = a\frac{\partial W_0(0,x)}{\partial t} - \frac{\partial^3 W_0(0,x)}{\partial t \partial x^2}. \end{gathered} \tag{32}$$

While constructing the functions $V_2, V_3, ..., V_n$ we use the following statement.

Lemma 2. *The functions V_j being the boundary layer type solutions of equations (10), satisfying the appropriate boundary conditions (14), are determined by the formula*

$$V_j = \left[b_{j0}(x) + b_{j1}(x)\tau + b_{j2}(x)\tau^2 + ... + b_{jj}(x)\tau^j\right] e^{-\tau}, \tag{33}$$

and the coefficients $b_{jk}(x)$ are uniformly expressed by the function

$$\frac{\partial^{1+2r} W_s(0,x)}{\partial t \partial x^{2r}};\ r+s \le j;\ j = 0,1,...,n. \tag{34}$$

Proof. The lemma is proved by the mathematical induction method. It was shown above that V_0, V_1 is determined from (33). Now suppose that the statement of the lemma is valid for $j \le k-1$, and prove that it is valid for $j = k$ as well. For $j = k$ from (10), (14) and (33) we have that V_k is a boundary layer solution of the following problem:

$$\frac{\partial^2 V_k}{\partial \tau^2} + \frac{\partial V_k}{\partial \tau} = \left\{ \sum_{S=0}^{k-1} \left[\left[b''_{k-1S}(x) - ab_{k-1S}(x)\right] \tau^S\right] \right\} e^{-\tau},\ \frac{\partial V_k}{\partial \tau} |_{\tau=0} = -\frac{\partial W_k}{\partial t} |_{t=0}.$$

Functions (34) are contained in the right hand side of the equation for V_k at $r+s \le k-1$.

Repeating the similar reasonings carried out in definition of the function V_1 we can confirm that V_k is also determined from (33), (34) for $j = k$.

Lemma 2 is proved.

By lemma 1 all the functions of (34), consequently the functions $b_{jS}(x)$, $s = 0,1,...,k;\ j = 0,1,...,n$ vanish for $x = 0$ and $x = 1$. Hence and from (5) it follows that in addition to initial conditions (12), (13), the sum $W + V$ satisfies also the boundary conditions

$$(W+V)|_{x=0} = (W+V)|_{x=1} = 0,\ (0 \le t \le T). \tag{35}$$

The difference of the generalized solution of problem (1), (2) and the constructed sum $\widetilde{U} = \sum\limits_{i=0}^{n} \varepsilon^i W_i + \sum\limits_{i=0}^{n} \varepsilon^{1+i} V_i$ denote by $z = U - \widetilde{U}$ and call z the remainder term.

Then we get the following asymptotic expansion in small parameter of the solution of problem (1), (2):

$$U=\sum_{i=0}^{n}\varepsilon^{i}W_{i}+\sum_{i=0}^{n}\varepsilon^{1+i}V_{i}+z. \tag{36}$$

Now, estimate the remainder term.

The following statement holds.

Lemma 3. *For the remainder term z in (36) it is valid the estimation*

$$\varepsilon\int_{0}^{1}\left(\frac{\partial z}{\partial t}\Big|_{t=T}\right)^{2}dx+\varepsilon^{2k}\iint_{D}\left(\frac{\partial z}{\partial t}\right)^{2k+2}dtdx+\iint_{D}\left(\frac{\partial z}{\partial t}\right)^{2}dtdx+$$
$$+\int_{0}^{1}\left(\frac{\partial z}{\partial x}\Big|_{t=T}\right)^{2}dx+\int_{0}^{1}\left(z\big|_{t=T}\right)^{2}dx\le C\varepsilon^{2(n+1)}, \tag{37}$$

where $C>0$ is independent of ε.

Proof. Putting together (3) and (7), we have that $\widetilde{U}$ satisfies the equation

$$L_{\varepsilon}\widetilde{U}=0(\varepsilon^{n+1}). \tag{38}$$

Subtracting (38) from (1), we get

$$\varepsilon\frac{\partial^{2}z}{\partial t^{2}}+\varepsilon^{2k}\left[\left(\frac{\partial U}{\partial t}\right)^{2k+1}-\left(\frac{\partial \widetilde{U}}{\partial t}\right)^{2k+1}\right]+\frac{\partial z}{\partial t}-\frac{\partial^{2}z}{\partial x^{2}}+az=0(\varepsilon^{n+1}). \tag{39}$$

It follows from (2), (12), (13), (35) that z satisfies the following initial and boundary conditions:

$$z\big|_{t=0}=-\varepsilon^{n+1}V_{n}\big|_{\tau=0},\ \frac{\partial z}{\partial t}\Big|_{t=0}=0,\ z\big|_{x=0}=z\big|_{x=1}=0. \tag{40}$$

In obtaining a uniform estimation for z the homogeneity of the first initial condition in (40) creates some difficulty. In this connection, consider the auxiliary function

$$z_{1}=\varepsilon^{n+1}\left[t^{2}x(1-x)-V_{n}\big|_{\tau=0}\right]. \tag{41}$$

Obviously, z_1 satisfies intial and boundary conditions (40) as well. Represent the remainder term z in the form

$$z=z_{1}+z_{2}. \tag{42}$$

Then the function z_2 will satisfy the homogeneous intial and boundary conditions

$$\begin{gathered}z_{2}\big|_{t=0}=0,\ \frac{\partial z_{2}}{\partial t}\Big|_{t=0}=0,\ (0\le x\le 1),\\ z_{2}\big|_{x=0}=z_{2}\big|_{x=1}=0,\ (0\le t\le T).\end{gathered} \tag{43}$$

Substituting the expression of z from (42) in (39), taking into account (41) and after some transformations we get the equation

$$\varepsilon\frac{\partial^{2}z_{2}}{\partial t^{2}}+\varepsilon^{2k}\left\{\left[\frac{\partial\left(\widetilde{U}+z_{1}+z_{2}\right)}{\partial t}\right]^{2k+1}-\left[\frac{\partial\left(\widetilde{U}+z_{1}\right)}{\partial t}\right]^{2k+1}\right\}+$$
$$+\frac{\partial z_{2}}{\partial t}-\frac{\partial^{2}z_{2}}{\partial x^{2}}+az_{2}=0(\varepsilon^{n+1}). \tag{44}$$

Multiplying (44) by $\dfrac{\partial z_2}{\partial t}$ and integrating by parts the both hand sides of the obtained equality with regard to initial and boundary conditions (43), after some transformations we get the validity of estimations (37) for z_2. From (41), (42) and from the estimation for z_2 it follows the validity of estimations (37) for z.

Lemma 3 is proved.

We can generalize the obtained results in the form of the following statement.

Theorem. *Let $f(t,x)$ be a function given in D, have continuous derivatives with respect to t up to the $(n+1)$-th order inclusively, and with respect to x up to the $(2n+6)$-th order inclusively and satisfy condition (15) for $p = n+2$. Then for the generalized solution of problem (1), (2) it is valid asymptotic expansion (36), where the functions W_i, are determined by the first iterative process, the boundary layer type functions V_i near the boundary S_0 are determined by the second iterative process, z is a remainder term, and estimation (37) is valid for it.*

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Received September 03, 2010; Revised December 22, 2010