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ON ASYMPTOTIC BEHAVIOR OF LOCAL PROBABILITIES OF LINEAR BOUNDARY CROSSING BY PERTURBED RANDOM WALK

Abstract

In the paper, a theorem on asymptotic behavior of the density of joint distribution of the first passage time and the overshoot of a perturbed random walk is proved.

Introduction. Let ξ_n, $n \geq 1$ be a sequence of independent identical random variables determined on some probability space (Ω, F, P), and let $\triangle(x), x \in R$ be a positive and continuous function.

Assume

$$S_n = \sum_{k=1}^{n} \xi_n, \quad \overline{S}_n = \frac{S_n}{n}$$

and

$$T_n = n\triangle(\overline{S}_n), \quad n \geq 1.$$

Consider the first passage moment

$$\tau_a = \inf \{n \geq 1 : \ T_n > a\}, \tag{1}$$

of the random process T_n for the level $a \geq 0$, and assume that $\inf \{\varnothing\} = \infty$.

Note that for some assumptions for the function $\triangle(x)$ and distribution of the random variable ξ_1, the sequece $T_n, \ n \geq 1$ is a perturbed random walk, i.e. this may be represented in the following form

$$T_n = Z_n + \varepsilon_n, \quad n \geq 1,$$

where $Z_n, \ n \geq 1$ is an ordinary random walk, and $\varepsilon_n, \ \ n \geq 1$ is a random perturbation [7].

If the random variable ξ_1 has the finite expectation ($E\left|\xi_1\right| < \infty$), and the function $\triangle(x)$ in continuously differentiable in some vicinity of the point $x = v = E\xi_1$, morever $\triangle'(v) \neq 0$ and $\mu = \triangle(v) > 0$, we have

$$T_n = n\triangle(\overline{S}_n) = Z_n + \varepsilon_n, \tag{2}$$

where

$$Z_n = \sum_{k=1}^{n} \eta_k, \quad n \geq 1, \quad \eta_k = \triangle(v) + \triangle'(v)(\xi_k - v),$$

$$\varepsilon_n = n\psi(\overline{S}_n) \ \text{ and } \ \psi(x) = \triangle(x) - \triangle(v) - \triangle'(v)(x - v).$$

Note that $E\eta_1 = \triangle(v) = \mu > 0$ and for each $n \geq 1$ the random variable ε_n is independent of random variables $\xi_k,\ k > n$. We also note that from the conditon $E|\xi_1| < \infty$ it follows

$$\frac{\varepsilon_n}{n} \overset{a.s.}{\to} \infty \quad \text{as} \quad n \to \infty \tag{3}$$

Really, we have

$$\varepsilon_n = n\left(\overline{S}_n - v\right)\delta_n,$$

where

$$\delta_n = \begin{cases} 0 & \text{if } \ \overline{S}_n = v \\ \dfrac{\psi(S_n)}{S_n - v}, & \text{if } \ S_n \neq v. \end{cases}$$

It is clear that $\psi(v) = 0$ and $\psi'(v) = 0$.

Then from the law of strong numbers it follows that on the set $\left\{\omega : \overline{S}_n \neq v\right\}$

$$\delta_n = \frac{\psi\left(\overline{S}_n\right) - \psi(v)}{\overline{S}_n - v} \overset{a.s.}{\to} \psi'(v) = 0 \quad \text{as} \quad n \to \infty.$$

Consequently, convergence (3) is valid.

It follows from (3) and (2) that

$$\frac{T_n}{n} \overset{a.s.}{\to} \mu \quad \text{as} \quad n \to \infty. \tag{4}$$

Note that the representation T_n in the form (2) is oftenly used in nonlinear renewal theory ([3], [6], [7]).

While studying asymptotic properties of distribution of the first passage moment τ_a of the form (1), local limit theorems under which we understand any statement on asymptotic behavior of the local probability $P(\tau_a = n)$ as $n \to \infty$ or $a = a(n) \to \infty$ as $n \to \infty$ [3], are of great interest.

The conditional probability $P\left(\tau_a \geq n \ / \ \overline{S}_n = x\right)$ of the boundary crossing, i.e. the probability $\tau_a \geq n$ provided $T_n \approx a$ ([6]), is of great importance for studying the asymptotic behavior of local probabilities in the boundary crossing problems.

The conditional probability of boundary crossing by a random walk was studied in the papers [1], [2], [6].

In the present paper, by means of the theorem from [2] on limit behavior of the conditional probability of boundary crossing we prove a theorem on asymptotic behavior of density of point distribution of the first passage moment τ_a and the overshoot $\chi_a = T_{\tau_a} - a$ as $a \to \infty$ for the case when the distribution of the random variable ξ_1 belongs to the domain of attraction of stable distribution with characteristical exponent $\alpha \in (1, 2]$. From this theorem, in particular, it follows the local limit theorem for τ_a.

2. Conditions and formulation of main results. We'll assume that the distribution of the random variable (the random walk step) ξ_1 belongs to the domain of attraction of the stable distribution $G_\alpha(x)$ with the patameter $\alpha \in (1,2]$, i.e. it holds

$$\lim_{n\to\infty} P\left(\frac{S_n - nv}{A(n)} \le x\right) = G_\alpha(x), \quad x \in R \tag{5}$$

where $v = E\xi_1$, $A(t) = t^{1/\alpha}L(t)$ and $L(t)$, $t > 0$ is some slowly changing function at infinity [5].

Further more, for the random variable ξ_1 we'll assume that its characteristic function $f(t)$ satisfies the condition

$$\int_{-\infty}^{\infty} |f(t)|^m = dt < \infty \tag{6}$$

for some $m \ge 1$.

It is well known that from condition (6) it follows that the sum S_n has a bounded density $P_n(x)$ for $n \ge m$ ([5]).

It is also well known that from (5), (6) it follows that the local limit theorem holds for S_n ([4]):

$$P_n(x) = \frac{1}{A(n)} g_a\left(\frac{x - nv}{A(n)}\right) + o\left(1/A(n)\right), \quad x \in R, \quad \text{as} \quad n \to \infty, \tag{7}$$

where $g_\alpha(x)$ is the density of the stable distribution $G_\alpha(x)$.

It is proved in the paper [6] that if there exists a final mathematical expectation $v = E\xi_1$, and the function $\triangle(x)$ has a derivative at the point $x = v$, moreover $\mu = \triangle(v) > 0$ and $\triangle'(v) \neq 0$, then the overshoot $\chi_a = T_{\tau_a} - a$ as $a \to \infty$ has a limit dstribution with the density

$$h(r) = \frac{1}{\mu} P\left(Z_k \ge r, \quad k \ge 1\right),$$

where Z_k is defined in (2).

Denote by $q_a(n,r)$ the density of the distribution of τ_a and χ_a, i.e.

$$q_a(n,r) = \frac{d}{dr} P\left(\tau_a = n, \ \chi_a \le r\right), \quad r > 0, \quad n \ge 1.$$

It holds

Theorem. *Let all the above conditions for the distribution of the random variable ξ_1 be fulfilled, and the function $\triangle(x)$ be strongly convex and continuously-differentiable in the vicinity of the point $v = E\xi_1$, moreover $\mu =$
$= \triangle(v) > 0$ and $\sigma = \triangle'(v) > 0$.*

Then, if

$$n = n_a = \frac{a}{\mu} + \theta_a A\left(\frac{a}{\mu}\right), \tag{8}$$

where $\theta_a \to \theta \in R$ as $a \to \infty$, then for $r > 0$

$$q_a\left(n,r\right) \sim \frac{\mu}{\sigma A(n)} g_\alpha\left(-\frac{\mu}{\delta}\theta\right) h(r)$$

as $a \to \infty$.

Corollary 1. *Let the conditions of the theorem be fulfilled. Then,*

$$\int_0^\infty |h_a(r) - h(r)|\, dr \to 0 \text{ as } a \to \infty,$$

where $h_a(r)$ *is a marginal density of the overshoot* χ_a, $a > 0$.

Corollary 2. *(local limit theorem for* τ_a*). Let the conditions of the theorem be fulfilled, and* $E\left\lfloor (\triangle(\xi_1))^+ \right\rfloor < \infty$. *Then*

$$P(\tau_a = n) \sim \frac{\mu}{\sigma A(n)} g_\alpha\left(-\frac{\mu}{\sigma}\theta\right), \ a \to \infty,$$

Corollary 3. *Let* $E\left\lfloor (\triangle(\xi_1))^+ \right\rfloor < \infty$, *and the conditions of the theorem be fulfilled. Then, for* $r > 0$.

$$h_a(r\,|n) \to h(r) \text{ as } a \to \infty,$$

where $h_a(r/n)$ *is the conditional density of the overshoot provided* $\tau_a = n$, *i.e.*

$$h_a(r\,|n) = \frac{d}{dr} P(\chi_a \le r\,|\tau_a = n).$$

3. Proof of the main results. We'll need some facts formulated in the form of the following lemma.

Lemma 1. *Let the function* $\triangle(x)$ *have a derivative at the point* $v = E\xi_1$, *moreover,* $\mu = \triangle(v) > 0$ *and* $\triangle'(v) \ne 0$. *Then*

1) $P(\tau_a < \infty) = 1$ *for* $a \ge 0$;

2) $\tau_a \overset{a.s.}{\to} \infty$ *as* $a \to \infty$;

3) $\dfrac{\tau_a}{a} \overset{a.s.}{\to} \dfrac{1}{\mu}$ *as* $a \to \infty$.

Lemma 2. *Let (5) be fulfilled, and the function* $\triangle(x)$ *have a derivative at the point* v, *moreover* $\mu = \triangle(v) > 0$ *and* $\triangle'(v) \ne 0$. *Then, for* $x \in R$

$$\lim_{a\to\infty} P\left(\frac{\tau_a - a/\mu}{\triangle'(v) A(a/\mu)} \le x\right) = 1 - G_a(-\mu x),$$

The statement of this lemma follows from theorem 1 of the paper [3].

Lemma 3. *Let the sequence of non-negative functions* $\varphi_n(x)$, $x \in R$ *converge almost everywhere in Lebesgue measure to the non-negative function* $\varphi(x)$ *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x) dx \to \int_{-\infty}^{\infty} \varphi_n(dx).$$

Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\varphi_n(x) dx - \varphi_n(x)|\, dx = 0.$$

Lemma 3 is a special case of Sheffe's known theorem [6].

Proof of theorem 1. Assume

$$M_n = M_n(a,r) = \{y{:}a < n\triangle(y) \le a+r\}.$$

It is easy to see that for rather large n we have

$$P(\tau_a = n,\ \chi_a \le r) = P\left(\tau_a = n,\ a \le n\triangle\left(\overline{S}_n\right) \le a+r\right) =$$

$$= P\left(\tau_a \ge n,\ \overline{S}_n \in M_n\right) = \int\limits_{M_n} l_a(n,y)\, nP_n(ny)\, dy, \tag{9}$$

where $l_a(n,y) = P\left(\tau_a \ge n \left|\overline{S}_n = y\right.\right)$, and $P_n(x)$ is the density of the sum $S_n = = \xi_1 + ... + \xi_n$.

The function $\triangle(x)$ has the inverse function $\triangle^{-1}(x)$ in some vicinity of the point $x = v$ since $\triangle'(v) > 0$. Therefore, for the rather large a from (9) we have

$$P(\tau_a = n,\ \chi_a \le r) = \int\limits_{\triangle^{-1}(a/n)}^{\triangle^{-1}\left(\frac{a+r}{n}\right)} l_a(n,y) nP_n(ny)dy \tag{10}$$

Applying the formula for differentiation of the integral with respect to parameter, and taking into account that $\triangle^{-1}\left(\dfrac{a+r}{n}\right) - \triangle^{-1}\left(\dfrac{a}{n}\right) = O(1/n)$ as $a \to \infty$, from (10) for rather large a we get

$$q_a(n,r) = \sum_{y:y=\triangle^{-1}\left(\frac{a+r}{n}\right)} \frac{1}{\triangle'(v)} l_a(n,y)P(ny)dy + O(1/n) =$$

$$= \sum_{y:y=\triangle(y)=a+r} \frac{1}{\triangle'(v)} l_a(n,y)P_n(ny)dy + O(1/n) \tag{11}$$

Note that from the strong convexity of the function $\triangle(x)$, the equation

$$n\triangle(y) = a + r \tag{12}$$

has at most two solutions for the fixed a, n and r.

For some number $\varepsilon > 0$, by the made assumptions we have

$$\triangle'(y) > 0 \text{ or } y \in [v-\varepsilon,\ v+\varepsilon]$$

or

$$n\triangle(v-\varepsilon) < a + r < n\triangle(v+\varepsilon)$$

Hence and from (8) it follows that for rather large a equation (12) has at least one solution $y = y_0$ that by (7) and the Taylor expansion of the function $\triangle(x)$ at the point $x = v$ has the following form

$$y_0 = v - \theta_a \frac{\triangle(v)}{\triangle'(v)} \frac{A(n)}{n} + o(A(n)/n) \quad \text{as } a \to \infty. \tag{13}$$

Note that some other solution of equation (12) is bounded and doesn't converge to v as $a \to \infty$.

Now, from (11) we have

$$q_a(n,r) = \frac{1}{\triangle'(v)} l_a(n, y_0) P_n(ny_0) + O(1/n) \tag{14}$$

Further, under the condiitons of the proved theorem, the statament of the main result of the paper [2] is fulfilled. By virtue of this from theorem 2.7 of the paper [1] we have

$$l_a(n, y_0) \to \mu h(r) \ \text{ as } \ a \to \infty, \tag{15}$$

where $n\triangle(y_0) - a = r > 0$.

(7) and (13) yield

$$P_n(ny_0) = \frac{1}{A(n)} g_\alpha \left(\frac{ny_0 - nv}{A(n)} \right) + o(1/A(n)) =$$

$$= \frac{1}{A(n)} g_\alpha \left(-\theta_a \frac{\triangle(v)}{\triangle'(v)} \right) + o(1/A(n)). \tag{16}$$

Substituting (15) and (16) in (14), we get the statement of the theorem.

Remark 1. Note that the course of the theorem proof shows that its statement is fulfilled uniformly with respect to n for which $\left| \dfrac{n - a/\mu}{A(a/\mu)} \right| \leq c < \infty$.

Proof of Corollary 1. Assume

$$\lambda_a = \frac{n - a/\mu}{A(a/\mu)}.$$

For $a > 0$ and $r > 0$ we have

$$h_a(r) = \sum_n q_a(n,r) = \sum_{n:|\lambda_a| \leq c} q_a(n,r) + \sum_{n:|\lambda_a|>c} q_a(n,r),$$

where $c > 0$ is some number.

Denote

$$h_{a,1}(r) = \sum_{n:|\lambda_a| \leq c} q_a(n,r) \ \text{ and } \ h_{a,2}(r) = \sum_{n:|\lambda_a|>c} q_a(n,r).$$

Then

$$h_a(r) = h_{a,1}(r) + h_{a,2}(r). \tag{17}$$

Taking into account remark 1, from theorem 1 we have as $a \to \infty$

$$h_{a,1}(r) \to h(r) \left[G_\alpha \left(\frac{\mu}{\sigma} c \right) - G_\alpha \left(-\frac{\mu}{\sigma} c \right) \right] \tag{18}$$

for each $c > 0$.

From relation (18) assuming $c = c_a \to \infty$ as $a \to \infty$, we get

$$h_{a,1}(r) = h(r) \quad \text{as} \quad a \to \infty. \tag{19}$$

Further, by lemma 2,

$$\int_0^\infty h_{a,2}(r)dr = P\left(\left|\frac{\tau_a - a/\mu}{A(a/\mu)}\right| > c\right) \to 0 \tag{20}$$

as $c = c_a \to \infty$ and $a \to \infty$.

Now, from (17), (19) and (20) we have

$$\int_0^\infty h_a(r)dr = \int_0^\infty h_{a,1}(r)dr + \int_0^\infty h_{a,2}(r)dr$$

or

$$\int_0^\infty h_a(r)dr \to 1 = \int_0^\infty h(r)dr$$

as $a \to \infty$ since $\int_0^\infty h_a(r)dr = 1$.

Then from lemma 3 it follows that $\int_0^\infty |h_{a,1}(r) - h(r)|\, dr \to 0$ as $a \to \infty$. Hence, taking into account (20), from the inequality

$$\int_0^\infty |h_a(r) - h(r)|\, dr \le \int_0^\infty |h_{a,1}(r) - h(r)|\, dr + \int_0^\infty h_{a,2}(r)dr$$

we get the statement of Corollary 1.

Proof of Corollary 2. For $c > 0$ we have

$$P(\tau_a = n) = \int_0^\infty q_a(n,r)dr = \int_0^c q_a(n,r)dr + \int_c^\infty q_a(n,r)dr.$$

Assume

$$q_{a,1}(n,c) = A(n)\int_0^c q_a(n,r)dr$$

and

$$q_{a,2}(n,c) = A(n)\int_c^\infty q_a(n,r)dr.$$

Then

$$A(n)P(\tau_a = n) = q_{a,1}(n,c) + q_{a,2}(n,c). \tag{21}$$

From the theorem in majorized convergence we get that for each $c > 0$

$$q_{a,1}(n,c) \to \frac{\mu}{\sigma} H(c) g_\alpha \left(-\frac{\mu}{\sigma}\theta\right), \ a \to \infty, \tag{22}$$

where $H(c) = \int\limits_0^c (r)dr$.

Taking into account $H(c) \to 1$ as $c \to \infty$, from (22) we find

$$q_{a,1}(n,c) \to \frac{\mu}{\sigma} g_\alpha \left(-\frac{\mu}{\sigma}\theta\right) \quad \text{as} \quad a \to \infty \quad \text{and} \quad c \to \infty. \tag{23}$$

Prove that

$$q_{a,2}(n,c) \to 0 \quad \text{as} \quad a \to \infty \quad \text{and} \quad c \to \infty. \tag{24}$$

From the convexity property of the function $\triangle(x)$ it follows that

$$T_n = n\triangle\left(\frac{S_n}{n}\right) = n\triangle\left(\frac{S_{n-1}}{n} + \frac{\xi_n}{n}\right) = n\triangle\left(\frac{n-1}{n}\overline{S}_{n-1} + \frac{\xi_n}{n}\right) \le$$

$$\le n\left(\frac{n-1}{n}\triangle\left(\overline{S}_{n-1}\right) + \frac{1}{n}\triangle\xi_n\right) = T_{n-1} + \triangle\left(\xi_n\right)$$

Then taking into account $\{\tau_a = n\} \subseteq \{T_{n-1} \le a\}$, we have

$$q_{a,2}(n,c) = A(n)P\left(\tau_a = n, \ T_n > c + a\right) \le$$

$$\le A(n)P(T_{n-1} \le a, \ T_{n-1} + \triangle\left(\xi_n\right) > a + c) =$$

$$= A(n)P\left(a + c - \triangle\left(\xi_n\right) < T_n \le a\right) = A(n)\int\limits_c^\infty P(a + c - s < T_{n-1} \le a)dQ(s), \tag{25}$$

where $Q(s) = P(\triangle\left(\xi_1\right) \le s)$.

Further, from the equality

$$\frac{\varepsilon_n}{A(n)} = \frac{S_n - nv}{A(n)}\delta_n,$$

we have

$$\frac{\varepsilon_n}{A(n)} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty, \quad \text{since we have} \quad \delta_n \xrightarrow{a.s.} 0.$$

Consequently,

$$\lim_{n\to\infty} P\left(\frac{T_n - n\mu}{\sigma A(n)} \le x\right) = \lim_{n\to\infty} P\left(\frac{Z_n - n\mu}{\sigma A(n)} \le x\right) =$$

$$= \lim_{n\to\infty} P\left(\frac{S_n - n\mu}{A(n)} \le x\right) = G_a(x).$$

Hence it follows that for any numbers $c, d \in (-\infty, \infty)$

$$P(c \le T_n \le d) - P\left(c \le S_n \le d\right) \to o \quad \text{as } n \to \infty.$$

Then, from the local limit theorem it follows that for rather large a there exists a constant $M > 0$ such that

$$P\left(a + c - s < T_n \leq a\right) \leq M(s - c)$$

Thus, from (25)

$$q_{a,2}(n,c) \leq M \int\limits_{c}^{\infty} (s-c)dQ(s) \leq M \int\limits_{c}^{\infty} sdQ(s) \tag{26}$$

From $E\left(\triangle\left(\xi_1\right)^+\right) < \infty$ it follows that $\int\limits_{c}^{\infty} sdQ(s) \to 0$ as $c \to \infty$.

Therefore, (24) follows from (26).

The statement of Corollary 3 by theorem 1 and Corollary 2 follows from the equality

$$h_a\left(r\left|n\right.\right) = \frac{q_a(n,r)}{P\left(\tau_a = n\right)}.$$

Remark 2. From Corollary 3 it follows that the conditional distribution χ_a provided $\tau_a = n$ strongly converges as $a \to \infty$ to the unconditional limit distribution $H(r) = \int\limits_{o}^{r} h(u)du = \lim\limits_{a\to\infty} P\left(\chi_a \leq r\right).$

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