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OPTIMAL CONTROL PROBLEM WITH CONTROL AT THE HIGH COEFFICIENTS FOR THE NONLINEAR HYPERBOLIC EQUATION WITHOUT UNIQUENESS THEOREM OF SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEM

Abstract

An optimal control problem with a control in coeffiicients for higher derivatives is considered for one non-linear hyperbolic equation without a uniqueness theorem on the solution of an initial-boundary value problem. Solvability of an optimal control problem is proved, necessary conditions of optimality for appropriate approximate solution of the problem is derived.

Optimal control problem for nonlinear hyperbolic equation leads to some difficulties, connected with boundedness, imposed by the theorems on single valued solvability of boundary-value problem. Some forms of approximate solution of extremal problems for elliptic type singular equations are investigated in [1], [2].

In [3] this method is used for study the optimal problems for one nonlinear equation of hyperbolic type without uniqueness theorem of solution of boundary-value problem and boundedness on parameters.

In the present paper the method from [1]-[3] is used for solution of the optimal control problem with control at the high coefficients for nonlinear hyperbolic equation.

1.Problem statement

Let's consider in the cylinder $Q=\Omega\times(0,T)$ the following problem

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon(x,t)\frac{\partial u}{\partial x_i}\right)+|u|^{\rho}u=f(x,t), \tag{1}$$

with the initial and boundary conditions

$$u(x,0)=\varphi_0(x),\quad \frac{\partial u(x,0)}{\partial t}=\varphi_1(x),\quad x\in\Omega,\quad u(x,t)=0,\quad (x,t)\in S, \tag{2}$$

where Ω is an open bounded domain of the space R^n with the smooth boundary Γ, $T>0$, $S=\Gamma\times(0,T)$ lateral surface of the cylinder Q, $u=u(x,t)$ system state function, $\upsilon=\upsilon(x,t)$ control, $f\in L^2(Q)$, $\varphi_0\in H_0^1(\Omega)\cap L^p(\Omega)$, $\varphi_1\in L^2(Q)$ are known functions, $p=\rho+2$, $\rho>0$.

The control is chosen from the set

$$V=\left\{\upsilon\in H^1(Q)\,|\,0<a\le\upsilon(x,t)\le b,\left|\frac{\partial\upsilon(x,t)}{\partial x_i}\right|\le M_i,\ i=1,...,n,\right.$$

$$\left.\left|\frac{\partial\upsilon(x,t)}{\partial t}\right|\le M\quad a.e.\ on\quad Q\right\},$$

where a, b, M_i, M are given numbers.

As in [4] we can show that for any $\upsilon \in V$ problem (1), (2) is solvable in the space $Y_0 = \left\{ u|\, u \in L^\infty\left(0,T; H_0^1(\Omega) \cap L^p(\Omega)\right),\ \frac{\partial u}{\partial t} \in L^\infty\left(0,T; L^2(\Omega)\right)\right\}$.

Note, that for each control from V the solution of boundary-value problem (1), (2) is taken in sense of integral identity.

Expressing from equality (1) the second derivative by t, let's construct the inclusion $\frac{\partial^2 u}{\partial t^2} \in Y_1$, where $Y_1 = L^2(Q) + L^\infty\left(0,T; H^{-1}(\Omega) \cap L^{p'}(\Omega)\right)$, $1/p + 1/p' = 1$. Thus, the solvability of the problem is showed in the space $Y = \left\{ u|\, u \in Y_0,\ \frac{\partial^2 u}{\partial t^2} \in Y_1\right\}$. Let's note, that the uniqueness of the solution of problem (1), (2) is guaranteed only at small values of the exponent of nonlinearity ρ and dimension n (see [4]). We'll not impose these limitations, admitting possible non-uniqueness of solution of problem (1), (2).

Let's give the set U_d of admissible pairs of the system (1), (2), consisting of such pairs $y = (\upsilon, u) \in V \times Y$, which satisfy to these equalities.

Let's determine the functional

$$I(\upsilon, u) = \frac{\nu}{2}\int_Q \left[\upsilon^2(x,t) + \left(\frac{\partial \upsilon(x,t)}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial \upsilon(x,t)}{\partial x_i}\right)^2\right] dxdt +$$

$$+\frac{1}{2}\int_Q \sum_{i=1}^n \left|\frac{\partial (u - u_d)}{\partial x_i}\right|^2 dxdt, \qquad (3)$$

where $\nu > 0$ is a given number, $u_d \in L^2\left(0,T; H_0^1(\Omega)\right)$ is a given function.

Consider the following optimal control problem: to find an admissible pair, minimizing the functional $I(\upsilon, u)$ on the set U_d. This problem we'll call the problem (1)-(3).

Theorem 1. *Problem (1)-(3) is solvable.*

Proof. By virtue of boundedness from below of the functional $I(\upsilon, u)$ for the given problem there exists the minimizing sequence, i.e., such sequence $\{y_k\} = (\upsilon_k, u_k)$ of the elements of the set U_d, that $\lim_{k\to\infty} I(\upsilon_k, u_k) = \inf_{(\upsilon,u)\in U_d} I(\upsilon, u)$. By virtue of coercitivity of the functional in the space $U = H^1(Q) \times L^2\left(0,T; H_0^1(\Omega)\right)$. Multiplying equality (1) at $(\upsilon, u) = (\upsilon_k, u_k)$ by the derivative $\frac{\partial u_k}{\partial t}$ and integrating the result on the domain Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(\frac{\partial u_k}{\partial t}\right)^2 dx + \int_\Omega \upsilon_k \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2 dx + \frac{2}{p}\int_\Omega \sum_{i=1}^n |u_k|^p\, dx\right] \le$$

$$\le \frac{M}{2}\int_\Omega \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2 dx + \int_\Omega \left| f \frac{\partial u_k}{\partial t}\right| dx.$$

Hence, after the elementary transformations we have

$$\int_\Omega \left[\left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2 + |u_k|^p\right] dx \le$$

$$\leq c\int\limits_{\Omega}\left[\varphi_1^2(x)+\sum_{i=1}^{n}\left(\frac{\partial\varphi_0}{\partial x_i}\right)^2+|\varphi_0|^p\right]dx+$$

$$+c\int\limits_{Q}f^2(x,t)\,dxdt+c\int\limits_0^t\int\limits_{\Omega}\left[\left|\frac{\partial u_k}{\partial t}\right|^2+\sum_{i=1}^{n}\left(\frac{\partial u_k}{\partial x_i}\right)^2+|u_k|^p\right]dxdt.$$

Hence, subject to Gronwall's lemma we'll show the boundedness of the sequence $\{u_k\}$ in the space Y_0. Then, after the choosing of subsequence, we'll prove that $\upsilon_k\to\upsilon$ is weak in $H^1(Q)$ and $u_k\to u$ $*$ - weakly in Y_0 as $k\to\infty$. Considering that the set V is convex and closed in $H^1(Q)$ we have $\upsilon\in V$. Using the compactness of the space embedding $H^1(Q)$ in $L^2(Q)$ and Y_0 in $L^2(Q)$, after choosing subsequence we obtain that $\upsilon_k\to\upsilon$ is strong in $L^2(Q)$ and $u_k\to u$ is strong in $L^2(Q)$ and a.e. on Q, and so in the space $|u_k|^\rho u_k\to|u|^\rho u$, Q is bounded in the space $L^{p'}(Q)$. Then by the lemma from [4, p.25], we conclude that $|u_k|^\rho u_k\to|u|^\rho u$ weakly in $L^{p'}(Q)$ as $k\to\infty$.

Passing to the limit in the integral identity, corresponding to the equation

$$\frac{\partial^2 u_k}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon_k\frac{\partial u_k}{\partial x_i}\right)+|u_k|^\rho u_k=f$$

one may see that $y=\{\upsilon,u\}$ satisfies (1), (2) and the inclusion $y\in U_d$ is true. Considering for the properties of the norm we come to the inequality $\underline{\lim}_{k\to\infty} I(y_k)\geq I(y)$, whence it follows that $y=\{\upsilon,u\}$ is a solution of problem (1)-(3). Theorem is proved.

Let's note, that at proving on existence of optimal pair it is not important is the solution of problem (1), (2) unique or not on this or that controls.

Note, that we don't use any boundedness on the parameters ρ and n.

Usually, nonlinear optimal control problems are solved approximately. Under the approximate solution of minimization problem of the functional $I(\upsilon,u)$ on the set U_d it is understood the element of this set, sufficiently close to the minimum point y_0 of the functional on this set. Along with this applies a weaker form of the approximate solution, when is guaranteed only proximity value of the functional to the point I_0- it's minimum on the set U_d. Thus, strong approximate solution of the problem is close near to the exact solution of the problem in sense of admissible pairs, and the weak in the sense of functional. In both cases, an approximate solution is an element of the set U_d. Since the problem is solved approximately, we can admit the fulfillment of given boundedness aren't inexact, but with some error. Following to [1], [2] under the weak approximate solution of minimization problem of the functional $I(\upsilon,u)$ on U_d we'll understand such point y_* from the small neighbourhood $\varepsilon>0$ of the set $|I(y_*)-I_0|\leq\varepsilon$ that for the sufficient small number U_d the inequality is true. Thus, the weak approximate solution is not obliged to be part of the set, but is sufficiently close to any of its elements, while the value of the functional on it is sufficiently close to it minimum on this set.

2.Approximate solution of the problem

To solve the problem we'll use the penalty function method [5]

$$I_m(\upsilon,u)=I(\upsilon,u)+\frac{1}{2\varepsilon_m}\int\limits_{\Omega}\left[\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon\frac{\partial u}{\partial x_i}\right)+|u|^{\rho}u-f\right]^2 dxdt, \tag{4}$$

where $\{\varepsilon_m\}$ is a such numerical sequence that $\varepsilon_m>0$ and $\varepsilon_m\to 0$ as $m\to\infty$.

The problem is: to find an admissible pair, minimizing the functional $I_m(\upsilon,u)$ on the set U_*, $\upsilon\in V$, $u\in Y$ at fulfillment of conditions (2). This problem we'll denote by P_m.

Theorem 2. *The problem P_m is solvable.*

Proof. Let $y_k=\{\upsilon_k,u_k\}$ be a minimazing sequence for the problem P_m. It is clear that it is bounded in the space $U=H^1(Q)\times L^2\left(0,T;H_0^1(\Omega)\right)$, moreover there exists bounded in $L^2(Q)$ sequence $\{q_k\}$, for which the equation

$$\frac{\partial^2 u_k}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon_k\frac{\partial u_k}{\partial x_i}\right)+|u_k|^{\rho}u_k=g_k \tag{5}$$

is true, with the corresponding boundary conditions. As at proving of theorem 1, we'll show the boundedness of the sequence $\{u_k\}$ in the space Y. Then after the choosing of subsequence, we'll construct the convergence $\upsilon_k\to\upsilon$ weakly in $H^1(Q),u_k\to u$ $*$ - weakly in Y and $g_k\to g$ weakly in $L^2(Q)$, moreover $\upsilon\in V$. Repeating the considerations from the proving of theorem 1, we conclude that $|u_k|^{\rho}u_k\to|u|^{\rho}u$ and weakly in $L^{p'}(Q)$ as $k\to\infty$. Then taking a limit, we'll construct the relation

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon\frac{\partial u}{\partial x_i}\right)+|u|^{\rho}u=g. \tag{6}$$

From equality (5) follows the inclusion $\dfrac{\partial^2 u_k}{\partial t^2}\in Y_1$, that means, $u_k\in Y$. Then multiplying equality (5) by sufficiently smooth function $\lambda(x,t)$, $\lambda(x,T)=0$, $\dfrac{\partial\lambda(x,T)}{\partial t}=0$ after integrating on the domain Q considering for the integrating formula by parts and initial conditions for the functions u_k we obtain

$$\int\limits_{Q}\frac{\partial^2\lambda}{\partial t^2}u_k dxdt+\int\limits_{Q}\left(-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon_k\frac{\partial u_k}{\partial x_i}\right)+|u_k|^{\rho}u_k-g_k\right)\lambda dxdt-$$

$$-\int\limits_{\Omega}\lambda(x,0)\varphi_1(x)\,dx+\int\limits_{\Omega}\frac{\partial\lambda(x,0)}{\partial t}\varphi_0(x)\,dx=0.$$

Passing here to the limit as $k\to\infty$ and again integrating by parts considering equality (6), we have

$$\int\limits_{\Omega}\lambda(x,0)\left[\frac{\partial u(x,0)}{\partial t}-\varphi_1(x)\right]dx+\int\limits_{\Omega}\frac{\partial\lambda(x,0)}{\partial t}\left[\varphi_0(x)-u(x,0)\right]dx=0.$$

At first, choosing here $\lambda(x,0)=0$, $\dfrac{\partial\lambda(x,0)}{\partial t}\neq 0$, we see $u(x,0)=\varphi_0(x)$, further assuming $\lambda(x,0)\neq 0$, $\dfrac{\partial\lambda(x,0)}{\partial t}=0$, we have $\dfrac{\partial u(x,0)}{\partial t}=\varphi_1(x)$. Thus, the function $u(x,t)$ satisfies the boundary conditions (2). Using the obtained above conditions, we have the inequality $\underline{\lim}_{k\to\infty} I_m(y_k)\geq I_m(y)$, whence it follows that y is a solution of the problem P_m. Theorem is proved.

Denote by $y_m=(\upsilon_m,u_m)$ the solution of the problem P_m, and by $y_0=(\upsilon_0,u_0)$ the solution of problem (1)-(3).

Theorem 3. *As $m\to\infty$ the convergence*

$$\lim_{m\to\infty} I_m(y_m)=\min_{(\upsilon,u)\in V\times Y}(\upsilon,u)\equiv \min I(U_d).$$

is valid.

Proof. The following relations

$$I_m(y_m)=\min I_m(U_*)\leq I_m(y_0)=I(y_0)=\min I(U_d). \tag{7}$$

are true.

Then from the definition of functional (4) follows the boundedness of the sequence $\{y_m\}$ in the space U. Besides, the equality

$$\frac{\partial^2 u_m}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon_m\frac{\partial u_m}{\partial x_i}\right)+|u_m|^{\rho}u_m=f+\sqrt{\varepsilon_m}f_m \tag{8}$$

is true, with corresponding boundary conditions, where the sequence $\{f_m\}$ is bounded in $L^2(Q)$. Hence, it follows the boundedness of the sequence $\{u_m\}$ in Y. Then by choosing of subsequence we prove the convergence $u_m\to u$ $*-$weakly in Y, $\upsilon_m\to\upsilon$ weakly in $H^1(Q)$, moreover $\upsilon\in V$. As a result of the transition to the limit in equality (8) we obtain equality (1), whence it follows that the point $y=(\upsilon,u)$ belongs to U_d.

It is clear that the inequality $I_m(y_m)\geq I(y_m)$ is true. Then, considering definition of the functional we have $\underline{\lim}_{m\to\infty} I_m(y_m)\geq \underline{\lim}_{m\to\infty} I(y_m)\geq I(y)$. Hence and from condition (7) we obtain $I(y)\leq \underline{\lim}_{m\to\infty} I_m(y_m)\leq \min I(U_d)$, and it means $I(y)=\min I(U_d)$. So, the inequalities

$$\min I(U_d)\leq \lim_{m\to\infty} I(y_m)\leq \underline{\lim}_{m\to\infty} I_m(y_m)\leq \overline{\lim}_{m\to\infty} I_m(y_m)\leq \min I(U_d),$$

are true, whence the statements of theorems are concluded.

In the process of proving of theorems it was also shown that the subsequence $\{y_m\}$ converges $*-$ weakly to admissible pair of system (1), (2). If at determining of weak approximate solution the neighbourhood understood in corresponding $*-$ weak topology, then we lead to the following statement.

Corollary. *At sufficiently large values m the solution y_m of the problem P_m is a weak approximate solution of problem (1)-(3).*

So, for finding the approximate solution of initial problem, it is enough to find the solution of the problem P_m.

The problem P_m is minimization problem for the smooth functional, that allows one to obtain for corresponding optimality condition.

Theorem 4. *The solution of problem P_m is determined by the relations*

$$\int\limits_Q \left\{ \nu \left[\upsilon_m (\upsilon - \upsilon_m) + \frac{\partial \upsilon_m}{\partial t} \frac{\partial (\upsilon - \upsilon_m)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \upsilon_m}{\partial x_i} \frac{\partial (\upsilon - \upsilon_m)}{\partial x_i} \right] - \right.$$

$$\left. -\psi_m \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left((\upsilon - \upsilon_m) \frac{\partial u_m}{\partial x_i} \right) \right\} dxdt \geq 0, \forall \upsilon \in V, \tag{9}$$

$$\frac{\partial^2 \psi_m}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\upsilon_m \frac{\partial \psi_m}{\partial x_i} \right) + (\rho + 1) |u_m|^\rho \psi_m = \sum_{i=1}^{n} \frac{\partial^2 (u_d - u)}{\partial x_i}, \ (x,t) \in Q, \tag{10}$$

$$\psi_m (x,t) = 0, \ (x,t) \in S, \ \psi_m (x,T) = 0, \ \frac{\partial \psi_m (x,T)}{\partial t} = 0, \ x \in \Omega, \tag{11}$$

$$\frac{\partial^2 u_m}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\upsilon_m \frac{\partial u_m}{\partial x_i} \right) + |u_m|^\rho u_m = f + \varepsilon \psi_m, \ (x,t) \in Q, \tag{12}$$

$$\begin{gathered} u_m (x,t) = 0, \ (x,t) \in S, \ u_m (x,0) = \varphi_0 (x), \\ \frac{\partial u_m (x,0)}{\partial t} = \varphi_1 (x), \ x \in \Omega. \end{gathered} \tag{13}$$

Proof. In order to the point y_* minimized the differentiable functional J on a convex subset U of a Banach space it is necessary that the variational inequality

$$\left\langle J' (y_*), y - y_* \right\rangle \geq 0, \ \forall y \in U, \tag{14}$$

is fulfilled, where by $J' (y_*)$ is denoted the derivative by the convex set of the functional $J (y)$ at the indicated point, and $< F, y >$ is a value of linear continuous functional F at the point y. For the problem P_m the minimazing functional depends on two arguments. Let's find its partial derivatives at the point $y_m = (\upsilon_m, u_m)$ from the following relations

$$\langle I_{m\upsilon} (y_m), \upsilon - \upsilon_m \rangle = \lim_{\sigma \to 0} \left[I_m (\upsilon_m + \sigma (\upsilon - \upsilon_m), u_m) - I_m (\upsilon_m, u_m) \right] / \sigma, \ \forall \upsilon \in V$$

$$\langle I_{mu} (y_m), g \rangle = \lim_{\sigma \to 0} \left[I_m (\upsilon_m, u_m + \sigma g) - I_m (\upsilon_m, u_m) \right] / \sigma, \ \forall g \in Y.$$

Using the expression of the functional $I_m (\upsilon, u)$ we find that

$$\langle I_{m\upsilon} (y_m), \upsilon - \upsilon_m \rangle =$$

$$= \int\limits_Q \left\{ \nu \left[\upsilon_m (\upsilon - \upsilon_m) + \frac{\partial \upsilon_m}{\partial t} \frac{\partial (\upsilon - \upsilon_m)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \upsilon_m}{\partial x_i} \frac{\partial (\upsilon - \upsilon_m)}{\partial x_i} \right] - \right.$$

$$\left. - \psi_m \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left((\upsilon - \upsilon_m) \frac{\partial u_m}{\partial x_i} \right) \right\} dxdt,$$

$$\langle I_{mu} (y_m), g \rangle = \int\limits_Q \left\{ \sum_{i=1}^{n} \frac{\partial (u_m - u_d)}{\partial x_i} \frac{\partial g}{\partial x_i} + \right.$$

$$+\psi_m \left[\frac{\partial^2 g}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\upsilon_m \frac{\partial g}{\partial x_i} \right) + (\rho + 1) \left| u_m \right|^{\rho} g \right] \right\} dxdt,$$

where

$$\psi_m = \left(\frac{\partial^2 u_m}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\upsilon_m \frac{\partial u_m}{\partial x_i} \right) + \left| u_m \right|^{\rho} u_m - f \right) / \varepsilon_m.$$

Hence, it follows that equations (12) is true, and condition (13) it fulfilled, since minimization I_m given between all functions, satisfying conditions (2). Considering the definition of the set U_d we conclude, that extremum condition (14) leads to the variational inequality for the derivative of the functional by the control and stationary condition for derivative by the function state. The first of the given relations is condition (9), and the second takes the following form

$$\int_Q \left\{ \sum_{i=1}^{n} \frac{\partial \left(u_m - u_d \right)}{\partial x_i} \frac{\partial g}{\partial x_i} + \right.$$

$$+\psi_m \left[\frac{\partial^2 g}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\upsilon_m \frac{\partial g}{\partial x_i} \right) + (\rho + 1) \left| u_m \right|^{\rho} g \right] \right\} dxdt = 0, \ \forall g \in Y \ . \qquad (15)$$

Considering the for definition of the function ψ_m we'll prove the inclution $\psi_m \in Y_1$. Then relation (15) determines the weak (in sense of indicated inclusion) solution of the boundary value problem (10), (11) ([5], pp.163-164).

Thus, for finding the solution of the problem P_m we obtain the system of optimality conditions of (9)-(13). Theorem is proved.

According to theorem 3, at sufficiently large values m the corresponding pair (υ_m, u_m) becomes a weak approximate solution of the problem, i.e., the value of the functional I on it will be enough close to its minimum on the set U_d, and relation (1), (2) will be fulfilled with a sufficiently high degree of accuracy. In particular, equality (12) at small ε_m we can understood as approximate form of the equation of relation (1).

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Received March 26, 2010; Revised May 31, 2010.