Zammedin ISMAILOV, M. EROL

THE SPECTRUM STRUCTURE OF NORMAL OPERATORS

Abstract

In this work the structure of the spectrum of a normal operator is investigated in terms of the spectra of its real and imaginary parts. Furthermore, it is established an asymptotical formula of the modules of the eigenvalues of the normal operators with discrete spectrum in the language of the asymptotical behaviour of the eigenvalues its real and imaginary parts.

1. Introduction

It is known that a densely defined closed linear operator A in the Hilbert space H with domain $D(A)$, $A : D(A) \subset H \to H$, is called a normal operator if $D(A) = D(A^*)$ and for each element $x \in D(A)$ the condition $\|Ax\|_H = \|A^*x\|_H$ holds. The general theory of the normal operators and its spectral theory have been studied in [1-11]. However, in these works spectral structure properties of normal operators have not been constructively investigated. For the latter investigation, let $\rho(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$, $H_\lambda(A)$ denote resolvent set, spectrum, point, continuous, residual spectrums, linear subspace of eigenvectors corresponding to $\lambda \in \sigma_p(A)$ of an operator$A$ respectively. It is known that for any linear normal operator A in the Hilbert space H $\sigma_r(A) = \emptyset$ [2].

On the other hand in many books and papers the spectral properties and asymptotical behavior of the eigenvalues of the linear densely defined self adjoint operators in any Hilbert space are well studied.

This work consists of two sections. In the first section a formula for the spectrum of the one subclass normal operators is given and in the second section an asymptotical formula for the modules of eigenvalues of such normal operators with discrete spectrum in the any Hilbert space is established.

2. Structure of spectrum of the normal operators

In this and next sections, let $A_R(\lambda_r)$ and $A_I(\lambda_i)$ denote respectively real and imaginary parts of an operator A (number $\lambda \in \mathbb{C}$) in any Hilbert space, i.e.

$$A_R = \overline{\frac{1}{2}(A + A^*)}, \qquad A_I = \overline{\frac{1}{2i}(A - A^*)}$$

$$\left(\lambda_r = \frac{1}{2}\left(\lambda + \overline{\lambda}\right)\right), \qquad \left(\lambda_i = \frac{1}{2i}\left(\lambda - \overline{\lambda}\right)\right).$$

Theorem 2.1. *If A is a normal operator in any Hilbert space H, then*

$$\sigma_p(A) = \sigma_p(A_R) \boxplus i\sigma_p(A_I),$$

where $\boxplus$ *denotes special algebraic sum of* $\lambda_r \in \sigma_p(A_R)$ *and* $\lambda_i \in \sigma_p(A_I)$ *such that* $H_{\lambda_r}(A_R) \cap H_{\lambda_i}(A_I) \neq \{0\}$.

Moreover, if at least one of the sets $\sigma_p(A_R)$ *and* $\sigma_p(A_I)$ *is empty, then* $\sigma_p(A)$ *is empty and is vice versa.*

Proof. If A is a normal operator in H, then it is clear that the operator $A + zE$ is normal for any number $z \in \mathbb{C}$. In this case it is easy to see that

$$\| (A - \lambda E) x\|_H^2 = \| (A_R - \lambda_r E) x\|_H^2 + \| (A_I - \lambda_i E) x\|_H^2$$

for every complex number $\lambda \in \mathbb{C}$ and for any element $x \in D(A)$.

From this if $\lambda = \lambda_r + i\lambda_i \in \sigma_p(A)$, then we get $\lambda_r \in \sigma_p(A_R)$ and $\lambda_i \in \sigma_p(A_I)$ and these two have the same eigenvector $x \in H_{\lambda_r}(A_R) \cap H_{\lambda_i}(A_I)$. On the other hand, if $\lambda_r \in \sigma_p(A_R)$ and $\lambda_i \in \sigma_p(A_I)$ with eigenvector $x \in H_{\lambda_r}(A_R) \cap H_{\lambda_i}(A_I)$, $x \neq 0$, then the above equality implies that $\lambda \in \sigma_p(A)$ with eigenvector x.

Example 2.2. Let us $H = L^2(0,1)$, $A_I = -i\frac{d}{dt}$, $A_R = E$, $D(A_I) = \{u : u' \in L^2(0,1), u(0) = u(1)\}$, $D(A_R) = L^2(0,1)$. In this case $A_R = A_R{}^*$, $A_I = A_I^*$ and an operator $A = \frac{d}{dt} + E$ is a normal in $L^2(0,1)$. It is clear that $\sigma(A_R) = \sigma_p(A_R) = \{1\}$, $H_1(A_R) = H$, $\sigma_p(A_I) = \{2n\pi : n \in \mathbb{Z}\}$, $H_{\lambda_n}(A_I) = H_{2n\pi}(A_I) = span\left(e^{2n\pi i}\right), n \in \mathbb{Z}$. Then by the above Theorem 2.1

$$\sigma_p(A) = \sigma_p(A_R) \boxplus i\sigma_p(A_I) = \{1 + 2n\pi i : n \in \mathbb{Z}\}.$$

Theorem 2.3. *The joint spectrum of two commuting operators is contained in the Cartesian product of their spectra.*

Proof. Let $A = A_R + iA_I$ be a normal operator in the Hilbert space H and $\lambda \in \sigma(A)$. Then for any $\varepsilon > 0$ there exists $x_\varepsilon \in D(A)$ such that

$$\| (A - \lambda E) x_\varepsilon\| \leq \varepsilon \|x_\varepsilon\|$$

(see [1]). Since the operator A is normal, then

$$\| (A_R - \lambda_r E) x_\varepsilon\|^2 + \| (A_I - \lambda_i E) x_\varepsilon\|^2 = \| (A - \lambda E) x_\varepsilon\|^2 \leq \varepsilon^2 \|x_\varepsilon\|^2$$

From this we obtain

$$\| (A_R - \lambda_r E) x_\varepsilon\| \leq \varepsilon \| x_\varepsilon\|, \quad \| (A_I - \lambda_i E) x_\varepsilon\| \leq \varepsilon \| x_\varepsilon\|.$$

Hence from last relation and [1] we have $\lambda_r \in \sigma(A_R)$ and$\lambda_i \in \sigma(A_I)$ .Therefore,

$$\sigma(A) \subset \sigma(A_R) + i\sigma(A_I).$$

Theorem 2.4. *If* A *is a normal operator in the Hilbert space* H *and* $\rho(A_R) \neq \emptyset$ $(\rho(A_I) \neq \emptyset)$*, then*

$$\rho(A_R) + i\mathbb{R} \subset \rho(A) \qquad (\mathbb{R} + i\rho(A_I) \subset \rho(A)).$$

Proof. Let $\lambda_r \in \rho(A_R)$. In this case for the number $\lambda = \lambda_r + i\mu, \quad \mu \in \mathbb{R}$,

$$\begin{aligned} A - \lambda E &= (A_R - \lambda_r E) + i (A_I - \mu E) \\ &= i (A_R - \lambda_r E) \left[(A_R - \lambda_r E)^{-1} (A_I - \mu E) - iE \right]. \end{aligned}$$

Now we show that the operator $T := (A_R - \lambda_R E)^{-1} (A_I - \mu E)$, $\lambda_r \in \rho(A_R)$, $\mu \in \mathbb{R}$ is closed in any Hilbert space H and investigate this case in general. We prove that if T and S are two linear closed operators in H and $S^{-1} \in L(H)$, then the operator $K := TS^{-1}$ is closed in H. Let an arbitrary sequence (x_n) be defined in $D(K)$ such that $x_n \xrightarrow{H} x$ and $Kx_n = TS^{-1}x_n \xrightarrow{H} y$ as $n \to \infty$. Because of $S^{-1} \in L(H)$ $S^{-1}x_n \xrightarrow{H} S^{-1}x$ as $n \to \infty$. On the other hand since $S^{-1}x_n \xrightarrow{H} S^{-1}x$ and $T\left(S^{-1}x_n\right) \xrightarrow{H} y$ as $n \to \infty$ and hence T is a closed operator in H, then $y = TS^{-1}x$ and $S^{-1}x$ is an element in $D(K)$. From these results we get $y = TS^{-1}x$ and the element x is in $D\left(TS^{-1}\right)$ and so the operator $K = TS^{-1}$ is closed in H. Hence we have $\lambda \in \rho(A)$. On the other hands, since $A_R A_I = A_I A_R$ then for every $\lambda := \lambda_r + i\mu \in \mathbb{C}$ the operator $A - \lambda E = (A_R - \lambda_r E) + i (A_I - \mu E)$ is normal. Hence

$$(A_R - \lambda_r E)(A_I - \mu E) = (A_I - \mu E)(A_R - \lambda_r E)$$

and for every $\lambda_r \in \rho(A_R)$

$$A_I - \mu E \supset (A_R - \lambda_r E)^{-1} (A_I - \mu E)(A_R - \lambda_r E)$$

is obtained. From the last equation

$$(A_I - \mu E)(A_R - \lambda_r E)^{-1} \supset (A_R - \lambda_r E)^{-1} (A_I - \mu E)$$

is hold. It means that these operators are commutative. Moreover

$$\begin{aligned} T^* &= \left[(A_R - \lambda_r E)^{-1} (A_I - \mu E) \right]^* \supset (A_I - \mu E)^* \left[(A_R - \lambda_r E)^{-1} \right]^* \\ &= (A_I - \mu E)(A_R - \lambda_r E)^{-1} \supset (A_R - \lambda_r E)^{-1} (A_I - \mu E) \\ &= T \end{aligned}$$

i.e. $T^* \supset T$. Hence T is symmetric.

Now we show that the operator $T = (A_R - \lambda_r E)^{-1} (A_I - \mu E) : D(T) \to H$ for any $\lambda_r \in \rho(A_R)$ and $\lambda_i \in \mathbb{R}$ is a self-adjoint. For that it is sufficient to show that the deficiency indices of T is $n_+ = n_- = 0$. Firstly, let us consider the equation

$$(A_I - \mu E)(A_R - \lambda_r E)^{-1} x \pm ix = 0, \; x \in D(T^*).$$

In this case

$$(A_I - \mu E)(A_R - \lambda_r E)^{-1} x \pm (A_R - \lambda_r E)(A_R - \lambda_r E)^{-1} x = 0$$

is hold. From this

$$[(A_I - \mu E) \pm i (A_R - \lambda_r E)] (A_R - \lambda_r E)^{-1} x = 0.$$

For $y := (A_R - \lambda_r E)^{-1} x$ the last equation can be written in form $(A_I - \mu E) y \pm i (A_R - \lambda_r E) y = 0$. Since A is a normal operator then

$$\|(A_I - \mu E) y\|^2 + \|(A_R - \lambda_r E) y\|^2 = 0.$$

From this $(A_I - \mu E) y = 0$ and $(A_R - \lambda_r E) y = 0$. Because of $\lambda_r \in \rho(A_R)$, $y = 0$ is obtained. It means that $(A_R - \lambda_r E)^{-1} x = 0$, i.e.

$$(A_R - \lambda_r E) (A_R - \lambda_r E)^{-1} x = (A_R - \lambda_r E)^{-1} 0 = 0.$$

Then $x = 0$ and therefore $n_+ = n_- = 0$ is hold. Hence for any numbers $\lambda_r \in \rho(A_R)$ and $\mu \in \mathbb{R}$ the operator T is a self adjoint in H and so $i \in \mathbb{C}$ is a regular point of this operator.

The second part of this theorem can be proved similarly.

In this section one of the main purpose is to establish the formula $\sigma(A) = \sigma(A_R) + i\sigma(A_I)$ for the one class of normal operators in any Hilbert space H.

Now we give an example which shows that this formula is not true in general for normal operators.

Example 2.5. Let B be a linear bounded selfadjoint operator in the Hilbert space$H$ such that $\pi, \frac{3\pi}{2} \in \sigma(B)$ and $A = e^{iB}$. Then A is a linear unitary operator in H [1] and $A = \cos B + i \sin B$, $A_R = \cos B, A_I = \sin B$. By the spectral mapping theorem [1] $\sigma(\sin B) = \sin(\sigma(B))$, $\sigma(\cos B) = \cos(\sigma(B))$ and $0 \in \sigma(\sin B)$, $0 \in \sigma(\cos B)$. But since A is the unitary operator in H, then $0 \notin \sigma(A)$, i.e. $0 = 0 + i0 \notin \sigma(A)$ Hence, the formula $\sigma(A) = \sigma(A_R) + i\sigma(A_I)$ is not valid in this simple case.

Now we will give a theorem in which the above formula given for the spectrum will be established. If A is a linear bounded normal operator in separable Hilbert space H, then there exists a linear self adjoint operator B and a bounded measurable function $f : \mathbb{R} \to \mathbb{C}$ such that $A = f(B)$ (for the purely point spectrum [13], for the general case [14]). This result has been proved for the unbounded linear normal operators by Y. Mimura [15]. In the both cases a measurable function $f : \mathbb{R} \to \mathbb{C}$ can be chosen continuously (see [16]).

Theorem 2.6. *Let $A \in L(H)$ be a normal operator in separable Hilbert space H and $A = f(B)$. Then the equality $\sigma(A) = \sigma(A_R) + i\sigma(A_I)$ holds if and only if for every $x, y \in \sigma(B)$ there is an element $z = z(x, y)$ in $\sigma(B)$ which satisfies $f_R(z) = f_R(x)$ and $f_I(z) = f_I(y)$, where $f = f_R + i f_I$.*

This claim does not depend on the description $A = f(B)$.

Proof. From the theorem 2.3 it is clear that the relation $\sigma(A) \subset \sigma(A_R) + i\sigma(A_I)$ is true. Now let us prove that $\sigma(A_R) + i\sigma(A_I) \subset \sigma(A)$. According to spectral mapping theorem this result is equal to a relation $f_R(\sigma(B)) + i f_I(\sigma(B)) \subset f(\sigma(B))$. The last relation means that for every $x, y \in \sigma(B)$ there is an element $z = z(x, y)$ in $\sigma(B)$ which satisfies $f_R(z) = f_R(x)$, $f_I(z) = f_I(y)$ and is vice versa.

Now let $f(x)$ and $h(x)$ be two continuous functions from $\mathbb{R}$ to $\mathbb{C}$ and B and C be two linear selfadjoint operators in H such that $A = f(B)$ and $A = h(C)$. Firstly, suppose that the equalities $A = f(B)$ and $\sigma(A) = \sigma(A_R) + i\sigma(A_I)$ are true, in other words, for all $x, y \in \sigma(B)$ there is an element z in $\sigma(B)$ which satisfies $f_R(z) = f_R(x)$ and $f_I(z) = f_I(y)$.

Now let $A = h(C)$, $C = C^*$, and arbitrary two elements α and β be in $\sigma(C)$. It is known that the equality $\sigma(A) = f(\sigma(B))$ and $\sigma(A) = h(\sigma(C))$. In this case, since $h(\alpha), h(\beta) \in \sigma(A)$, then there are two elements x_α and y_β in $\sigma(B)$ such that $h(\alpha) = f(x_\alpha)$ and $h(\beta) = f(y_\beta)$, i.e.

$$h(\alpha) = h_R(\alpha) + ih_I(\alpha) = f_R(x_\alpha) + if_I(x_\alpha) \tag{1}$$

$$h(\beta) = h_R(\beta) + ih_I(\beta) = f_R(y_\beta) + if_I(y_\beta). \tag{2}$$

Hence, since the equality $\sigma(A) = \sigma(A_R) + i\sigma(A_I)$ holds and the elements x_α and y_β are in $\sigma(B)$, then there exists an element $z_{\alpha,\beta} = z(x_\alpha, y_\beta) = z(\alpha, \beta)$ in $\sigma(B)$ such that

$$f_R(x_\alpha) = f_R(z_{\alpha,\beta}),$$

$$f_I(y_\beta) = f_I(z_{\alpha,\beta}).$$

Therefore, (2.1) and (2.2) imply that

$$\left.\begin{array}{l} h_R(\alpha) = f_R(x_\alpha) = f_R(z_{\alpha,\beta}) \\ h_I(\beta) = f_I(y_\beta) = f_I(z_{\alpha,\beta}) \end{array}\right\} \tag{3}$$

and also there is an element $\gamma_{\alpha,\beta}$ in $\sigma(C)$ for the element $z_{\alpha,\beta}$ such that $f(z_{\alpha,\beta}) = h(\gamma_{\alpha,\beta})$. Hence,

$$f_R(z_{\alpha,\beta}) = h_R(\gamma_{\alpha,\beta}),$$

$$f_I(z_{\alpha,\beta}) = h_I(\gamma_{\alpha,\beta}).$$

It follows from the previous equations and (2.3) that for every $\alpha, \beta \in \sigma(C)$ there exists an element $\gamma = \gamma_{\alpha,\beta}$ in $\sigma(C)$ such that

$$\begin{array}{l} h_R(\alpha) = h_R(\gamma_{\alpha,\beta}) \\ h_I(\alpha) = h_I(\gamma_{\alpha,\beta}). \end{array}$$

which proves the theorem.

Remark 2.7. If at least one of the functions f_R, $f_I : \mathbb{R} \to \mathbb{R}$ is constant on $\sigma(B)$, then conditions in the theorem 2.6 hold.

Example 2.8. Let us consider an operator $A : L^2(0,1) \to L^2(0,1)$ $Af(t) := f(t) + i\int\limits_0^1 k(t,s)f(s)ds$ where $k(t,s) := \begin{cases} t(1-s), & if \quad 0 \le t \le s \le 1 \\ s(1-t), & if \quad 0 \le s \le t \le 1 \end{cases}$. It is easy to see that $Bf(t) := \int\limits_0^1 k(t,s)f(s)ds$ is a compact selfadjoint operator in $L^2(0,1)$ Hilbert space and

$$\sigma(B) = \sigma_p(B) \cup \sigma_c(B) = \left\{ \frac{1}{\pi^2 n^2} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Moreover $A = E + iB$ can be written. In this case $f_R(\lambda) = 1$ and $f_R(\lambda) = \lambda, \quad \lambda \in \mathbb{R}$. Because the conditions of theorem 2.6 hold, the relation $\sigma(A) = \sigma(A_R) + i\sigma(A_I)$ is true, i.e.

$$\sigma(A) = \left\{1 + i\frac{1}{\pi^2 n^2} : n \in \mathbb{N}\right\} \cup \{1\}.$$

The following results can be proved easily.

Corollary 2.9. *If U is a unitary operator in any Hilbert space H and $U = e^{iC}$, $C = C^*$, then the equality $\sigma(C) = \sigma(\cos C) + i\sigma(\sin C)$ is true if and only if $x + y = n\pi$ or $x - y = m\pi$, $n, m \in \mathbb{Z}$, is satisfied, where $x, y \in \sigma(C)$.*

Corollary 2.10. *If $U = e^{iC}$, $C = C^*$, is a unitary operator in any Hilbert space H and $\sigma_c(C) \neq \emptyset$, then the equality $\sigma(C) = \sigma(\cos C) + i\sigma(\sin C)$ is not true (see [2], p.312).*

3. Asymptotical behavior of the modules of the eigenvalues of normal operators

In this section we will investigate discreteness of the spectrum and asymptotical behavior of the modules of eigenvalues of normal operators in any Hilbert space H.

We denote by $\mathfrak{S}_p(H)$, $p \geq 1$, the Schatten-von Neumann class of operators and $B(H)$ the space of linear bounded operators in the Hilbert space H [1].

Theorem 3.1. *Let A be a normal operator in H. If $\left(A_R - \lambda_r^0 E\right)^{-1} \in \mathfrak{S}_p(H)$, $p \geq 1$, for some $\lambda_r^0 \in \mathbb{R}$, then for any $\lambda = \lambda_r^0 + i\lambda_i$, $\lambda_i \in \mathbb{R}$, $(A - \lambda E)^{-1} \in \mathfrak{S}_p(H)$, $p \geq 1$.*

Similarly, if for some $\lambda_r^0 \in \mathbb{R}$, $\left(A_I - \lambda_i^0 E\right)^{-1} \in \mathfrak{S}_p(H)$, $p \geq 1$ then for any $\lambda = \lambda_r + i\lambda_i^0$, $\lambda_r \in \mathbb{R}$, $(A - \lambda E)^{-1} \in \mathfrak{S}_p(H)$, $p \geq 1$.

Proof. For any $\lambda = \lambda_r^0 + i\lambda_i$, $\lambda_i \in \mathbb{R}$ we can write

$$A - \lambda E = \left(A_R - \lambda_r^0 E\right) + i\left(A_I - \lambda_i E\right) =$$

$$= i\left(A_R - \lambda_r^0 E\right)\left[\left(A_R - \lambda_r^0 E\right)^{-1}\left(A_I - \lambda_i E\right) - iE\right].$$

According to the stimulus in the proof of Theorem 2.4 the operator $\left(A_R - \lambda_r^0 E\right)^{-1}\left(A_I - \lambda_i E\right)$ is closed in Hilbert space H. Since the operator $\left(A_R - \lambda_r^0 E\right)^{-1}\left(A_I - \lambda_i E\right)$ is a self adjoint operator, then the point $i \in \mathbb{C}$ is their regular point, i.e. $\left(\left(A_R - \lambda_r^0 E\right)^{-1}\left(A_I - \lambda_i E\right) - iE\right)^{-1} \in B(H)$. Hence from the last equality we get

$$(A - \lambda E)^{-1} = (-i)\left(A_R - \lambda_r^0\right)^{-1}\left(\left(A_R - \lambda_r^0 E\right)^{-1}\left(A_I - \lambda_i E\right) - iE\right)^{-1} \in \mathfrak{S}_p(H).$$

In a similar manner, one can prove the second part of the theorem.

Definition 3.2. *Let (a_n) and (b_n) be two sequences of real numbers. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$, then we will write $a_n \sim b_n$, as $n \to \infty$.*

Theorem 3.3. *Let $A_R^{-1}, A_I^{-1} \in \mathfrak{S}_\infty(H)$, $\lambda_n(A_R) \sim an^\alpha$, $\lambda_n(A_I) \sim bn^\beta$, $0 < a, b, \alpha, \beta < +\infty$, as $n \to \infty$ and $\lambda_m(A) = \lambda_{p_m}(A_R) + i\lambda_{q_m}(A_I) \in \sigma_p(A)$, $q_m =$*

$q_m(p_m)$, $m \geq 1$ such that $\lambda_{p_m}(A_R) \sim cm^{\gamma}$, $\lambda_{q_m}(A_I) \sim dm^{\sigma}$, $0 < c, d, \gamma, \sigma < +\infty$, then $\lambda_m(A) \sim em^{\max(\gamma,\sigma)}$ where $0 < e < +\infty$, as $m \to \infty$.

Proof. From Theorem 3.1 we have $A^{-1} \in \mathfrak{S}_{\infty}(H)$. On the other hand, by Theorem 2.1 it is clear that $\lambda_{k,l}(A) = \lambda_k(A_R) + i\lambda_l(A_I)$ $k, l \geq 1$ and $l = l(k)$.

Therefore,

$$\begin{aligned} |\lambda_m(A)| &= \left(\left| \lambda_{p_m}(A_R) \right|^2 + \left| \lambda_{q_m}(A_I) \right|^2 \right)^{\frac{1}{2}} \sim \left(c^2 m^{2\gamma} + d^2 m^{2\sigma} \right)^{\frac{1}{2}} \\ &= m^{\max(\gamma,\sigma)} \left(c^2 m^{2\gamma - 2\max(\gamma,\sigma)} + d^2 m^{2\sigma - 2\max(\gamma,\sigma)} \right) \sim em^{\max(\gamma,\sigma)} \end{aligned}$$

as $m \to \infty, 0 < e < +\infty$, which completes the proof.

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Zammedin Ismailov, M. Erol
Karadeniz Technical University,
Faculty of Arts and Sciences
Department of Mathematics,
61080 Trabzon, Turkey
E-mail: zameddin@yahoo.com, meltemaysev@hotmail.com
Tel.: (99412) 439 47 20 (off.).

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