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# A MODEL OF MOTION OF TWO INTERACTING PARTICLES

**Abstract**

*The paper is devoted to construction and investigation of mathematical models (asymmetric) of $S = 2$ moving particles on a ring without overtaking with arbitrary parameters. Necessary and sufficient conditions are found when the separately considered particle performs a random binomial motion. A class of distributions of distances between the moving particles is described.*

**1. Introduction.** In connection with wide use of mathematical models of moving particles in different applications (transport problems, service nets, computer nets and etc.), there is a great interest to investigation of such models. In 1969, Yu.K. Belyayev [1] constructed a simplified mathematical model of motion of two particles on a straight line without overtaking describing behavior of transport systems. It was unexpectedly revealed that in stationary motion a separately considered particle performs random binomial walk. In [2], this result was generalized for models with a great number of particles when the particles motion depends on distance between them. This effect allowed to calculate some characteristics of transport flows and reveal undesirable phenomena as jams in transport systems. More complicated mathematical models arise in investigating motion of particles on closed contrours, for example on a ring, since each particle may brake the motion of other particle.

Asymmetric mathematical model of motion of two particles on a ring (when one particle is leading and the motion of another one may be braken by a leader) is constructed and studied in [3], where invariant character of binomial walk of separately considered particle is proved. In [4], similar effect was revealed for a symmetric model of motion of $S > 2$ particles on a ring, where a class of distributions arising between moving particles is also described. The above mentioned models are discrete and motion happens at discrete time. Continuous motion models were investigated in [5], where the motion diagram determining steady state mode of motion in these models is found.

The models and the method stated in [1] were used in [6] for planning transport motion of Moscow in annular road.

The paper is devoted to construction and investigation of mathematical models (asymmetric) of $S = 2$ moving particles on a ring without overtaking with arbitrary parameters. Necessary and sufficient conditions are found when a separately considered particle performs a random binomial walk. A class of distributions of distances between the moving particles is described.

**2. Model's description.** Let's consider a unit radius circle and clockwise numbered two particles that move (counterclokwise) on equidistant points of the circle. The motion happens at discrete time $t \in T = \{0, h, 2h, ...\}$, $h > 0$. Each particle may perform a jump per a distance unit in the direction of the motion at moment $t$ with some probability (below we'll give law of motion of particles) or remain at its place.

Introduce the denotation:

$\xi_{i,t}$ is a coordinate of the particle of number $i = 1, 2$ at time $t$, $\rho_{i,t}$ is a distance from the $i$-th particle to the next particle in the direction of the motion at time $t$, i.e. $\rho_{i,t} = \xi_{2,t} - \xi_{1,t}$ if $\xi_{2,t} > \xi_{1,t}$ and $\rho_{i,t} = n + \xi_{2,t} - \xi_{1,t}$ , if $\xi_{2,t} < \xi_{1,t}$.

$\varepsilon_{i,t} = \left|\xi_{i,t+h} - \xi_{i,t}\right|$ determines the particle's motion, i.e. $\varepsilon_{i,t} = 1$ if the $i$-th particle performs a jump at time $t$, and $\varepsilon_{i,t} = 0$ if the $i$-th particle stands at its place.

Let the parameters of motion of one particle $r$ and $l\,(r + l = 1,\ \ 0 < r < 1)$, and the motion parameters of another particle $\overline{r}$ and $\overline{l}\left(\overline{r} + \overline{l} = 1,\ \ 0 < \overline{r} < 1\right)$, i.e. the motion happens by he following law:

$$P\left\{\varepsilon_{1,t} = 1\left|\rho_{1,t} = k\right.\right\} = r,\ P\left\{\varepsilon_{1,t} = 0\left|\rho_{1,t} = k\right.\right\} = l\ \ k = \overline{2, n-1},\ r + l = 1;$$

$$P\left\{\varepsilon_{1,t} = 1\left|\rho_{1,t} = 1\ ,\ \varepsilon_{2,t} = 1\right.\right\} = r,\ P\left\{\varepsilon_{1,t} = 0\left|\rho_{1,t} = 1\ ,\ \varepsilon_{2,t} = 1\right.\right\} = l;$$

$$P\left\{\varepsilon_{1,t} = 1\left|\rho_{1,t} = 1\ ,\ \varepsilon_{2,t} = 0\right.\right\} = 0,\ P\left\{\varepsilon_{1,t} = 0\left|\rho_{1,t} = 1\ ,\ \varepsilon_{2,t} = 0\right.\right\} = 1;$$

$$P\left\{\varepsilon_{1,t} = 0, \varepsilon_{2,t} = 0\left|\rho_{1,t} = 1\right.\right\} = \overline{l};\ P\left\{\varepsilon_{1,t} = 1,\ \varepsilon_{2,t} = 1\left|\rho_{1,t} = 1\right.\right\} = \overline{r}r;$$

$$P\left\{\varepsilon_{1,t} = 0, \varepsilon_{2,t} = 0\left|\rho_{1,t} = 1\right.\right\} = l\overline{r};\ P\left\{\varepsilon_{1,t} = 1,\ \varepsilon_{2,t} = 0\left|\rho_{1,t} = 1\right.\right\} = 0; \qquad (1)$$

$$P\left\{\varepsilon_{2,t} = 1\left|\rho_{1,t} = n-k\right.\right\} = \overline{r},\ P\left\{\varepsilon_{2,t} = 0\left|\rho_{1,t} = n-k\right.\right\} = \overline{l}\ \ k = \overline{2, n-1},\ \overline{r}+\overline{l} = 1;$$

$$P\left\{\varepsilon_{2,t} = 1\left|\rho_{1,t} = n-1\ ,\ \varepsilon_{1,t} = 1\right.\right\} = \overline{r},\ P\left\{\varepsilon_{2,t} = 0\left|\rho_{1,t} = n-1\ ,\ \varepsilon_{1,t} = 1\right.\right\} = \overline{l};$$

$$P\left\{\varepsilon_{2,t} = 1\left|\rho_{1,t} = n-1\ , \varepsilon_{1,t} = 0\right.\right\} = 0,\ P\left\{\varepsilon_{2,t} = 0\left|\rho_{1,t} = n-1\ , \varepsilon_{1,t} = 0\right.\right\} = 1.$$

It follows from these relations that the first particle is leading since none of the particles may impede its motion.

**Theorem 1.** *For a model defined by relations* (1) *there is a unique stationary distribution*

$$a_k = \lim_{t\to\infty} P\left\{\rho_{2,t} = k\right\},\ \ \sum_{k=1}^{n-1} a_k = 1. \qquad (2)$$

*for which it holds the recurrent formula*

$$a_k r\overline{l} = a_{k+1}\overline{r}l. \qquad (3)$$

*Whence $a_k$ is determined as*

$$a_k = \frac{A_k}{A},\ \ A_k = \left(\frac{r}{l}\right)^{k-l}\left(\frac{\overline{l}}{\overline{r}}\right)^{k-l},\ A_1 = 1,\ \ A = \sum_{j=1}^{n-1} A_j\ . \qquad (4)$$

**Proof.** The random variables $\rho_{2,t}$ form Markov's ergodic chain with finite number of states. Consequently, [7, p.550] these exists a unique stationary distribution of $\rho_{2,t}$. Using the total probabilities formula and relation (1), for stationary probabilities of distribution of distance between the particles we write the recurrent equations.

$$P\left\{\rho_{2,t+h} = 1\right\} = P\left\{\rho_{2,t} = 1\right\}\left[P\left\{\varepsilon_{1,t} = 1\left|\rho_{2,t} = 1\right.\right\}\times\right.$$

$$\left.\times P\left\{\varepsilon_{2,t} = 1\left|\rho_{2,t} = 1\right.\right\} + P\left\{\varepsilon_{1,t} = 0\left|\rho_{2,t} = 1\right.\right\}\right] +$$

$$+P\left\{\rho_{2,t} = 2\right\}P\left\{\varepsilon_{1,t} = 0\left|\rho_{1,t} = 2\right.\right\}P\left\{\varepsilon_{2,t} = 1\left|\rho_{1,t} = 2\right.\right\};$$

$$P\left\{\rho_{2,t+h}=k\right\}=P\left\{\rho_{2,t}=k-1\right\}P\left\{\varepsilon_{1,t}=1\left|\rho_{2,t}=k-1\right.\right\}\times$$
$$\times P\left\{\varepsilon_{2,t}=0\left|\rho_{2,t}=k-1\right.\right\}+P\left\{\rho_{2,t}=k\right\}\times$$
$$\times\left[P\left\{\varepsilon_{1,t}=1\left|\rho_{2,t}=k\right.\right\}P\left\{\varepsilon_{2,t}=1\left|\rho_{2,t}=k\right.\right\}+\right.$$
$$\left.+P\left\{\varepsilon_{2,t}=0\left|\rho_{2,t}=k\right.\right\}P\left\{\varepsilon_{1,t}=0\left|\rho_{2,t}=k\right.\right\}\right]+ \qquad (5)$$
$$+P\left\{\rho_{2,t}=k+1\right\}P\left\{\varepsilon_{1,t}=0\left|\rho_{2,t}=k+1\right.\right\}P\left\{\varepsilon_{2,t}=1\left|\rho_{2,t}=k+1\right.\right\};\ \ k=\overline{2,n-2};$$
$$P\left\{\rho_{2,t+h}=n-1\right\}=P\left\{\rho_{2,t}=n-2\right\}P\left\{\varepsilon_{1,t}=1\left|\rho_{2,t}=n-2\right.\right\}\times$$
$$\times P\left\{\varepsilon_{2,t}=0\left|\rho_{1,t}=n-2\right.\right\}+P\left\{\rho_{1,t}=n-1\right\}\times$$
$$\times\left[P\left\{\varepsilon_{2,t}=1\left|\rho_{2,t}=n-1\right.\right\}P\left\{\varepsilon_{1,t}=1\left|\rho_{2,t}=n-1\right.\right\}+P\left\{\varepsilon_{2,t}=0\left|\rho_{2,t}=n-1\right.\right\}\right].$$

Passing to limit as $t\to\infty$ and using the fact that there exists a stationary mode, from (5) we get

$$a_1=a_1\left(r\overline{r}+l\right)+a_2 l\overline{r};$$
$$a_k=a_{k+1}l\overline{r}+a_k\left(r\overline{r}+l\overline{l}\right)+a_{k-1}l\overline{r},\quad \left(k=\overline{2,n-2}\right); \qquad (6)$$
$$a_{n-1}=a_{n-2}\overline{l}r+a_{n-1}\left(r\overline{r}+\overline{l}\right);$$

Hence we get:

$$\begin{cases} a_1=a_1,\\ a_2=a_1\dfrac{r}{l}\dfrac{\overline{l}}{\overline{r}},\\ a_3=a_2\dfrac{r}{l}\dfrac{\overline{l}}{\overline{r}},\\ a_4=a_3\dfrac{r}{l}\dfrac{\overline{l}}{\overline{r}},\\ .\\ .\\ a_k=a_{k-1}\dfrac{r}{l}\dfrac{\overline{l}}{\overline{r}}.\end{cases} \qquad (7)$$

Hence, recurrent formula (3) follows.

From (7) we get:

$$\begin{cases} a_1=a_1,\\ a_2=a_1\dfrac{r}{l}\dfrac{\overline{l}}{\overline{r}},\\ a_3=a_1\left(\dfrac{r}{l}\right)^2\left(\dfrac{\overline{l}}{\overline{r}}\right)^2,\\ a_4=a_1\left(\dfrac{r}{l}\right)^3\left(\dfrac{\overline{l}}{\overline{r}}\right)^3,\\ .\\ .\\ .\\ a_k=a_1\left(\dfrac{r}{l}\right)^{k-1}\left(\dfrac{\overline{l}}{\overline{r}}\right)^{k-1},\quad \left(k=\overline{1,n-1}\right).\end{cases}$$

If we denote $A_1 = 1,\ A_k = \left(\frac{r}{l}\right)^{k-1}\left(\frac{\bar{l}}{\bar{r}}\right)^{k-1}$, $A = \sum_{j=1}^{n-1} A_j$, then

$$\begin{cases} a_1 = a_1 A_1, \\ a_2 = a_1 A_2, \\ a_3 = a_1 A_3, \\ a_4 = a_1 A_4, \\ . \\ . \\ . \\ a_k = a_1 A_k, \quad k = \overline{1, n-1}. \end{cases}$$

Hence,

$$\sum_{k=1}^{n-1} a_k = 1 \Longrightarrow 1 = a_1\left(A_1 + A_2 + ... + A_{n+1}\right) = a_1 \sum_{j=1}^{n-1} A_j \Longrightarrow 1 = a_1 A \Longrightarrow a_1 = \frac{1}{A}.$$

We get that $a_k = \frac{A_k}{A}$ is a solution of (6).

This proves theorem 1.

Denote $r_{n-1} = P\left\{\varepsilon_{2,t} = 1 \left|\rho_{2,t} = n-1\right.\right\}$

**Lemma.** *In order that*

$$P\left\{\varepsilon_{1,t} = 1\right\} = P\left\{\varepsilon_{2,t} = 1\right\} = r$$

*be fulfilled for a model described by relation (1) it is necessary and sufficient that the condition* $r_{n-1} = 1$ *be satisfied.*

**Proof.** Let

$$r = r\sum_{k=1}^{n-1}\left(\overline{r} + \bar{l}\right)a_k = \sum_{k=1}^{n-1} r\overline{r}a_k + \sum_{k=1}^{n-1} r\bar{l}a_k = \sum_{k=1}^{n-1} r\overline{r}a_k + r\bar{l}a_{n-1} + \sum_{k=1}^{n-2} r\bar{l}a_k =$$

$$= r\bar{l}a_{n-1} + \sum_{k=1}^{n-1} r\overline{r}a_k + \sum_{k=1}^{n-2} l\overline{r}a_{k+1} = r\bar{l}a_{n-1} + r\overline{r}a_1 + \sum_{k=2}^{n-1} r\overline{r}a_k + \sum_{k=2}^{n-1} l\overline{r}a_k =$$

$$= r\bar{l}a_{n-1} + r\overline{r}a_1 + \sum_{k=2}^{n-1} \overline{r}a_k.$$

On the other hand, for probabilities of jumps of particles we have

$$P\left\{\varepsilon_{1,t} = 1\right\} = \sum_{k=2}^{n-1} P\left\{\varepsilon_{2,t} = 1, \rho_{2,t} = k\right\} + P\left\{\varepsilon_{2,t} = 1, \rho_{2,t} = 1, \varepsilon_{1,1} = 1\right\} =$$

$$= \sum_{k=2}^{n-1} P\left\{\varepsilon_{2,t} = 1 \left|\rho_{2,t} = k\right.\right\} P\left\{\rho_{2,t} = k\right\} + P\left\{\varepsilon_{2,t} = 1 \left|\rho_{2,t} = 1\right., \varepsilon_{1,t} = 1\right\} \times$$

$$\times P\left\{\rho_{2,t} = 1, \varepsilon_{1,t} = 1\right\} = \sum_{k=2}^{n-1} \overline{r}a_k + \overline{r}ra_1\ .$$

$$r = P\left\{\varepsilon_{2,t} = 1\right\} + r\bar{l}a_{n-1}. \tag{8}$$

Since

$$P\left\{\varepsilon_{1,t} = 1\right\} = \sum_{k=1}^{n-2} P\left\{\varepsilon_{1,t} = 1, \rho_{2,t} = k\right\} + P\left\{\varepsilon_{1,t} = 1, \rho_{2,t} = n-1, \varepsilon_{2,t} = 1\right\} =$$

$$= \sum_{k=1}^{n-2} P\left\{\varepsilon_{1,t} = 1 \left|\rho_{2,t} = k\right.\right\} P\left\{\rho_{2,t} = k\right\} +$$

$$+P\left\{\varepsilon_{1,t} = 1 \middle| \rho_{2,t} = n-1, \varepsilon_{2,t} = 1\right\} P\left\{\varepsilon_{2,t} = 1 \left|\rho_{2,t} = n-1\right.\right\} P\left\{\rho_{2,t} = n-1\right\} =$$

$$= \sum_{k=1}^{n-2} ra_k + r\overline{r}a_{n-1} = r\left(1 - a_{n-1}\right) + r\overline{r}a_{n-1} = r - r\bar{l}a_{n-1}$$

$$r = P\left\{\varepsilon_{1,t} = 1\right\} + ra_{n-1}\bar{l}. \tag{9}$$

From equalities (8) and (9) we get the statement of the lemma.

**Theorem 2.** *In order* $b\left(\varepsilon_1, ..., \varepsilon_m\right) = P\left\{\varepsilon_{i,t+h} = \varepsilon_1, ..., \varepsilon_{i,t+mh} = \varepsilon_m\right\} = r^{\varepsilon_m^+} l^{\varepsilon_m^-}$, $(i = 1, 2)$ *where* $\varepsilon_j = 0$ *or* $1$ *(*$\varepsilon_j$ *takes the value* $1$ *if the particle performs a jump, otherwise it takes the value* $0$*),* $\varepsilon_m^+ = \sum_{j=1}^{m} \varepsilon_j$, $\varepsilon_m^- = m - \varepsilon_m^+$*, it is necessary and sufficient that the conditions* $r_{n-1} = 1$ *be satisfied.*

**Proof.** We'll prove it by the mathematical induction method. For $m = 1$, the theorem's proof was obtained in expressions (8),(9).

Let the statement be true for $m$ steps. Prove it for $m + 1$,

$$b\left(\varepsilon_1, ..., \varepsilon_m\right) = \sum_{k=1}^{n-1} P\left\{\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+(m+1)h} = 1, \rho_{2,t+(m+1)h} = k\right\} =$$

$$= \sum_{k=1}^{n-1} P\left\{\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m, \varepsilon_{2,t+(m+1)h} = \right.$$

$$\left. = 1, \rho_{2,t+(m+1)h} = k, \varepsilon_{1,t+(m+1)h} = 1\right\} +$$

$$+ \sum_{k=1}^{n-2} P\left\{\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m, \varepsilon_{2,t+(m+1)h} = 1, \right.$$

$$\left. \rho_{2,t+(m+1)h} = k + 1, \varepsilon_{1,t+(m+1)h} = 0\right\} =$$

$$= \sum_{k=1}^{n-1} P\left\{\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m\right\} P\left\{\rho_{2,t+(m+1)h} = \right.$$

$$\left. = k \left|\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m\right.\right\} \times$$

$$\times P\left\{\varepsilon_{1,t+(m+1)h} = 1 \left|\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m\right., \rho_{2,t+(m+1)h} = k\right\} \times$$

$$\times P\left\{\varepsilon_{2,t+(m+1)h} = 1 \left|\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{1,t+mh} = \varepsilon_m\right.,\right.$$

$$\rho_{2,t+(m+1)h} = k, \varepsilon_{1,t+(m+1)h} = 1\Big\} +$$

$$\times \sum_{k=1}^{n-2} P\left\{\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m\right\} P\Big\{\rho_{2,t+(m+1)h} =$$

$$= k+1 \left|\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m\right\} \times$$

$$\times P\Big\{\varepsilon_{1,t+(m+1)h} = 0 \left|\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m \right., \rho_{2,t+(m+1)h} = k+1\Big\} \times$$

$$\times P\left\{\varepsilon_{2,t+(m+1)h} = 1 \left|\varepsilon_{2,t+h} = \varepsilon_1, ..., \varepsilon_{2,t+mh} = \varepsilon_m \right.,\right.$$

$$\rho_{2,t+(m+1)h} = k+1, \varepsilon_{1,t+(m+1)h} = 0\Big\} =$$

$$= \sum_{k=1}^{n-1} \bar{r} r a_k b\left(\varepsilon_1, ..., \varepsilon_m\right) + \sum_{k=1}^{n-1} \bar{r} l a_{k+1} b\left(\varepsilon_1, ..., \varepsilon_m\right) =$$

$$= b\left(\varepsilon_1, ..., \varepsilon_m\right) \left(\sum_{k=1}^{n-1} \bar{r} r a_k + \sum_{k=1}^{n-2} \bar{r} a_{k+1} l\right) = b\left(\varepsilon_1, ..., \varepsilon_m\right) \left(\sum_{k=1}^{n-1} \bar{r} r a_k + \sum_{k=1}^{n-2} r a_k \bar{l}\right) =$$

$$= b\left(\varepsilon_1, ..., \varepsilon_m\right) \left(r \bar{r} a_{n-1} + \sum_{k=1}^{n-2} r a_k\right) = b\left(\varepsilon_1, ..., \varepsilon_m\right) P\left\{\varepsilon_{1,t} = 1\right\} = b\left(\varepsilon_1, ..., \varepsilon_m\right) r.$$

The case $\varepsilon_{m+1} = 0$ is proved similarly.

Theorem 2 is proved.

This result may be interpreted in the following way.

If we make one perticle visible, the other one invisible, then the visible particle performs a random walk with parameters $r, l$.

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