Hidayat M. HUSEYNOV, Ahmad H. JAMSHIDIPOUR

ON JOST SOLUTIONS OF STURM-LIOUVILLE EQUATIONS WITH SPECTRAL PARAMETER IN DISCONTINUITY CONDITION

Abstract

Integral representations for the Jost solutions are obtained for one-dimensional Sturm-Liouville equation with discontinuity conditions at some point.

Consider the differential equation

$$-y'' + q(x)\,y = \lambda^2 y, \quad x \in (-\infty, +\infty)\,, \tag{1}$$

with the conditions

$$y(a+0) = y(a-0)\,, \tag{2}$$

$$y'(a+0) - y'(a-0) = \lambda\beta y(a)\,, \tag{3}$$

where $\beta, a \in (-\infty, +\infty)$, $\beta \neq 0$, λ is a complex parameter, $q(x)$ is a real-valued function and satisfies the condition

$$\int_{-\infty}^{+\infty} (1+|x|)\,|q(x)|\,dx < +\infty. \tag{4}$$

We can understand problem (1)-(3) as one of the treatments of the equation

$$-y'' + \lambda p(x)\,y + q(x)\,y = \lambda^2 y, \quad -\infty < x < +\infty, \tag{5}$$

when $p(x) = \beta\delta(x-a)$. When the function $p(x)$ is sufficiently smooth, the functions $p(x)$ and $q(x)$ are real-valued and decrease quite rapidly, the inverse scattering problem for equation (5) is completely solved in the papers [1]-[4].

In order to solve such a problem in the case $p(x) = \beta\delta(x-a)$, in the present report we prove the existence of Jost type solutions for problem (1)-(3) and investigate their properties.

The functions $e^{\pm}(x,\lambda)$ satisfying equation (1), conditions (1)-(3) and the condition at infinity

$$\lim_{x\to\pm\infty} e^{\pm}(x,\lambda)\cdot e^{\mp i\lambda x} = 1, \tag{$6_\pm$}$$

are called the Jost solutions.

It is easy to show that if $q(x) \equiv 0$, the functions

$$e_0^{\pm}(x,\lambda) = \begin{cases} e^{\pm i\lambda x}, & \pm x > \pm a, \\ \left(1+\frac{i\beta}{2}\right) e^{\pm i\lambda x} - \frac{i\beta}{2} e^{\pm i\lambda(2a-x)}, & \pm x < \pm a. \end{cases}$$

are the Jost solutions.

Introduce the following denotation:

$$c = 1 + |\beta|, \quad \sigma_1^{\pm}(x) = \pm \int_x^{\pm\infty} t\,|q(t)|\,dt.$$

The basic result of the paper is the following

Theorem. *Let the real-valued function* $q(x)$ satisfy condition (4). *Then for all* λ, *there* exist Jost solutions $e^{\pm}(x,\lambda)$ of (1)-(3) *from the upper half-plane, they are unique and represented in the form*

$$e^{\pm}(x,\lambda) = e_0^{\pm}(x,\lambda) \pm \int_x^{\pm\infty} K^{\pm}(x,t)\,e^{\pm i\lambda t}dt, \qquad (7_{\pm})$$

where for each fixed $x \neq a$ *the kernel* $K^{+}(x,\cdot)$ $(K^{-}(x,\cdot))$ *belongs to the space* $L_1(x,+\infty)$ $(L_1(-\infty,x))$ *and the estimations*

$$\pm \int_x^{\pm\infty} \left|K^{\pm}(x,t)\right| dt \le e^{c\sigma_1^{\pm}(x)} - 1. \qquad (8_{\pm})$$

are fulfilled.

Furthermore,

$$K^{\pm}(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} q(\xi)\,d\xi, \quad \pm x > \pm a,$$

$$K^{\pm}(x,x) = \pm\frac{1}{2}\left(1+\frac{i\beta}{2}\right)\int_x^{\pm\infty} q(\xi)\,d\xi, \quad \pm x < \pm a, \qquad (9_{\pm})$$

$$K^{\pm}(x,2a-x+0) - K^{\pm}(x,2a-x-0) =$$
$$= \frac{i\beta}{4}\left\{\int_x^{a} q(\xi)\,d\xi - \int_a^{+\infty} q(\xi)\,d\xi\right\}, \quad \pm x < \pm a.$$

Remark. When no discontinuity condition exists, i.e. when in condition (3) $\beta = 0$, the representation of the Jost solution for Sturm-Liouville equation is first obtained in the paper [5] (see also [6]). Such a problem for the equation $-y'' + q(x)\,y = \lambda p(x)\,y$, when $p(x)$ is a piecewise-constant real function, is solved in the paper [7].

Theorem's proof. Having rewritten equation (1) in the form $y'' + \lambda^2 y = q(x)\,y$ and assuming the right hand side to be known, for finding the solution $e^{+}(x,\lambda)$ of this equation we can apply the arbitrary constants variation method. As a result, we get the integral equation

$$e^{+}(x,\lambda) = e_0^{+}(x,\lambda) + \int_x^{+\infty} S_0(t,x,\lambda)\,q(t)\,e^{+}(t,\lambda)\,dt, \qquad (10)$$

where

$$S_0(t,x,\lambda)=\begin{cases}\frac{\sin\lambda(t-x)}{\lambda}, & t>x>a \text{ or } a>t>x,\\ \frac{\sin\lambda(t-x)}{\lambda}-\frac{\beta}{2}\cdot\frac{\cos\lambda(t-x)-\cos\lambda(t-2a+x)}{\lambda}, & t>a>x\end{cases} \tag{11}$$

It is easy to show that the solution $e^+(x,\lambda)$ of integral equation (10) is the Jost solution of problem (1)-(3), (6_+). Well look for the solution of equation (10) in the form (7_+). In order such kind function satisfy equation (10), the equality

$$\int_x^{+\infty}K^+(x,t)\,e^{i\lambda t}dt=\int_x^{+\infty}S_0(t,x,\lambda)\,q(t)\,e_0^+(t,\lambda)\,dt+$$

$$+\int_x^{+\infty}S_0(t,x,\lambda)\,q(t)\int_t^{+\infty}K^+(t,s)\,e^{i\lambda s}dsdt, \tag{12}$$

should be fulfilled. And vice-versa, if $K^+(x,t)$ satisfies this equality for all $\lambda\,(\operatorname{Im}\lambda\geq 0)$, the function $e^+(x,\lambda)$ is the Jost solution of problem (1)-(3), (6_+).

Transform each term in the right side of (12) so that they have the form of the Fourier transform of some functions.

At first we consider the first term. It $x<a$, we have

$$\int_x^{+\infty}S_0(t,x,\lambda)\,q(t)\,e_0^+(t,\lambda)\,dt=$$

$$=\int_x^{a}\frac{\sin\lambda(t-x)}{\lambda}q(t)\left[\left(1+\frac{i\beta}{2}\right)e^{i\lambda t}-\frac{i\beta}{2}e^{i\lambda(2a-t)}\right]dt+$$

$$+\int_a^{+\infty}\left[\frac{\sin\lambda(t-x)}{\lambda}-\frac{\beta}{2}\cdot\frac{\cos\lambda(t-x)-\cos\lambda(t-2a+x)}{\lambda}\right]q(t)\,e^{i\lambda t}dt=$$

$$=\left(1+\frac{i\beta}{2}\right)\int_x^{a}\left(\frac{1}{2}\int_x^{2t-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt-\frac{i\beta}{2}\int_x^{a}\left(\frac{1}{2}\int_{x-2t+2a}^{2a-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt+$$

$$+\int_a^{+\infty}\left(\frac{1}{2}\int_x^{2t-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt-\frac{i\beta}{2}\int_a^{+\infty}\left(\frac{1}{2}\int_{2t-2a+x}^{2t-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt+$$

$$+\frac{i\beta}{2}\int_a^{+\infty}\left(\frac{1}{2}\int_x^{2a-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt.$$

Changing the integration order and then in the obtained equality changing the denotation for integration variables, we get (for $x<a$)

$$\int_x^{+\infty}S_0(t,x,\lambda)\,q(t)\,e_0^+(t,\lambda)\,dt=\frac{1}{2}\left(1+\frac{i\beta}{2}\right)\int_x^{2a-x}\left(\int_{\frac{t+x}{2}}^{a}q(\xi)\,d\xi\right)e^{i\lambda t}dt-$$

$$-\frac{i\beta}{4}\int\limits_{x}^{2a-x}\left(\int\limits_{\frac{x+2a-t}{2}}^{a}q(\xi)\,d\xi\right)e^{i\lambda t}dt+\frac{1}{2}\int\limits_{x}^{2a-x}\left(\int\limits_{a}^{+\infty}q(\xi)\,d\xi\right)e^{i\lambda t}dt+$$

$$+\frac{1}{2}\int\limits_{2a-x}^{+\infty}\left(\int\limits_{\frac{t+x}{2}}^{+\infty}q(\xi)\,d\xi\right)e^{i\lambda t}dt-\frac{i\beta}{4}\int\limits_{x}^{2a-x}\left(\int\limits_{a}^{\frac{t+2a-x}{2}}q(\xi)\,d\xi\right)e^{i\lambda t}dt-$$

$$-\frac{i\beta}{4}\int\limits_{2a-x}^{+\infty}\left(\int\limits_{\frac{t+x}{2}}^{\frac{t+2a-x}{2}}q(\xi)\,d\xi\right)e^{i\lambda t}dt+\frac{i\beta}{4}\int\limits_{x}^{2a-x}\left(\int\limits_{a}^{+\infty}q(\xi)\,d\xi\right)e^{i\lambda t}dt=$$

$$=\frac{1}{2}\int\limits_{x}^{+\infty}\left(\int\limits_{\frac{t+x}{2}}^{+\infty}e^{i\lambda\xi}d\xi\right)e^{i\lambda t}dt+\frac{i\beta}{4}\int\limits_{x}^{2a-x}\left(\int\limits_{\frac{t+x}{2}}^{+\infty}q(\xi)\,d\xi-\int\limits_{\frac{x+2a-t}{2}}^{\frac{t+2a-x}{2}}q(\xi)\,d\xi\right)e^{i\lambda t}dt-$$

$$-\frac{i\beta}{4}\int\limits_{2a-x}^{+\infty}\left(\int\limits_{\frac{t+x}{2}}^{\frac{t+2a-x}{2}}q(\xi)\,d\xi\right)e^{i\lambda t}dt. \tag{13}$$

For $x>a$ we behave in the similar way and have

$$\int\limits_{x}^{+\infty}S_0(t,x,\lambda)\,q(t)\,e_0^{+}(t,\lambda)\,dt=\int\limits_{x}^{+\infty}\frac{\sin\lambda(t-x)}{\lambda}q(t)\,e^{i\lambda t}dt=$$

$$=\int\limits_{x}^{+\infty}\left(\frac{1}{2}\int\limits_{x}^{2t-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt=\frac{1}{2}\int\limits_{x}^{+\infty}\left(\int\limits_{\frac{x+t}{2}}^{+\infty}q(\xi)\,d\xi\right)e^{i\lambda t}dt. \tag{14}$$

Now, transform the second term from the right hand side of relation (12). For $x<a$ we have

$$\int\limits_{x}^{+\infty}S_0(\xi,x,\lambda)\,q(\xi)\int\limits_{\xi}^{+\infty}K^{+}(x,u)\,e^{i\lambda u}du\;ds=$$

$$=\int\limits_{x}^{+\infty}\frac{\sin\lambda(\xi-x)}{\lambda}q(\xi)\int\limits_{\xi}^{+\infty}K^{+}(\xi,u)\,e^{i\lambda u}dud\xi-$$

$$-\frac{\beta}{2}\int\limits_{a}^{+\infty}\frac{\cos\lambda(\xi-x)-\cos\lambda(\xi-2a+x)}{\lambda}q(\xi)\int\limits_{\xi}^{+\infty}K^{+}(\xi,u)\,e^{i\lambda u}dud\xi=$$

$$=\frac{1}{2}\int\limits_{x}^{+\infty}q(\xi)\int\limits_{\xi}^{+\infty}K^{+}(\xi,u)\left\{\int\limits_{x-\xi+u}^{\xi-x+u}e^{i\lambda t}dt\right\}dud\xi-$$

$$-\frac{i\beta}{4}\int\limits_a^{+\infty} q(\xi)\int\limits_\xi^{+\infty} K^+(\xi,u)\left\{\int\limits_{\xi-2a+x+u}^{\xi-x+u} e^{i\lambda t}dt\right\}dud\xi+$$

$$+\frac{i\beta}{4}\int\limits_a^{+\infty} q(\xi)\int\limits_\xi^{+\infty} K^+(\xi,u)\left\{\int\limits_{x+u-\xi}^{2a-x+u-\xi} e^{i\lambda t}dt\right\}dud\xi.$$

Continuing the function $K^+(\xi,u)$ by a zero for $u<\xi$, for all $t\geq x$ we find

$$\int\limits_\xi^{+\infty} K^+(\xi,u)\int\limits_{x-\xi+u}^{\xi-x+u} e^{i\lambda t}dtdu=\int\limits_{-\infty}^{+\infty} K^+(\xi,u)\int\limits_{x-\xi+u}^{\xi-x+u} e^{i\lambda t}dtdu=$$

$$=\int\limits_{-\infty}^{+\infty}\left(\int\limits_{t-\xi+x}^{t+\xi-x} K^+(\xi,u)\,du\right)e^{i\lambda\xi}d\xi=\int\limits_x^{\infty}\left(\int\limits_{t-\xi+x}^{t+\xi-x} K^+(\xi,u)\,du\right)e^{i\lambda t}dt, \tag{15}$$

since for $t<x$

$$\int\limits_{t-\xi+x}^{t+\xi-x} K^+(\xi,u)\,du=0.$$

Behaving in the same way, for all $t\geq a$ we have

$$\int\limits_\xi^{+\infty} K^+(\xi,u)\left\{\int\limits_{\xi-2a+x+u}^{\xi-x+u} e^{i\lambda t}dt\right\}du=\int\limits_{-\infty}^{+\infty}\left\{\int\limits_{\xi-2a+x+u}^{\xi-x+u} e^{i\lambda t}dt\right\}K^+(\xi,u)\,du=$$

$$=\int\limits_{-\infty}^{+\infty}\left\{\int\limits_{t-\xi+x}^{t-\xi+2a-x} K^+(\xi,u)\,du\right\}e^{i\lambda t}dt=\int\limits_x^{+\infty}\left\{\int\limits_{t-\xi+x}^{t-\xi+2a-x} K^+(\xi,u)\,du\right\}e^{i\lambda t}dt, \tag{16}$$

$$\int\limits_\xi^{+\infty} K^+(\xi,u)\left\{\int\limits_{x+u-\xi}^{2a-x+u-\xi} e^{i\lambda t}dt\right\}du=\int\limits_{-\infty}^{+\infty}\left\{\int\limits_{x+u-\xi}^{2a-x+u-\xi} e^{i\lambda t}dt\right\}K^+(\xi,u)\,du=$$

$$=\int\limits_{-\infty}^{+\infty}\left\{\int\limits_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi,u)\,du\right\}e^{i\lambda t}dt=\int\limits_x^{+\infty}\left\{\int\limits_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi,u)\,du\right\}e^{i\lambda t}dt. \tag{17}$$

Here, we used the fact that for $t<x$

$$\int\limits_{x+t-\xi}^{t+2a-x-\xi} K^+(\xi,u)\,du=0,\qquad \int\limits_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi,u)\,du=0.$$

It follows from formulae (13)-(17) that equality (12) is fulfilled if the function $K^+(x,t)$ satisfies the equation

$$K^+(x,t) = K_0^+(x,t) + \frac{1}{2}\int_x^{+\infty} q(\xi) \int_{t-\xi+x}^{t+\xi-x} K^+(\xi,u)\,du d\xi -$$

$$-\frac{i\beta}{4} H^+(x) \int_a^{+\infty} q(\xi) \int_{t-\xi+x}^{t-\xi+2a-x} K^+(\xi,u)\,du d\xi +$$

$$+\frac{i\beta}{4} H^+(x) \int_a^{+\infty} q(\xi) \int_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi,u)\,du d\xi, \tag{18}$$

where

$$H^+(x) = \begin{cases} 1, & x < a, \\ 0, & x > a, \end{cases}$$

$$K_0^+(x,t) = \frac{1}{2}\int_{\frac{t+x}{2}}^{+\infty} q(s)\,ds - \frac{i\beta}{4} H(x) \times$$

$$\times \begin{cases} \int_{\frac{x+2a-t}{2}}^{\frac{t+2a-x}{2}} q(s)\,ds - \int_{\frac{x+t}{2}}^{+\infty} q(s)\,ds, & x < t < 2a - x, \\ \int_{\frac{t+x}{2}}^{\frac{t+2a-x}{2}} q(s)\,ds, & 2a - x < t < \infty. \end{cases} \tag{19}$$

Thus, in order to complete the proof of existence of the solution $e^+(x,\lambda)$ it is enough to show that for each fixed $x \in (-\infty, a) \cup (a, +\infty)$ equation (18) has the solution $K^+(x,\cdot) \in L_1(x,+\infty)$ satisfying inequality (8_+) and condition (9_+).

Assume

$$K_n^+(x,t) = \frac{1}{2}\int_x^{+\infty} q(\xi) \int_{t-\xi+x}^{t+\xi-x} K_{n-1}^+(\xi,u)\,du d\xi -$$

$$-\frac{i\beta}{4} H(x) \left\{ \int_a^{+\infty} q(\xi) \int_{t-\xi+x}^{t-\xi+2a-x} K_{n-1}^+(\xi,u)\,du d\xi - \right.$$

$$\left. - \int_a^{+\infty} q(\xi) \int_{t-2a+\xi+x}^{t-x+\xi} K_{n-1}^+(\xi,u)\,du d\xi \right\}, \quad n = 1, 2, ..., \tag{20}$$

where $K_0^+(x,t)$ is determined by formula (19).

Show that

$$\int_x^{+\infty} \left| K_n^+(x,t) \right| dt \le \frac{c^{n+1}\sigma_1^{n+1}(x)}{(n+1!)}, \tag{21}$$

whence, it will follow that the series $K^{+}(x,\cdot)=\sum\limits_{n=0}^{+\infty}K_n^{+}(x,\cdot)$ converges in the space $L_1(x,+\infty)$, its sum $K^{+}(x,t)$ is a solution of integral equation (18) and satisfies estimation (8_+).

It follows from the definition of $K_n^{+}(x,t)$ (see formula 20) that

$$\left|K_n^{+}(x,t)\right|\le c\int\limits_x^{+\infty}|q(\xi)|\int\limits_{t-\xi+x}^{t+\xi-x}\left|K_{n-1}^{+}(\xi,u)\right|dud\xi,$$

consequently,

$$\int\limits_x^{+\infty}\left|K_n^{+}(x,t)\right|dt\le c\int\limits_x^{+\infty}\xi|q(\xi)|\int\limits_{\xi}^{+\infty}\left|K_{n-1}^{+}(\xi,u)\right|dud\xi. \tag{22}$$

Now, for establishing inequality (21), apply the mathematical induction method. For $n=0$ use (19), change the integration order and have

$$\int\limits_x^{+\infty}\left|K_0^{+}(x,t)\right|dt\le\int\limits_x^{+\infty}(\xi-x)|q(\xi)|d\xi+\frac{|\beta|}{2}H(x)\int\limits_x^{a}(\xi-x)|q(\xi)|d\xi+$$

$$+\int\limits_a^{2a-x}(2a-x-\xi)|q(\xi)|d\xi+\int\limits_x^{a}(\xi-x)|q(\xi)|d\xi+\int\limits_a^{+\infty}(a-x)|q(\xi)|d\xi+$$

$$+\int\limits_a^{2a-x}(\xi-a)|q(\xi)|d\xi+\int\limits_{2a-x}^{+\infty}(a-x)|q(\xi)|d\xi=\int\limits_x^{+\infty}(\xi-x)|q(\xi)|d\xi+$$

$$+\beta H(x)\left\{\int\limits_x^{a}(\xi-a)|q(\xi)|d\xi+\int\limits_a^{+\infty}(a-x)|q(\xi)|d\xi\right\}\le c\int\limits_x^{+\infty}\xi|q(\xi)|d\xi=c\sigma_1(x).$$

Thus, estimation (21) is true for $n=2$ and if it is true for $\|K_n^{+}(x,\cdot)\|_{L_1(x,+\infty)}$, then using inequality (22), we have

$$\int\limits_x^{+\infty}\left|K_{n+1}^{+}(x,t)\right|dt\le c\int\limits_x^{+\infty}\xi|q(\xi)|\frac{c^{n+1}\sigma_1^{n+1}(\xi)}{(n+1)!}d\xi=\frac{c^{n+2}\sigma_1^{n+2}(x)}{(n+2)!}.$$

Validity of relations (9) follows directly from (18)-(19).

From the integral equation

$$e^{-}(x,\lambda)=e_0^{-}(x,\lambda)+\int\limits_{-\infty}^{x}S_0(x,t,\lambda)q(t)e^{-}(t,\lambda)dt,$$

the proof of the theorem statement related with the solution $e^{-}(x,\lambda)$ is carried out in the similar way. Here we notice only the integral equation for the kernel $K^{-}(x,t)$:

$$K^{-}(x,t)=K_0^{-}(x,t)+\frac{1}{2}\int_{-\infty}^{x}q(\xi)\int_{t-x+\xi}^{t-\xi+x}K^{-}(\xi,u)\,dud\xi-$$

$$-\frac{i\beta}{4}H^{-}(x)\left\{\int_{-\infty}^{a}q(\xi)\int_{t-2a+x+\xi}^{t-\xi+x}K^{-}(\xi,u)\,dud\xi-\int_{-\infty}^{a}q(\xi)\int_{t-x+\xi}^{t-x+2a-\xi}K^{-}(\xi,u)\,dud\xi\right\},$$

where

$$K_0^{-}(x,t)=\frac{1}{2}\int_{-\infty}^{\frac{t+x}{2}}q(\xi)\,d\xi-\frac{i\beta}{4}H^{-}(x)\begin{cases}\int_{\frac{t-x+2a}{2}}^{\frac{t+x}{2}}q(\xi)\,d\xi, & -\infty<t<2a-x,\\ \int_{\frac{t-x+2a}{2}}^{\frac{x+2a-t}{2}}q(\xi)\,d\xi-\int_{-\infty}^{\frac{x+t}{2}}q(\xi)\,d\xi, & 2a-x<t<x.\end{cases}$$

$$H^{-}(x)=\begin{cases}1, & x>a,\\ 0, & x<a.\end{cases}$$

The theorem is proved.

References

[1]. Jaulent M. *On an ineverse scattering problem with an energy-dependet potential.* -Ann. Inst. Henri Paunkare, 1972, v. 7, No 4, pp. 363-378.

[2]. Jaulent M. *On the ineverse problem for the Schrodinger equation with an energy-dependet potential.* -Comptes rendus Akad. Sci. Paris, 1975, v. 280, serie A., pp. 1467-1470.

[3]. Jaulent M., Jean C. *Inverse problem for the one dimensional Schrodinger equation with an energy-dependet potential.* -Ann. Inst. Henri Paunkare, 1976, v. 25, No 2, pp. 105-137.

[4]. Maksudov F.G., Huseynov G. Sh. *To the solution of the inverse scattering problem for a quadratic bundle of Schrodinger one-dimensional operators on the axis.* DAN SSSR, 1486, v.289, No 1, pp. 42-46 (Russian).

[5]. Levin B. Ya. *Fourier and Laplace type transformation by means of the solutions of a second order differential equation.* DAN SSSR, 1956, vol 106, No 2, pp. 187-190 (Russian).

[6].Marchenko V.A. *Sturm-Liouville operators and their applications.* Kiev, Naukova Dumka, 1977, p 331 (Russian).

[7] Huseynov I.M. Pashaev R.T. *On an inverse problem for a second order differential equation.* Uspekhi matem, Nauk, 20, issue 7, pp. 143-148 (Russian).

Hidayat M. Huseynov, Ahmad H. Jamshidipour
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: (+99412) 439 47 20 (off.).

Received February 25, 2010; Revised April 30, 2010.