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ON A TWO-WEIGHT CRITERION FOR HARDY TYPE OPERATOR IN THE VARIABLE LEBESGUE SPACES WITH MEASURES

Abstract

The main purpose of this paper is to prove the boundedness of multidimensional Hardy type operator in variable Lebesgue spaces with measures.

It is well known that the variable exponent Lebesgue space in the literature for the first time already in a 1931 was studied by Orlicz [24]. In [24] the Hölder's inequality for variable exponent discrete Lebesgue space was proved. Orlicz also considered the variable exponent Lebesgue space on the real line, and proved the Hölder inequality in this setting.

However, after this one paper, Orlicz abandoned the study of variable exponent Lebesgue spaces, to concentrate on the theory of the Orlicz spaces (see also [21]). Further development of this theory was connection with theory of modular function spaces. The first systematic study of modular spaces is due to Nakano [22]. In the appendix, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see. Musielak [20]).

The next step in the investigation of variable exponent spaces was the paper by Sharapu-dinov [27],[28] and Kováčik and Rǎkosník in [14]. This paper established many basic properties of variable exponent Lebesgue and Sobolev spaces. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [2], [30], [25]).

In this paper a necessary and sufficient condition for the pair of measures ensuring the validity of inequality of strong type for Hardy type operator are found. We also investigated the corresponding problem for the dual operator.

Let R^n be the n-dimensional Euclidean spaces of points $x=(x_1,\dots,x_n)$, $|x|=\left(\sum\limits_{i=1}^{n}x_i^2\right)^{1/2}$ and (R^n,Σ,μ) be a σ-finite, complete measure spaces. By $\mathcal{P}(R^n)$ we define the set of μ-measurable functions such that $p: R^n\mapsto[1,\infty)$. The functions $p\in\mathcal{P}(R^n)$ are called exponents on R^n. Let $\underline{p}=\text{ess}\inf\limits_{x\in R^n}p(x)$ and $\overline{p}=\text{ess}\sup\limits_{x\in R^n}p(x)$. By $\mathcal{Q}(R^m)$ we define the set of ν-measurable functions such that $r: R^m\mapsto[1,\infty)$. Let $p'(x)$ is the conjugate exponent function defined by $\dfrac{1}{p(x)}+\dfrac{1}{p'(x)}=1$ and $x\in R^n$. Obviously, $ess\sup\limits_{x\in R^n}p'(x)=\overline{p}'=\dfrac{\underline{p}}{\underline{p}-1}$ and $ess\inf\limits_{x\in R^n}p'(x)=\underline{p}'=\dfrac{\overline{p}}{\overline{p}-1}$.

Definition. *Let* $p\in\mathcal{P}(R^n)$. *By* $L_{p(x),\,\mu}(R^n)$ *we denote the space of* μ*-measurable* *functions f on R^n such that*

$$\int\limits_{R^n} |f(x)|^{p(x)}\, d\mu(x) < \infty.$$

Under the condition $1 \le p(x) \le \overline{p} < +\infty$, the space $L_{p(x),\,\mu}(R^n)$ is a Banach space (see [8]) with respect to the norm

$$\|f\|_{L_{p(x),\,\mu}(R^n)} = \|f\|_{p(\cdot),\,\mu} = \inf\left\{\lambda > 0 : \int\limits_{R^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

For absolutely continuous measures the spaces $L_{p(x),\,\mu}(R^n)$ coincides with the weighted variable Lebesgue space $L_{p(x),\,\omega}(R^n)$, where ω is a weight function on R^n.

The following theorem is valid.

Theorem 1. *Let $1 \le \underline{p} \le p(x) \le q(y) \le \overline{q} < \infty$ for almost every $x \in R^n$ and $y \in R^m$, $p \in \mathcal{P}(R^n)$ and $q \in \mathcal{Q}(R^m)$. If $p(x) \in C(R^n)$, then the inequality*

$$\Big\|\|f\|_{L_{p(\cdot),\,\mu}(R^n)}\Big\|_{L_{q(\cdot),\,\nu}(R^m)} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q}-\underline{p}}{\overline{q}}\right)^{\frac{2}{\underline{p}}} \Big\|\|f\|_{L_{q(\cdot),\,\nu}(R^m)}\Big\|_{L_{p(\cdot),\,\mu}(R^n)}$$

is valid, where $\underline{q} = ess \inf\limits_{R^m} q(x)$, $\overline{q} = ess \sup\limits_{R^m} q(x)$ and $C(R^n)$ is space of continuous functions in R^n and $f : R^n \times R^m \to R$ is $\mu \otimes \nu$-measurable function such that

$$\Big\|\|f\|_{L_{q(\cdot),\,\nu}(R^m)}\Big\|_{L_{p(\cdot),\,\mu}(R^n)} =$$

$$= \inf\left\{\delta > 0 : \int\limits_{R^n} \left(\frac{\|f(x,\cdot)\|_{L_{q(\cdot),\,\nu}(R^m)}}{\delta}\right)^{p(x)} d\mu(x) \le 1\right\} < \infty.$$

The proof of Theorem 1 is similar to the case of Lebesgue measures μ and ν.

Remark 1. Note that in the case $p(x) = 1$, and when the measure μ and ν is Lebesgue measures Theorem 1 is the analog of generalized Minkowski type inequality and was proved in [26].

Now we prove a criteria on boundedness of multidimensional Hardy type operator in variable Lebesgue spaces with measure.

Theorem 2. *Let μ and ν nonnegative Borel measure on R^n and ν^* is absolutely continuous part of measure ν. Let $p(x) = p = const$, $q \in \mathcal{P}(R^n)$ and $1 < p \le q(x) \le \overline{q} < \infty$. Then the inequality*

$$\|Hf\|_{L_{q(\cdot),\,\mu}(R^n)} \le C\,\|f\|_{L_{p,\,\nu}(R^n)} \qquad (1)$$

holds, for every $f \ge 0$ if and only if there exists $\alpha \in (0,1)$ such that

$$A(\alpha, p, q) =$$

$$= \sup_{t>0} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy}\right]^{1-p'} dy\right)^{\frac{\alpha}{p'}} \left\|\left(\int\limits_{|y|<|\cdot|} \left[\frac{d\nu^*}{dy}\right]^{1-p'} dy\right)^{\frac{1-\alpha}{p'}}\right\|_{L_{q(\cdot),\,\mu}(|x|>t)} < \infty. \qquad (2)$$

Moreover, if $C > 0$ *is the best possible constant in (1) then*

$$\sup_{0<\alpha<1} \frac{p'\,A(\alpha,p,q)}{(1-\alpha)\left[\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)}\right]^{1/p}} \le C \le$$

$$\le \left(\frac{p}{\underline{q}} + \frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}} \inf_{0<\alpha<1} \frac{A(\alpha,p,q)}{(1-\alpha)^{1/p'}}.$$

Proof. Sufficiency. Let $f = 0$ on the support of singular part of measure ν. Then the inequality (1) is equivalent to the inequality

$$\|Hf\|_{L_{q(\cdot),\,\mu}(R^n)} \le C \left(\int_{R^n} |f(x)|^p \,\frac{d\nu^*}{dx}\, dx \right)^{1/p}.$$

Passing to the polar coordinates , we have

$$h(y) = \left(\int_{|z|<|y|} \left[\frac{d\nu^*}{dz}\right]^{1-p'} dz \right)^{\frac{\alpha}{p'}} = \left(\int_{|z|<|y|} [\omega(z)]^{-p'}\, dz \right)^{\frac{\alpha}{p'}} =$$

$$= \left(\int_0^{|y|} s^{n-1} \left(\int_{|\xi|=1} [\omega(s\xi)]^{-p'}\, d\xi \right) ds \right)^{\frac{\alpha}{p'}},$$

where $\dfrac{d\nu^*}{dz} = \omega^p(z)$ and $d\xi$ the surface element on the unit sphere. Obviously, $h(y) = \beta(|y|)$, i.e., $h(y)$ is a radial function.

Applying Hölder's inequality for $L_p(R^n)$ spaces and after some standard transformations, we have

$$\|Hf\|_{L_{q(\cdot),\,\mu}(R^n)} = \left\| \int_{|y|<|\cdot|} f(y)\, dy \right\|_{L_{q(\cdot),\,\mu}(R^n)} =$$

$$= \left\| \int_{|y|<|\cdot|} [f(y)h(y)\omega(y)]\,[h(y)\omega(y)]^{-1}\, dy \right\|_{L_{q(\cdot),\,\mu}(R^n)} \le$$

$$\le \left\| \|f\,h\,\omega\|_{L_p(|y|<|\cdot|)} \left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)} \right\|_{L_{q(\cdot),\,\mu}(R^n)} =$$

$$= \left\| \left\| f\,h\,\omega\,\chi_{\{|\cdot|<|y|\}}(\cdot) \left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)} \right\|_{L_p(R^n)} \right\|_{L_{q(\cdot),\mu}(R^n)}.$$

Applying Theorem 1, we have

$$\left\|\left\|f\,h\,\omega\,\chi_{\{|\cdot|<|y|\}}(\cdot)\left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)}\right\|_{L_p(R^n)}\right\|_{L_{q(\cdot),\mu}(R^n)}\le$$

$$\le\left(\frac{p}{\underline{q}}+\frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}}\left\|\left\|f h\,\omega\,\chi_{\{|\cdot|<|y|\}}(\cdot)\left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)}\right\|_{L_{q(\cdot),\,\mu}(R^n)}\right\|_{L_p(R^n)}=$$

$$=\left(\frac{p}{\underline{q}}+\frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}}\left\|f\,h\,\omega\left\|\chi_{\{|\cdot|<|y|\}}(\cdot)\left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)}\right\|_{L_{q(\cdot),\,\mu}(R^n)}\right\|_{L_p(R^n)}=$$

$$=\left(\frac{p}{\underline{q}}+\frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}}\left\|f\,h\,\omega\left\|\left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)}\right\|_{L_{q(\cdot),\,\mu}(|\cdot|>|y|)}\right\|_{L_p(R^n)}.$$

Passing to polar coordinates in R^n, we get

$$\left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|x|)}=\left(\int\limits_{|y|<|x|}[h(|y|)\,\omega(y)]^{-p'}\,dy\right)^{1/p'}=$$

$$=\left(\int\limits_0^{|x|}r^{n-1}\,[h(r)]^{-p'}\left[\int\limits_{|\xi|=1}[\omega(r\xi)]^{-p'}\,d\xi\right]dr\right)^{1/p'}=$$

$$=\left(\int\limits_0^{|x|}\left[\int\limits_0^{r}s^{n-1}\left(\int\limits_{|\xi|=1}[\omega(s\xi)]^{-p'}d\xi\right)ds\right]^{-\alpha}\left(\int\limits_{|\xi|=1}[\omega(r\xi)]^{-p'}\,d\xi\right)r^{n-1}dr\right)^{1/p'}=$$

$$=\frac{1}{(1-\alpha)^{1/p'}}\left(\int\limits_0^{|x|}\frac{d}{dr}\left\{\left(\int\limits_0^{r}s^{n-1}\left(\int\limits_{|\xi|=1}[\omega(s\xi)]^{-p'}d\xi\right)ds\right)^{1-\alpha}\right\}dr\right)^{1/p'}=$$

$$=\frac{1}{(1-\alpha)^{1/p'}}\left(\int\limits_0^{|x|}s^{n-1}\left(\int\limits_{|\xi|=1}[\omega(s\xi)]^{-p'}d\xi\right)ds\right)^{\frac{1-\alpha}{p'}}=$$

$$=\frac{1}{(1-\alpha)^{1/p'}}\left(\int\limits_{|y|<|x|}[\omega(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}.$$

Therefore by the condition (2), we obtain

$$\left\|f\,h\,\omega\left\|\left\|[h\,\omega]^{-1}\right\|_{L_{p'}(|y|<|\cdot|)}\right\|_{L_{q(\cdot),\,\mu}(|\cdot|>|y|)}\right\|_{L_p(R^n)}=$$

$$= \frac{1}{(1-\alpha)^{1/p'}} \left\| f\,\omega\,h \left\| [h(|\cdot|)]^{\frac{1-\alpha}{\alpha}} \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>|y|)} \right\|_{L_p(R^n)} \le$$

$$\le \frac{A(\alpha,p,q)}{(1-\alpha)^{1/p'}} \, \|f\,\omega\|_{L_p(R^n)} = \frac{A(\alpha,p,q)}{(1-\alpha)^{1/p'}} \, \|f\|_{L_{p,\,\nu}(R^n)}\,.$$

Necessity. Let $f \in L_{p,\nu}(R^n)$, $f \ge 0$ and the inequality (1) is valid. We choose the test function as

$$f(x) = \frac{p'}{1-\alpha} [g(t)]^{-\frac{\alpha}{p'}-\frac{1}{p}} \left[\frac{d\nu^*}{dx}\right]^{1-p'} \chi_{\{|x|<t\}}(x) + [g(|x|)]^{-\frac{\alpha}{p'}-\frac{1}{p}} \left[\frac{d\nu^*}{dx}\right]^{1-p'} \chi_{\{|x|>t\}}(x),$$

where $t>0$ is a fixed number and

$$g(t) = \int\limits_{|y|<t} \left[\frac{d\nu^*}{dx}\right]^{1-p'} dy = \int\limits_0^t s^{n-1} \left(\int\limits_{|\eta|=1} \omega^{-p'}(s\eta)\,d\eta \right) ds.$$

It is obvious that $\dfrac{dg}{dt} = t^{n-1} \displaystyle\int\limits_{|\eta|=1} \omega^{-p'}(t\eta)\,d\eta$. Passing to polar coordinates from the right hand side of inequality (1) we get that

$$\|f\|_{L_{p,\nu}(R^n)} = \left[\int\limits_{|x|<t} \left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{-\alpha(p-1)-1} \omega^{-p'}(x)\,dx + \right.$$

$$\left. + \int\limits_{|x|>t} [g(|x|)]^{-\alpha(p-1)-1} \omega^{-p'}(x)\,dx \right]^{1/p} =$$

$$= \left[\left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{\alpha(1-p)} + \int\limits_t^\infty r^{n-1} [g(r)]^{-\alpha(p-1)-1} \left(\int\limits_{|\xi|=1} \omega^{-p'}(r\xi)\,d\xi \right) dr \right]^{1/p} =$$

$$= \left[\left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{\alpha(1-p)} - \frac{1}{\alpha(p-1)} \int\limits_t^\infty \frac{d}{dr} [g(r)]^{-\alpha(p-1)}\,dr \right]^{1/p} =$$

$$= \left[\left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{\alpha(1-p)} + \frac{1}{\alpha(p-1)} \left\{ [g(t)]^{-\alpha(p-1)} - \left[\int\limits_{R^n} \omega^{-\overline{p}'}(y)\,dy \right]^{-\alpha(p-1)} \right\} \right]^{1/p} \le$$

$$\le \left[\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p} [g(t)]^{-\frac{\alpha}{p'}} = \left[\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p} [h(t)]^{-1}.$$

After some calculations from the left hand side of inequality (1), we have

$$\|Hf\|_{L_{q(\cdot),\,\mu}(R^n)} = \left\| \int\limits_{|y|<|\cdot|} f(y)\,dy \right\|_{L_{q(\cdot),\,\mu}(R^n)} \ge \left\| \int\limits_{|y|<|\cdot|} f(y)\,dy \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)} =$$

$$= \left\| \frac{p'}{1-\alpha} \int\limits_{|y|<t} [g(t)]^{-\frac{\alpha}{p'}-\frac{1}{p}} \omega^{-\overline{p}'}(y)\, dy + \int\limits_{t<|y|<|\cdot|} [g(|y|)]^{-\frac{\alpha}{p'}-\frac{1}{p}} \omega^{-p'}(y)\, dy \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)} =$$

$$= \left\| \frac{p'}{1-\alpha}[g(t)]^{\frac{1-\alpha}{p'}} + \int\limits_t^{|\cdot|} r^{n-1}\, [g(r)]^{-\frac{\alpha}{p'}-\frac{1}{p}} \left(\int\limits_{|\eta|=1} \omega^{-p'}(r\eta)\, d\eta \right) dr \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)} =$$

$$= \left\| \frac{p'}{1-\alpha}[g(t)]^{\frac{1-\alpha}{p'}} + \frac{p'}{1-\alpha} \int\limits_t^{|\cdot|} \frac{d}{dr}[g(r)]^{\frac{1-\alpha}{p'}}\, dr \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)} =$$

$$\left\| \frac{p'}{1-\alpha}[g(t)]^{\frac{1-\alpha}{p'}} + \frac{p'}{1-\alpha} \left([g(|\cdot|)]^{\frac{1-\alpha}{p'}} - [g(t)]^{\frac{1-\alpha}{p'}} \right) \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)} =$$

$$= \frac{p'}{1-\alpha} \left\| [g(\cdot)]^{\frac{1-\alpha}{p'}} \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)}.$$

Hence, implies that

$$\frac{p'}{1-\alpha} \left[\left(\frac{p'}{1-\alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{-1/p} [g(t)]^{\frac{\alpha}{p'}} \left\| [g(\cdot)]^{\frac{1-\alpha}{p'}} \right\|_{L_{q(\cdot),\,\mu}(|\cdot|>t)} \le C,$$

i.e., $\dfrac{p'\, A(\alpha,p,q)}{(1-\alpha)\, \left[\left(\frac{p'}{1-\alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p}} \le C$ for all $\alpha \in (0,1)$.

This completes the proof of Theorem 2.

Corollary. *Let $q(x) = q = const$ and μ and ν satisfies the condition of Theorem 2. Then condition*

$$\ell = \sup_{t>0} \left(\int\limits_{|y|>t} d\mu(y) \right)^{\frac{1}{q}} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1}{p'}} < \infty, \tag{3}$$

is equivalent to the condition (2) and there exists a constant $\alpha \in (0,1)$ such that the inequalities

$$\ell \le A(\alpha,p,q) \le (1/\alpha)^{1/q}\, \ell \tag{4}$$

hold.

Proof. Let the condition (2) is valid. It is obvious that

$$A(\alpha,p,q) = \sup_{t>0} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|<|\cdot|} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q,\,\mu}(|x|>t)} \ge$$

$$\ge \sup_{t>0} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q,\,\mu}(|x|>t)} =$$

$$= \sup_{t>0} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \cdot \left(\int\limits_{|y|<|\cdot|} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1-\alpha}{p'}} \left(\int\limits_{|y|>t} d\mu(y) \right)^{\frac{1}{q}} =$$

$$= \sup_{t>0} \left(\int\limits_{|y|>t} d\mu(y) \right)^{\frac{1}{q}} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1}{p'}} .$$

Now we prove the right hand side of inequality (4). Let the condition (3) is valid. We have

$$\left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|<|\cdot|} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q,\,\mu}(|x|>t)} \leq$$

$$\leq \ell^{1-\alpha} \cdot \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|>|\cdot|} d\mu(y) \right)^{\frac{\alpha-1}{q}} \right\|_{L_{q,\,\mu}(|x|>t)} =$$

$$= \ell^{1-\alpha} \cdot \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \left(\int\limits_{|x|>t} \left[\int\limits_{|y|>|x|} d\mu(y) \right]^{\alpha-1} d\mu(x) \right)^{1/q} =$$

$$= \ell^{1-\alpha} \cdot \frac{1}{\alpha^{1/q}} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\alpha}{p'}} \left(\int\limits_{|x|>t} d\mu(x) \right)^{\frac{\alpha}{q}} \leq$$

$$\leq \frac{1}{\alpha^{1/q}} \ell^{1-\alpha} \cdot \ell^{\alpha} = \frac{1}{\alpha^{1/q}} \ell .$$

The Theorem below is proved analogously.

Theorem 3. *Let μ and ν nonnegative Borel measure on R^n and ν^* is absolutely continuous part of measure ν. Let $p(x) = p = const$, $q \in \mathcal{P}(R^n)$ and $1 < p \leq q(x) \leq \overline{q} < \infty$. Then the inequality*

$$\left\| \int\limits_{|y|>|\cdot|} f(y)\, dy \right\|_{L_{q(\cdot),\,\mu}(R^n)} \leq C \, \|f\|_{L_{p,\,\nu}(R^n)} \tag{5}$$

holds, for every $f \geq 0$ if and only if there exists $\beta \in (0,1)$ such that

$$B(\beta, p, q) =$$

$$= \sup_{t>0} \left(\int\limits_{|y|<t} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{\beta}{p'}} \left\| \left(\int\limits_{|y|<|\cdot|} \left[\frac{d\nu^*}{dy} \right]^{1-p'} dy \right)^{\frac{1-\beta}{p'}} \right\|_{L_{q(\cdot),\,\mu}(|x|>t)} < \infty. \quad (2)$$

Moreover, if $C>0$ is the best possible constant in (5) then

$$\sup_{0<\beta<1}\frac{p'\,B(\beta,p,q)}{(1-\beta)\left[\left(\frac{p'}{1-\beta}\right)^p+\frac{1}{\beta(p-1)}\right]^{1/p}}\leq C\leq$$

$$\leq\left(\frac{p}{\underline{q}}+\frac{\overline{q}-p}{\overline{q}}\right)^{\frac{2}{p}}\inf_{0<\beta<1}\frac{B(\beta,p,q)}{(1-\beta)^{1/p'}}.$$

Remark. Note that the Theorem 2 and Theorem 3 at $n=1$, $p(x)=p=const$ and $q(x)=q=const$, for $x\in(0,\infty)$ and under condition (3) were proved in [18]. Recently the Theorem 2 in the case of absolutely continuous measures at $n=1$, $p(x)=p=const$ and $q(x)=q=const$, $x\in(0,\infty)$ and under condition (2) (more exactly at $\alpha=\dfrac{s-1}{p-1}$, where $s\in(1,p)$) was proved in [29]. Also, the sufficiency parts of Theorem 2 and Theorem 3 in the case of absolutely continuous measures at $n=1$, $x\in(0,\infty)$ were proved in [1]. Further development in the direction of the boundedness of Hardy operator was given in the paper [5]-[8], [10]-[12] and [16]. Two-weight criterion for Hardy operator at $x\in[0,1]$ was proved in [13]. Also, other type two-weight criterion for multidimensional Hardy operator was proved in [17]. In the case $p(x)=p=const$ and $q(x)=q=const$ at $x\in(0,\infty)$ for absolutely continuous measures in classical Lebesgue spaces the various variants of Theorem 2 and Theorem 3 were proved in [4], [9], [11], [15], [19], [23] and etc. Recently the Theorem 2 and Theorem 3 in the case of absolutely continuous measures was proved in [32](see also [31]).

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