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ON ASYMPTOTIC PROPERTIES OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATION

Abstract

The solutions of a nonlinear elliptic equation in cylindrical domain, satisfying the Neumann boundary condition is considered. Asymptotics of such solutions is obtained in the vicinity of infinity.

Let G be a bounded domain in R^n with a Lipschits boundary.

Denote: $\Pi_{a,b} = G \times (a,b)$, $\Pi_{a,\infty} = \Pi_a$, $\Gamma_{a,b} = \partial G \times (a,b)$, $\Gamma_{a,\infty} = \Gamma_a$.

We'll investigate the behavior of the solution to the equation

$$u_{tt} + \Delta u - |u|^{\sigma} = 0 \text{ in } \Pi_0, \tag{1}$$

satisfying the condition

$$\frac{\partial u}{\partial n} = 0 \text{ in } \Gamma_0, \tag{2}$$

as $t \to +\infty$, where $\sigma > 1, n$ is a unit vector of an external normal to ∂G.

Notice that the similar problem with a nonlinearity of the form $|u|^{\sigma-1} \cdot u$ was investigated in the papers [1], [2].

As a solution of problem (1), (2) we understand a generalized soltion. The function $u(x,t)$ is said to be a generalized solution of equation (1), satisfying condition (2) if $u(x,t) \in W_2^1(\Pi_{a,b}) \cap L_\infty(\Pi_{a,b})$ for any $0 < a$, $b < \infty$ and it holds the equality

$$\int\limits_{\Pi_{a,b}} u_t \cdot \varphi_t dxdt + \sum_{i=1}^{n} \int\limits_{\Pi_{a,b}} u_{x_i} \cdot \varphi_{x_i} dxdt + \int\limits_{\Pi_{a,b}} |u|^{\sigma} \cdot \varphi dxdt = 0 \tag{3}$$

for any function $\varphi(x,t) \in W_2^1(\Pi_{a,b})$ such that $\varphi(x,a) = \varphi(x,b) = 0$.

Prove some auxiliary facts.

Lemma 1. *For any $\sigma > 1$ problem (1), (2) has no negative solutions.*

Proof. In definition of the solution, as a test function we take $\varphi(x,t) = t \cdot \psi(t)$, where $\psi(t) \in C_0^\infty(R)$, $\psi(t) = \begin{cases} 1, t \le R \\ 0, t \ge 2R \end{cases}$.

Then we have

$$\int\limits_{\Pi_{0,2R}} |u|^{\sigma} t \cdot \psi dtdx = -\int\limits_{\Pi_{0,2R}} u_t \left(t\psi' + \psi\right) dtdx = \int\limits_{\Pi_{0,2R}} u \left(t\psi'' + 2\psi'\right) dtdx +$$

$$+ \int\limits_G u(x,0)\,dx \le \left(\int\limits_{\Pi_{0,2R}} |u|^{\sigma} t \cdot \psi(t)\,dtdx\right)^{\frac{1}{\sigma}} \cdot \left(\int\limits_{\Pi_{0,2R}} \frac{\left|t \cdot \psi'' + 2\psi'\right|^q}{t^{q-1}\psi^{q-1}} dtdx\right)^{\frac{1}{q}} +$$

$$+ \int\limits_G u(x,0)\,dx \le \frac{\varepsilon}{\sigma} \int\limits_{\Pi_{0,2R}} |u|^{\sigma} t \cdot \psi(t)\,dtdx +$$

$$+\frac{1}{\varepsilon^{q-1}\cdot q}\int\limits_{\Pi_{0,2R}}\frac{\left|t\cdot\psi''+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dtdx+\int\limits_G u\left(x,0\right)dx,$$

where $\frac{1}{\sigma}+\frac{1}{q}=1$.

Hence we get

$$\left(1-\frac{\varepsilon}{\sigma}\right)\cdot\int\limits_{\Pi_{0,2R}}|u|^\sigma\, t\cdot\psi\left(t\right)dtdx\le\frac{1}{\varepsilon^{q-1}\cdot q}\times$$

$$\times\int\limits_{\Pi_{0,2R}}\frac{\left|t\psi''+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dtdx+\int\limits_G u\left(x,0\right)dx. \qquad (4)$$

Make the substitution $\tau=\frac{t}{R}$. Take $\psi\left(t\right)$ in the form $\psi\left(t\right)=\psi\left(\tau R\right)=\left(\varphi_0\left(\tau\right)\right)^\mu=\theta\left(\tau\right)$ where $\varphi_0\left(\tau\right)=\begin{cases}1 \text{ as } \tau\le1,\\ 0 \text{ as } \tau\ge2,\end{cases}$ $\varphi_0\left(\tau\right)\in C_0^\infty$, μ is a sufficiently great number in modulus. Estimate the first integral in the right hand side of inequality (4):

$$\int\limits_{\Pi_{0,2R}}\frac{\left|t\psi''+2\psi'\right|^q}{t^{q-1}\psi^{q-1}}dtdx=\int\limits_G\int\limits_{1\le\tau\le2}\frac{\left|\tau\cdot R^{-1}\theta''+2R^{-1}\theta'\right|^q}{R^{q-1}\tau^{q-1}\theta^{q-1}}Rd\tau dx=$$

$$=R^{2(1-q)}\cdot mesG\int\limits_{1\le\tau\le2}\frac{\left|\tau\cdot\theta''+2\theta'\right|^q}{\tau^{q-1}\theta^{q-1}}d\tau=R^{2(1-q)}\cdot mesG\times$$

$$\times\int\limits_{1\le\tau\le2}\frac{\left|\tau\cdot\mu\cdot\varphi_0^{\mu-1}\cdot\varphi_0''+\tau\cdot\mu\left(\mu-1\right)\cdot\varphi_0^{\mu-2}\cdot\varphi_0'^2+2\mu\cdot\varphi_0^{\mu-1}\varphi_0'\right|^q}{\tau^{q-1}\varphi_0^{\mu(q-1)}}d\tau=$$

$$=R^{2(1-q)}\cdot A\left(\varphi_0\right),$$

where

$$A\left(\varphi_0\right)=mesG\times$$

$$\times\int\limits_{1\le\tau\le2}\frac{\left|\tau\cdot\mu\cdot\varphi_0^{\mu-1}\cdot\varphi_0''+\tau\cdot\mu\left(\mu-1\right)\cdot\varphi_0^{\mu-2}\cdot\varphi_0'^2+2\mu\cdot\varphi_0^{\mu-1}\varphi_0'\right|^q}{\tau^{q-1}\varphi_0^{\mu(q-1)}}d\tau.$$

We can chose μ, φ_0 so that $A\left(\varphi_0\right)<\infty$.

If we take into account all these facts in (4), then:

$$\left(1-\frac{\varepsilon}{\sigma}\right)\cdot\int\limits_{\Pi_{0,2R}}|u|^\sigma\, tdtdx\le\left(1-\frac{\varepsilon}{\sigma}\right)\cdot\int\limits_{\Pi_{0,2R}}|u|^\sigma\, t\cdot\psi\left(t\right)dtdx\le$$

$$\le\frac{1}{\varepsilon^{q-1}}\cdot R^{2(1-q)}\cdot A\left(\varphi\right)+\int\limits_G u\left(x,0\right)dx. \qquad (5)$$

Since $q=\frac{\sigma}{\sigma-1}>1$, if $\int\limits_G u\left(x,0\right)dx\le0$ then as $R\to\infty$ from (5) we get

$$\int\limits_{\Pi_{0,2R}}|u|^\sigma\, tdtdx=0.$$

Hence we have $u \equiv 0$ in Π_0 if $\int_G u(x,0)\,dx \le 0$. This proves lemma 1.

We obtain that if $u(x,t)$ is a nontrivial solution of problem (1), (2), then $\int_G u(x,0)\,dx > 0$. If as a test function we take $\varphi(x,t) = \begin{cases} (t-t_0)\,\psi(t), & t \ge t_0 \\ 0, & t \le t_0, \end{cases}$, then for any nontrival solution $u(x,t)$ $\int_G u(x,t_0)\,dx > 0$.

Lemma 2. *If $u(x,t)$ is a solution of problem (1), (2), then*

$$\lim_{t\to\infty} u(x,t) = 0.$$

Proof. At first prove that any solution of problem (1), (2) is bounded. If $u(x,t)$ is a solution of equation (1), then $u(x,t)$ is a subsolution of the equation

$$u_{tt} + \Delta u - |u|^{\sigma-1} u = 0. \qquad (6)$$

Indeed:

$$u_{tt} + \Delta u - |u|^{\sigma-1} u \ge u_{tt} + \Delta u - |u|^{\sigma} = 0.$$

Eqution (6) has a strong positive solution $\omega(t)$ satisfying the relations $\omega(t_0) = 1$, $\omega'(t_0) = 0$ in the form of a parabola with asymptotes at the points $t_0 \pm T$ (where T is independent of t_0). Then for sufficiently large t from the maximum principle, the subsolution is less than the solution, i.e. $u(x,t) \le \omega(t)$ in Π_{t_0-T,t_0+T}. Thus, $u(x,t)$ is upper bounded, since for large t is less than the value at the top of the parabola.

The function $v(x,t) = u(x,t) - C_0 \cdot t^{-\frac{2}{\sigma-1}}$, where $C_0 = \left[\frac{2(\sigma+1}{(\sigma-1)^2}\right]^{\frac{1}{\sigma-1}}$ is also an upper bounded subsolution of equation (6). Then

$$v_{tt} + \Delta v - a(x,t)\,v \ge 0, \qquad (7)$$

where $a(x,t) \ge 0$.

Consider the function $v - \varepsilon t$. This function also satisfies inequality (7) and is negative for $t = 0$. There exists such $T_0(\varepsilon)$ that for $T \ge T_0(\varepsilon)$ $v - \varepsilon T \le 0$. Then it follows from the maximum principle that $v - \varepsilon t \le 0$ for $t \ge 0$. Tending ε to zero, we get $v \le 0$.

So,

$$u(x,t)^+ \le C_0 \cdot t^{-\frac{2}{\sigma-1}}. \qquad (8)$$

Making in (1) the substitution $v = -v$, consider the equation

$$v_{tt} + \Delta v + |v|^{\sigma} = 0.$$

Since $|v| = v^+ - v^-$, $v = v^+ + v^-$ then

$$\int_G |v|\,dx \le -2\int_G v^- dx \le 2\int_G C_0 \cdot t^{-\frac{2}{\sigma-1}} dx = C_1 \cdot t^{-\frac{2}{\sigma-1}}.$$

If $\sigma < 3$, then

$$\int_1^{\infty}\int_G |v|\,dxdt \le C_1 \int_1^{\infty} t^{-\frac{2}{\sigma-1}} dt = -C_2 \cdot t^{-\frac{\sigma-3}{\sigma-1}}\Big|_1^{\infty} = C_2.$$

If $\sigma > 3$, then for large T

$$\int\limits_{\Pi_{T-2,T+2}} |u|\, dxdt \le C_3,$$

where C_3 is independent of T. Indeed:

$$\int\limits_{\Pi_{T-2,T+2}} |v|\, dxdt \le C_1 \int\limits_{T-2}^{T+2} t^{-\frac{2}{\sigma-1}} dt = C_1 \frac{\sigma-1}{\sigma-3}\left((T+2)^{\frac{\sigma-3}{\sigma-1}} - (T-2)^{\frac{\sigma-3}{\sigma-1}}\right) =$$

$$= 4C_1 (T-2+\xi\cdot 4)^{-\frac{2}{\sigma-1}} = 4C_1 \frac{1}{(T-2+\xi\cdot 4)^{\frac{2}{\sigma-1}}} \le 4C_1,$$

if $T > 3$. There $0 < \xi < 1$.

For $\sigma = 3$, similarly we get

$$\int\limits_{\Pi_{T-2,T+2}} |v|\, dxdt \le C_1 \int\limits_{T-2}^{T+2} t^{-1} dt = C_1 \ln T \Big|\begin{matrix} T+2 \\ T-2 \end{matrix} =$$

$$= C_1 \frac{4}{(T-2+4\xi)^{\frac{2}{\sigma-1}}} \le 4C_1,$$

if $T > 3$.

From the theory of linear differential equations we know that [see3]

$$\max_{\Pi_{T-1,T+1}} |u| \le C \int\limits_{\Pi_{T-2,T+2}} |u|\, dxdt \le C_3 \text{ as } T > 3.$$

So, everywhere $|u| < C$.

From (5) we get

$$\int\limits_{\Pi_{1,\infty}} |u|^{\sigma}\, dxdt \le C_4. \tag{9}$$

Then, for each T_ε, there exists such a point $(x_\varepsilon, t_\varepsilon) \in \Pi_{T_\varepsilon-1,T_\varepsilon+1}$ and such C that

$$|u(x_\varepsilon, t_\varepsilon)| \le \frac{C}{2mesG} \int\limits_{\Pi_{T_\varepsilon-1,T_\varepsilon+1}} |u|\, dxdt \to 0 \tag{10}$$

as $T_\varepsilon \to +\infty$.

This is easily proved by contradiction. Using this, prove that $u(x,t) \to 0$ as $t \to \infty$.

If $u(x,t)$ is a solution of equation (1), then $v = -u$ is a solution of equation

$$v_{tt} + \Delta v + |v|^{\sigma} = 0. \tag{11}$$

Write it as follows

$$v_{tt} + \Delta v + |v|^{\sigma-1} signv \cdot v = 0.$$

Denote $q(x,t) = |v|^{\sigma-1}\, signv$. Since $|v| = |u| < C$, then $|q(x,t)| < C_1$. Consider the function

$$W(x,t) = v(x,t) + C_0 t^{-\frac{2}{\sigma-1}}. \tag{12}$$

If follows from (8) that $W(x,t) \geq 0$.

$W(x,t)$ satisfies the equation

$$W_{tt} + \Delta W + q(x,t)\, W = -C_0 \frac{2(\sigma+1)}{(\sigma-1)^2} t^{-\frac{2\sigma}{\sigma-1}} - q \cdot C_0 \cdot t^{-\frac{2}{\sigma-1}}.$$

Then by the Harnack inequality [see 4] we have:

$$\max_{\Pi_{T-1,T+1}} W(x,t) \leq C_1 \int\limits_{\Pi_{T-1,T+1}} W(x,t) + C_2 \cdot \|f\|_{L_{q/2}(\Pi_{T-2,T+2})} \leq$$

$$\leq C_1 \cdot \inf_{\Pi_{T-1,T+1}} W(x,t) + C_2 \times$$

$$\times \left(\int\limits_{\Pi_{T-2,T+2}} t^{-\frac{q}{\sigma-1}} \left[-C_0 \frac{2(\sigma+1)}{(\sigma-1)} t^{-2} - q \cdot C_0 \right]^{q/2} dxdt \right)^{\frac{2}{q}} \leq$$

$$\leq C_1 \cdot \inf_{\Pi_{T-1,T+1}} W(x,t) + C_2 \cdot C_3 \left(\int\limits_{\Pi_{T-2,T+2}} t^{-\frac{q}{\sigma-1}} dt \right)^{\frac{2}{q}} \to 0 \text{ as } T \to \infty,$$

by (10) and (12).

Hence it follows that $u = -v = C_0 \cdot t^{-\frac{2}{\sigma-1}} - W \to 0$ as $t \to +\infty$. This proves lemma 2.

Now, prove that $u(x,t) = O\left(t^{-\frac{2}{\sigma-1}}\right)$. If $u(x,t)$ is a non-negative solution, this is obvious.

Make the substitution $v = -u$.

Then

$$v_{tt} + \Delta v + |v|^{\sigma} = 0. \tag{13}$$

Since $u \leq C \cdot t^{-\frac{2}{\sigma-1}}$, then $v \geq -C \cdot t^{-\frac{2}{\sigma-1}}$. Denote $h(t) = -C \cdot t^{-\frac{2}{\sigma-1}}$, $z = v - h(t)$. Then, $z \geq 0$ and

$$z_{tt} + \Delta v + |z+h|^{\sigma} = h_{tt}.$$

Write it as follows

$$z_{tt} + \Delta z + \frac{|h+z|^{\sigma} - |h|^{\sigma}}{z} z = C_1 \cdot t^{-\frac{2\sigma}{\sigma-1}} + C_2 \cdot t^{-\frac{2\sigma}{\sigma-1}} = O\left(t^{-\frac{2\sigma}{\sigma-1}}\right).$$

Hence,

$$z_{tt} + \Delta z + B(x,t)\, z = C \cdot t^{-\frac{2\sigma}{\sigma-1}}, \tag{14}$$

where $B(x) = C \cdot t^{-2} + \frac{o(z)}{z}$ tends to zero as $z \to 0$.

Since $z \geq 0$, applying the Harnack inequality to (14), we get:

$$\max_{\Pi_{T-1,T+1}} |z(x,t)| \leq C_1 \min_{\Pi_{T-1,T+1}} |z(x,t)| + C_2 \cdot \|f\|_{L_{q/2}(\Pi_{T-2,T+2})}, \; q > n+1.$$

Let at first $T = t_\varepsilon$. Then by the Harnack inequality we have:

$$\max_{\Pi_{T-1,T+1}} |z(x,t)| \le C_1 t_\varepsilon^{-\frac{2}{\sigma-1}} + C_2 \cdot t^{-\frac{2}{\sigma-1}}.$$

$$t_\varepsilon^{-\frac{2}{\sigma-1}} = C_3 (t_\varepsilon + 1 + t_\varepsilon - 1)^{-\frac{2}{\sigma-1}} = C_3 (t_\varepsilon + 1)^{-\frac{2}{\sigma-1}} \left(1 + \frac{t_\varepsilon - 1}{t_\varepsilon + 1}\right)^{-\frac{2}{\sigma-1}} =$$

$$= C_3 (t_\varepsilon + 1)^{-\frac{2}{\sigma-1}} \left(1 + \frac{1 - \frac{1}{t_\varepsilon}}{1 + \frac{1}{t_\varepsilon}}\right)^{-\frac{2}{\sigma-1}} \le C_4 (t_\varepsilon + 1)^{-\frac{2}{\sigma-1}} \le$$

$$\le C_4 (T+1)^{-\frac{2}{\sigma-1}} \le C_4 t^{-\frac{2}{\sigma-1}},$$

if $T - 1 \le t \le T + 1$. So,

$$|z(x,t)| \le C_4 \cdot t^{-\frac{2}{\sigma-1}}, \text{ if } T - 1 \le t \le T + 1.$$

Having taken successively $T = t_\varepsilon + 1, t_\varepsilon + 2$ and etc., we get

$$|z(x,t)| \le C \cdot t^{-\frac{2}{\sigma-1}}, \text{ for } t \ge T_0.$$

Then

$$|v| = |z + h| \le |z| + |h| \le C \cdot t^{-\frac{2}{\sigma-1}},$$

$$|u| = |v| = O\left(t^{-\frac{2}{\sigma-1}}\right).$$

$$|u|^{\sigma-1} = O\left(t^{-2}\right).$$

The following theorem is the basic result.

Theorem.

I. *For any $\sigma > 1$ there is no solution of equation (1) satisfying condition (2), negative in Π_a, $a > 0$.*

II. *Let $u(x,t) > 0$ be a solution of equation (1) satisfying condition (2). Then, $u(x,t) = O\left(t^{-\frac{2}{\sigma-1}}\right)$.*

III. *Let $u(x,t)$ be a solution of equation (1) satisfying condition (2) that changes sign at each domain Π_a, $a > 0$. Then, $u(x,t) = O\left(e^{-ht}\right)$, where h is independent of $u(x,t)$.*

Proof.

Above we proved **I** and **II**. Prove **III**. Write equation (1) in the form

$$u_{tt} + \Delta u - q(x,t)\, u = 0, \tag{15}$$

where $q(x,t) = |u|^{\sigma-1} \cdot signu$.

Since $\lim\limits_{t\to\infty} |u(x,t)| = 0$, there exist such t_0 that for any $t \ge t_0$, $|u(x,t)|^{\sigma-1} < \varepsilon$.

Take $\theta(t) \in C^\infty$ such that $\theta(t) = 1$ for $t > t_0 + 1$, $\theta(t) = 0$ for $t \le t_0$ and $0 \le \theta(t) \le 1$.

Assume

$$v(x,t) = \theta(t) \cdot u(x,t).$$

The function $v(x,t)$ satisfies the equation

$$v_{tt} + \Delta v - q(x,t)\, v = F(x,t) \tag{16}$$

and boundary conditions

$$\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma, \tag{17}$$

where

$$q(x,t) = \begin{cases} |u|^{\sigma-1}\, sign\ u & \text{for } t \geq t_0 + 1, \\ 0 & \text{for } t \leq t_0, \end{cases}$$

$$F(x,t) = (\theta_t \cdot u)_t + \theta_t \cdot u_t.$$

Obviously, the function $F(x,t)$ has a compact support.

Show that $|v(x,t)| \leq C \cdot \exp\{-ht\}$, $C = const$. It follows from the theory of linear equations [see 4.5] that problem (16), (17) has the solution $v_1(x,t)$ such that

$$v_1(x,t) = \begin{cases} 0\left(e^{-ht}\right) & \text{as } t \to +\infty \\ at + b + 0\left(e^{ht}\right) & \text{as } t \to -\infty. \end{cases} \tag{18}$$

The function $\omega(x,t) = v_1(x,t) - v(x,t)$ satisfies the equation

$$\omega_{tt} + \Delta\omega - q(x,t)\,\omega = 0 \tag{19}$$

and boundary condition

$$\frac{\partial \omega}{\partial n} = 0 \quad \text{on } \Gamma,$$

$\omega(x,t) \to 0$ as $t \to +\infty$ and $\omega = at + b + O\left(e^{ht}\right)$ as $t \to -\infty$.

It we prove $\omega \equiv 0$, then this will prove theorem. Show $a = 0$, $b = 0$. Assume $a > 0$. So, $\omega(x,t) < 0$ for $t < -T$, where T_1 is a sufficiently large positive number. Prove that $\omega < 0$ for $t > -T_1$. Since $q(x,t) = |u|^{\sigma-1}\, sign\ u$ for $t \geq t_0 + 1$, then $q(x,t) = O\left(t^{-2}\right)$ for $t \to +\infty$.

Denote $k = \max\limits_{t=T} \omega(x,t)$ and $W(x,t) = (\omega - k)^+$, where T is a sufficiently large positive number. Obviously, $W(x,t) = 0$ for $t = T_1$ and for $t = T$.

It is obvious that

$$W(x,t) \in W_2^1\left(Q_{T_1,T}\right).$$

Then, from the definition of the solution we have:

$$\int\limits_{A_k^+} |\omega_t|^2\, dxdt + \int\limits_{A_k^+} |\nabla\omega|^2\, dxdt = -\int\limits_{A_k^+} q(x,t)\,\omega\,(\omega - k)^+\, dxdt, \tag{20}$$

where $A_k^+ = \{(x,t), W > 0\}$.

Estimate the right hand side using the inequality [see 3],

$$\|u\|_{\frac{2n}{n-2}} \leq C\, \|\nabla u\|_{2,\Omega}\,, \tag{21}$$

where C is a constant independent of the dimension of n. Then,

$$-\int\limits_{A_k^+} q(x,t)\,\omega\,(\omega - k)^+\, dxdt \leq \int\limits_{A_k^+} |q(x,t)|\,(\omega - k + k)\,(\omega - k)\, dxdt =$$

$$= \int\limits_{A_k^+} |q(x,t)| \cdot |\omega - k|^2 \, dxdt + k \int\limits_{A_k^+} |q(x,t)| \cdot |\omega - k| \, dxdt \leq$$

$$\leq \int\limits_{\substack{A_k^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - k|^2 \, dxdt + k \int\limits_{\substack{A_k^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - k| \, dxdt. \qquad (22)$$

At first we estimate the first summand

$$F_1 = \int\limits_{\substack{A_k^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - k|^2 \, dxdt \leq \left(\int\limits_{\substack{A_k^+ \\ t>t_0}} |q(x,t)|^{\frac{2(n+1)}{n-1}} dxdt \right)^{\frac{n-1}{n+1}} \times$$

$$\times \left(\int\limits_{\substack{A_k^+ \\ t>t_0}} |q(x,t)|^{\frac{n+1}{2}} dxdt \right)^{\frac{2}{n+1}} \leq \left(\int\limits_{A_k^+ \cap Q_{T_1,T_2}} |\omega - k|^{\frac{2(n+1)}{n-1}} dxdt \right)^{\frac{n-1}{n+1}} \times$$

$$\times \left(\int\limits_{t>t_0} |q(x,t)|^{\frac{n+1}{2}} dxdt \right)^{\frac{2}{n+1}} \leq \left[\left(\int\limits_{A_k^+ \cap Q_{T_1,T_2}} |\omega - k|^{\frac{2(n+1)}{n-1}} dxdt \right)^{\frac{n-2}{2(n+1)}} \right]^2 \times$$

$$\times \left(\int\limits_{A_k^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} dxdt \right)^{\frac{2}{n+1}} \leq C \cdot \left(\int\limits_{A_k^+ \cap Q_{T_1,T_2}} |\nabla(\omega - k)|^2 \, dxdt \right) \cdot I_2, \quad (23)$$

where $I_2 = \left(\int\limits_{A_k^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} dxdt \right)^{\frac{2}{n+1}}$.

Now, estimate I_2.

$$I_2 = \left(\int\limits_{A_k^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} dxdt \right)^{\frac{2}{n+1}} \leq C_1 \cdot \left(\int\limits_{A_k^+ \cap \{t>t_0\}} t^{-(n+1)} dxdt \right)^{\frac{2}{n+1}} \leq$$

$$\leq C_1 \cdot \left(\int\limits_{A_k^+ \cap \{t>t_0\}}^{T} t^{-(n+1)} dxdt \right)^{\frac{2}{n+1}} \leq C_2 \cdot \left(\frac{t^{-n}}{-n} \Big|_{t_0}^{T} \right)^{\frac{2}{n+1}} =$$

$$= C_2 \cdot \left(\frac{T^{-n}}{-n} + \frac{t_0^{-n}}{n} \right)^{\frac{2}{n+1}} = C_3 \cdot \left(t_0^{-n} - T^{-n} \right)^{\frac{2}{n+1}}$$

take t_0 so that $|u(x,t)| < \varepsilon$ and $C_3 \cdot t_0^{-\frac{2n}{n+1}} < \frac{1}{C\cdot 4}$. Then, we get

$$I_2 \leq \frac{1}{C \cdot 4}.$$

Then from (23) we get

$$F_1 \leq \frac{1}{4} \cdot \int\limits_{A_k^+ \cap Q_{T_1,T_2}} |\nabla (\omega - k)|^2 \, dxdt. \tag{24}$$

Estimate the second summand in the right hand side of (22)

$$F_2 = k \cdot \int\limits_{A_k^+} |q(x,t)| \cdot |\omega - k| \, dxdt \leq k \cdot \left(\int\limits_{A_k^+} |q(x,t)|^{p_1} \, dxdt \right)^{\frac{1}{p_1}} \times$$

$$\times \left(\int\limits_{A_k^+} |\omega - k|^{\frac{2(n+1)}{n-1}} \, dxdt \right)^{\frac{n-1}{2(n+1)}} \leq k \cdot C_1 \left(\int\limits_{\substack{t>t_0 \\ A_k^+}} t^{-2p_1} dt \right)^{\frac{1}{p_1}} \times$$

$$\times \left(\int\limits_{A_k^+} |\nabla (\omega - k)|^2 \, dxdt \right)^{\frac{1}{2}} \leq \frac{1}{4} \int\limits_{A_k^+} |\nabla (\omega - k)|^2 \, dxdt + k^2 \cdot C_2 \left(\int\limits_{\substack{t>t_0 \\ A_k^+}} t^{-2p_1} dt \right)^{\frac{2}{p_1}}, \tag{25}$$

here $\frac{1}{p_1} + \frac{n-1}{2(n+1)} = 1$.

Hence $p_1 = 1 + \frac{n-1}{n+3}$. Combining (24) and (25), we get

$$\int\limits_{A_k^+} |\omega_t|^2 \, dxdt + \int\limits_{A_k^+} |\nabla \omega|^2 \, dxdt \leq \frac{1}{2} \int\limits_{A_k^+} |\omega_t|^2 \, dxdt +$$

$$+ \frac{1}{2} \int\limits_{A_k^+} |\nabla \omega|^2 \, dxdt + k^2 \cdot C_2 \left(\int_{t_0}^{T} t^{-2p_1} dt \right)^{\frac{2}{p_1}}.$$

As a result, for $n > 1$ we have

$$\frac{1}{2} \cdot \int\limits_{A_k^+} |\omega_t|^2 \, dxdt + \frac{1}{2} \cdot \int\limits_{A_k^+} |\nabla \omega|^2 \, dxdt \leq k^2 \cdot C_2 \left(\int_{t_0}^{T} t^{-2p_1} dt \right)^{\frac{2}{p_1}}. \tag{26}$$

From $k(T) \to 0$ as $T \to 0$ and from the convergence of the integral $\int_{t_0}^{T} t^{-2p_1} dt$ we obtain $mes A_k^+ = 0$.

So, $\omega - k \leq 0$. Having taken T sufficiently large, we get that k tends to zero. Hence, it follows that $\omega < 0$.

We can similarly prove that if $a < 0$ then $\omega(x,t) > 0$.

Show that $a = b = 0$. Assume $a > 0$. So, $\omega(x,t) < 0$ for $t > t_1$. The function $\omega_1 = -t^{\beta}$ will be a supersolution of equation (9) for sufficiently large in modulus negative β.

Indeed:

$$L = \omega_{1tt} + \Delta w_1 - q(x,t)\,\omega_1 = -\beta(\beta-1)\,t^{\beta-2} + q(x,t)\,t^{\beta} =$$

$$= -t^{\beta-2}\left(\beta(\beta-1) - qt^{-2}\right) < 0.$$

Let t_2 be sufficiently great. Take A such small positive number that $-At_2^{\beta} \geq \omega(x,t_2)$.

Then, from $W = \omega(x,t_2) + At_2^{\beta} \leq 0$, $\omega(x,t) + At^{\beta} \to 0$ as $t \to +\infty$ and

$$LW \geq 0.$$

As above, we can prove

$$\omega(x,t) + At^{\beta \leq 0} \text{ as } t \geq t_2.$$

Consider a points set, where $v = u < 0$, for them we have

$$-A \cdot t^{\beta} \geq \omega(x,t) \geq v_1 - C_1 e^{-ht}.$$

This contradiction shows that a may not be positive. Similarly, we can show that a may not be negative and that $b = 0$. So, $\omega \to \pm\infty$ as $\omega \equiv 0$ and consequently $\omega \equiv 0$.

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