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ON REGULAR SOLVABILITY OF A BOUNDARY PROBLEM WITH OPERATOR BOUNDARY CONDITION

Abstract

In this paper, regular solvability conditions of some boundary value problem are indicated for a third order operator-differential equation whose boundary condition contains some operator. These conditions are expressed by the properties of coefficients of the operator-differential equation and operator participating in one of boundary conditions.

In a separable Hilbert space H consider the boundary value problem

$$P(d/dt)\,u(t) = u'''(t) - A^3 u(t) + \sum_{j=0}^{2} A_{3-j} u(j)(t) = f(t), \quad t \in R_+ = (0; +\infty), \quad (1)$$

$$u(0) = K_0 u, \quad u'(0) = 0, \quad (2)$$

where the derivatives are understood in the sense of theory of distributions [1], $f(t), u(t)$ are vector functions with values in H, A and A_j $\left(j = \overline{1,3}\right)$ are linear operators in H.

Let A be a positive-definite self-adjoint operator. Denote by H_γ $(\gamma \geq 0)$ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$. For $\gamma = 0$ we assume $H_0 = H$.

Determine the following Hilbert spaces

$$L_2(R_+; H) = \left\{ f : \|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|^2 dt \right)^{1/2} < \infty \right\},$$

$$W_2^3(R_+; H) = \left\{ u : u''', A^3 u \in L_2(R_+; H), \|u\|_{W_2^3(R_+;H)} = \right.$$
$$\left. = \left(\|u'''\|^2_{L_2(R_+;H)} + \|A^3 u\|^2_{L_2(R_+;H)} \right)^{1/2} \right\}.$$

$$\overset{\circ}{W}{}_2^3(R_+; H; K_0) = \left\{ u : u \in W_2^3(R_+; H), u(0) = K_0 u, \ u'(0) = 0 \right\}.$$

In the sequel, we'll assume that the operators in the problem (1), (2) satisfy the following conditions:

1) $B_j = A_j A^{-j}$ $\left(j = \overline{1,3}\right)$ are bounded operators in H;

2) the operator $K_0 : W_2^3(R_+; H) \to H_{5/2}$ is bounded and has the norm k_0, i.e.

$$K_0 \in L\left(W_2^3(R_+; H), H_{5/2}\right) \text{ and } \|K_0\|_{W_2^3(R_+;H) \to H_{5/2}} = k_0.$$

Definition 1. *If for any* $f(t) \in L_2(R_+;H)$ *there exists a vector-function* $u(t) \in W_2^3(R_+;H)$ *that satisfies equation (1) in* R_+ *almost everywhere, we'll call it a regular solution of equation (1).*

Definition 2. *If for any* $f(t) \in L_2(R_+;H)$ *there exists a regular solution of equation (1) that satisfies the boundary conditions in the sense of convergence* $\lim\limits_{t\to 0}\|u(t) - K_0 u\|_{5/2} = 0$, $\lim\limits_{t\to 0}\|u'(t)\|_{3/2} = 0$ *and it holds the estimation*

$$\|u(t)\|_{W_2^3(R_+;H)} \le const\, \|f\|_{L_2(R_+,H)},$$

problem (1), (2) is said to be regularly solvable.

In the paper we find conditions on the coefficients A, A_j $(j = 1,2,3)$ and K_0 that provide regular solvability of problem (1), (2). Notice that for $K_0 = 0$ this problem was investigated in the papers [3,4,5] in different situations.

At first we prove the following statement.

Theorem 1. *Let* A *be a positive-definite self-adjoint operator, condition 2) be fulfilled and the norm* $k_0 < \frac{1}{\sqrt{2}}$. *Then the operator* $P_0 \equiv \frac{d^3}{dt^3} - A^3$ *isomorphically maps the space* $\overset{\circ}{W}{}_2^3(R_+;H;K_0)$ *on to* $L_2(R_+;H)$.

Proof. At first show that the equation $P_0 u = 0$ has only a zero solution from the space $\overset{\circ}{W}{}_2^3(R_+;H;K_0)$.

Really, the general solution of the equation $P_0(d/dt)\,u(t) = 0$ from the space $W_2^3(R_+;H)$ is of the form:

$$u_0(t) = e^{\omega_1 tA}x_1 + e^{\omega_2 tA}x_2, \quad \omega_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \; \omega_2 = \overline{\omega}_1,$$

where x_1 and x_2 are any vectors from the space $H_{5/2}$ [2]. Taking into account the boundary conditions, for x_1 we get the equation

$$x_1 - \Re x_1 = 0,$$

where

$$\frac{1}{\sqrt{3}i}K_0\left(\left(\omega_2 e^{\omega_1 tA} - \omega_1 e^{\omega_2 tA}\right)x_1\right) \equiv \Re x_1. \tag{3}$$

Show that $\|\Re\|_{H_{5/2}\to H_{5/2}} < 1$. Since

$$\left(\omega_2 e^{\omega_1 tA} - \omega_1 e^{\omega_2 tA}\right)x_1 = 2ie^{-\frac{1}{2}tA}\sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}tA\right)x_1,$$

then taking into account the equality $\omega_1^3 = \omega_2^3 = 1$, from expression (3) we have

$$\left\|\left(\omega_2 e^{\omega_1 tA} - \omega_1 e^{\omega_2 tA}\right)x_1\right\|^2_{W_2^3(R_+;H)} =$$

$$= 8\left\|A^3 e^{-\frac{1}{2}tA}\sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}tA\right)x_1\right\|^2_{L_2(R_+;H)}. \tag{4}$$

For proving the theorem, at first we prove the following statement.

Lemma 1. *Let A be a positive-definite self-adjoint operator $\alpha>0$, $\beta>0$. Then for any $x\in H_{5/2}$ it holds the inequality*

$$\left\|A^3e^{-\alpha tA}\sin\left(\frac{\pi}{3}+\beta tA\right)x\right\|^2_{L_2(R_+;H)}\le$$

$$\le\left(\frac{1}{4\alpha}+\frac{1}{2}\left(\frac{\alpha}{4}+\frac{\beta\sqrt{3}}{4}\right)\right)\frac{1}{\alpha^2+\beta^2}\|x\|^2_{H_{5/2}}. \tag{5}$$

Proof. Let $A^{5/2}x=y\in H$. Then

$$\left\|A^3e^{-\alpha tA}\sin\left(\frac{\pi}{3}+\beta tA\right)x\right\|^2_{L_2(R_+;H)}=$$

$$=\int_0^{+\infty}\left(Ae^{-2\alpha tA}\sin^2\left(\frac{\pi}{3}+\beta tA\right)y,y\right)dt. \tag{6}$$

Using the spectral expansion of the operator A, from (6) we have:

$$\int_0^{+\infty}\left(Ae^{-2\alpha tA}\sin^2\left(\frac{\pi}{3}+\beta tA\right)y,y\right)dt=$$

$$=\int_{\mu_0}^{+\infty}\sigma\left(\int_{\mu_0}^{+\infty}e^{-2\alpha t\sigma}\sin^2\left(\frac{\pi}{3}+\beta t\sigma\right)dt\right)(dE_\sigma y,y). \tag{7}$$

Calculate the inner integral in expression (7)

$$\int_0^{\infty}e^{-2\alpha t\sigma}\sin^2\left(\frac{\pi}{3}+\beta t\sigma\right)dt=\frac{1}{2}\int_0^{\infty}e^{-2t\sigma\alpha}dt-\frac{1}{2}\int_0^{\infty}e^{-2t\sigma\alpha}\cos\left(\frac{2\pi}{3}+2\beta t\sigma\right)dt.$$

Applying the integration by parts formula several times, we get

$$\int_0^{\infty}e^{-2t\sigma\alpha}\cos\left(\frac{2\pi}{3}+2\beta t\sigma\right)dt=\frac{\alpha^2}{a^2+\beta^2}\left(-\frac{1}{4\sigma\alpha}-\frac{\beta}{2\alpha^2\sigma}\cdot\frac{\sqrt{3}}{2}\right).$$

Consequently,

$$\int_0^{\infty}e^{-2\alpha t\sigma}\sin^2\left(\frac{2\pi}{3}+\beta t\sigma\right)dt=\left(\frac{1}{4\sigma\alpha}+\frac{1}{2}\left(\frac{\alpha}{4\sigma}+\frac{\beta\sqrt{3}}{4\sigma}\right)\right)\frac{1}{\alpha^2+\beta^2}. \tag{8}$$

Allowing for (8) from equality (7) we get:

$$\left\|A^3e^{-\alpha tA}\sin\left(\frac{\pi}{3}+\beta tA\right)x\right\|^2_{L_2(R_+;H)}\le$$

$$\leq \left(\frac{1}{4\alpha}+\frac{1}{2}\left(\frac{\alpha}{4}+\frac{\beta\sqrt{3}}{4}\right)\right)\cdot\frac{1}{\alpha^2+\beta^2}\left\|x\right\|^2_{H_{5/2}}.$$

The lemma is proved.

The following Corollary follows from this lemma.

Corollary. *For* $\alpha=\frac{1}{2},\ \beta=\frac{\sqrt{3}}{2}$

$$\left\|\left(\omega_2 e^{\omega_1 tA}-\omega_1 e^{\omega_2 tA}\right)x_1\right\|_{L_2(R_+;H)}\leq\sqrt{6}\left\|x_1\right\|_{H_{5/2}}. \tag{9}$$

Now, continue the proof of the theorem. Taking into account the Corollary, we get

$$\left\|\Re x_1\right\|_{H_{5/2}}\equiv\left\|\frac{1}{\sqrt{3}i}K_0\left(\left(\omega_2 e^{\omega_1 tA}-\omega_1 e^{\omega_2 tA}\right)x_1\right)\right\|_{H_{5/2}}\leq$$

$$\leq\frac{1}{\sqrt{3}}\left\|K_0\right\|\left\|\left(\omega_2 e^{\omega_1 tA}-\omega_1 e^{\omega_2 tA}\right)x_1\right\|_{W_2^3(R_+;H)}\leq$$

$$\leq\frac{1}{\sqrt{3}}k_0\sqrt{6}\left\|x_1\right\|_{H_{5/2}}=\sqrt{2}k_0\left\|x_1\right\|_{H_{5/2}}.$$

Since $k_0<\frac{1}{\sqrt{2}}$, the operator $E-\Re$ is invertible and $x_1=0$, $x_2=0$.

Consequently, $u_0(t)=0$.

Now, show that the image of the operator P_0 coincides with the space $L_2(R_+;H)$, i.e. for any $f(t)\in L_2(R_+;H)$ the equation $P_0u=f$ has a regular solution from the space $\overset{\circ}{W}{}_2^3(R_+;H;K_0)$.

To this end, denote by $f_1(t)=\begin{cases} f(t), & t>0,\\ 0, & t<0.\end{cases}$ and consider the equation

$$P_0(d/dt)\,u_1(t)=f_1(t),\quad t\in R.$$

After the Fourier transformation we have

$$u_1(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}P_0^{-1}(i\xi)\,\widehat{f}_1(\xi)\,e^{i\xi t}d\xi=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}P_0^{-1}(i\xi)\left(\int_{-\infty}^{+\infty}f_1(s)\,e^{-ist}dt\right)e^{i\xi t}d\xi.$$

Show that $u_1(t)\in W_2^3(R;H)$.

Denote by $\overline{u}_1(t)$ a contraction of the vector-function $u_1(t)$ on $[0;+\infty)$ i.e. $\overline{u}_1(t)=u_1(t)\big|_{[0;+\infty)}$. Obviously, $\overline{u}_1(t)\in W_2^3(R_+;H)$. Therefore, by the theorem on traces $\overline{u}_1(0)\in H_{5/2}$, $\overline{u}_1'(0)\in H_{3/2}$, $\overline{u}_1''(0)\in H_{1/2}$. We'll look for the solution of the equation $P_0u=f$ in the form

$$u(t)=\overline{u}_1(t)+e^{\omega_1 tA}x_1+e^{\omega_2 tA}x_2,$$

where $\omega_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $\omega_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $x_1, x_2 \in H_{5/2}$ are the unknown vectors to be determined. It follows from the condition $u(t) \in \overset{\circ}{W}{}_2^3 (R_+; H; K_0)$ that

$$\overline{u}_1 (0) + x_1 + x_2 = K_0 u, \tag{10}$$

$$\overline{u}_1' (0) + \omega_1 A x_1 + \omega_2 A x_2 = 0,$$

$$x_2 = -\frac{1}{\omega_2} \left(\omega_1 x_1 + A^{-1} \overline{u}_1' (0) \right), \tag{11}$$

Taking this expression into account in (10), we get

$$\overline{u}_1 (0) + \frac{1}{\omega_2} K_0 \left(e^{\omega_2 t A} A^{-1} \overline{u}_1' (0) \right) -$$

$$-\frac{1}{\omega_2} A^{-1} \overline{u}_1' (0) - K_0 \overline{u}_1 (t) = \frac{\omega_2 - \omega_1}{\omega_2} (\Re - E) x_1,$$

i.e.

$$\frac{\omega_1 - \omega_2}{\omega_2} (E - \Re) x_1 = \overline{u}_1 (0) + \frac{1}{\omega_2} K_0 \left(e^{\omega_2 t A} A^{-1} \overline{u}_1' (0) \right) -$$

$$-\frac{1}{\omega_2} A^{-1} \overline{u}_1' (0) - K_0 \overline{u}_1 (t) .$$

Hence,

$$(E - \Re) x_1 = \psi,$$

where $\Re$ is determined from (3), and

$$\psi = \frac{\omega_2}{\omega_1 - \omega_2} \left[\overline{u}_1 (0) + \frac{1}{\omega_2} A^{-1} K_0 \left(\overline{u}_1' (0) e^{\omega_2 t A} \right) - \right.$$

$$\left. -\frac{1}{\omega_2} A^{-1} \overline{u}_1' (0) - K_0 \overline{u}_1 (t) \right] \in H_{5/2}.$$

As we showed, $\|\Re\|_{H_{5/2} \to H_{5/2}} < 1$, therefore $x_1 = (E - \Re)^{-1} \Psi \in H_{5/2}$. Now, we can find the vector

$$x_2 = -\frac{1}{\omega_2} \left(\omega_1 x_1 + A^{-1} \overline{u}_1' (0) \right) \in H_{5/2}.$$

Thus, $u \in W_2^3 (R_+; H; K_0)$ and $P_0 u = f$. But on the other hand,

$$\|P_0 u\|_{L_2(R_+;H)}^2 = \left\| P_0 \left(\frac{d}{dt} \right) u \right\|_{L_2(R_+;H)}^2 =$$

$$= \left\| \frac{d^3 u}{dt^3} - A^3 u \right\|_{L_2(R_+;H)}^2 \leq 2 \|u\|_{W_2^3(R_+;H)}^2 .$$

Therefore, by the Banach theorem there exists the inverse operator P_0^{-1} and it is bounded. Hence it follows that $\|u\|_{W_2^3(R_+;H)} \leq const \|f\|_{L_2(R_+;H)}$. The theorem is proved.

Lemma 2. *Let A be a positive-definite self-adjoint operator, condition 2) be fulfilled and $k_0 < \frac{3^{1/4}}{2^{5/3}}$. Then for any $u \in W_2^3\left(R_+;H;K_0\right)$ it holds the inequality*

$$\|P_0u\|^2_{L_2(R_+;H)} \geq \left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)\|u\|^2_{W_2^3(R_+;H)}. \tag{12}$$

Proof. Let $u \in W_2^3\left(R_+;H;K_0\right)$. Then

$$\|P_0u\|^2_{L_2(R_+;H)} = \left\|-u'''A^3u\right\|^2_{L_2(R_+;H)} =$$

$$= \left\|u'''\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)} - 2\operatorname{Re}\left(u''',A^3u\right)_{L_2(R_+;H)}.$$

On the other hand,

$$\left(u''',A^3u\right)_{L_2(R_+;H)} = \int_0^\infty \left(u''',A^3u\right)dt = -\left(A^{1/2}u''(0),A^{5/2}u(0)\right)-$$

$$-\left(A^{3/2}u'(0),A^{3/2}u'(0)\right)-$$

$$-\left(A^{5/2}u(0),A^{1/2}u''(0)\right)-\int_0^\infty\left(A^3u,u'''\right)dt.$$

Taking into account the boundary condition $u'(0)=0$, we get

$$2\operatorname{Re}\left(u''',A^3u\right)_{L_2(R_+;H)} = -2\operatorname{Re}\left(A^{5/2}u(0),A^{1/2}u''(0)\right) =$$

$$= -2\operatorname{Re}\left(A^{5/2}K_0u,A^{1/2}u''(0)\right).$$

Consequently,

$$\|P_0u\|^2_{L_2(R_+;H)} = \|u\|^2_{L_2(R_+;H)} + 2\operatorname{Re}\left(A^{5/2}K_0u,A^{1/2}u''(0)\right) \geq$$

$$\geq \|u\|^2_{W_2^3(R_+;H)} - 2k_0\|u\|_{W_2^3(R_+;H)}\left\|A^{1/2}u''(0)\right\|_H. \tag{13}$$

Thus, we should estimate $\left\|A^{1/2}u''(0)\right\|_H$. Since

$$\left\|A^{1/2}u''(0)\right\|^2_H = -2\operatorname{Re}\int_0^\infty\left(\frac{d^3u}{dt^3},A\frac{d^2u}{dt^2}\right)dt \leq$$

$$\leq 2\left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}\cdot\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}, \tag{14}$$

we estimate the norm $\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}$. It is obvious that for $u \in \overset{\circ}{W}{}_2^3\left(R_+;H;K_0\right)$ $(u'(0)=0)$

$$\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} = \int_0^\infty\left(A\frac{d^2u}{dt^2},A\frac{d^2u}{dt^2}\right) \leq$$

$$\leq \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)} \cdot \left\| A^2 \frac{du}{dt} \right\|_{L_2(R_+;H)} . \qquad (15)$$

Similarly we have

$$\left\| A^2 \frac{du}{dt} \right\|_{L_2(R_+;H)}^2 \leq \left\| A^3 u \right\|_{L_2(R_+;H)} \cdot \left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} . \qquad (16)$$

Taking into account (16) in (15), we have

$$\left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)}^2 \leq \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)} \cdot \left\| A^3 \frac{du}{dt} \right\|_{L_2(R_+;H)}^{1/2} \cdot \left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)}^{1/2} ,$$

or

$$\left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)}^{3/2} \leq \left\| \frac{d^3u}{dt^3} \right\| \cdot \left\| A^3 u \right\|_{L_2(R_+;H)}^{1/2} .$$

Hence we have

$$\left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} \leq \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^{2/3} \cdot \left\| A^3 u \right\|_{L_2(R_+;H)}^{1/3} ,$$

or for any $\delta > 0$

$$\left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)}^2 \leq \left(\left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 \right)^{2/3} \cdot \left(\left\| A^3 u \right\|_{L_2(R_+;H)}^2 \right)^{1/3} =$$

$$= \left(\delta \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 \right)^{2/3} \cdot \left(\frac{1}{\delta^2} \left\| A^3 u \right\|_{L_2(R_+;H)}^2 \right)^{1/3} \leq$$

$$\leq \frac{2}{3} \delta \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 + \frac{1}{3\delta^2} \left\| A^3 u \right\|_{L_2(R_+;H)}^2 .$$

Choose δ so that $\frac{2}{3}\delta = \frac{1}{3\delta^2}$, whence $\delta = \frac{1}{\sqrt[3]{2}}$. Then we get that

$$\left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} \leq \frac{2^{1/3}}{3^{1/2}} \cdot \left\| u \right\|_{W_2^3(R_+;H)} . \qquad (17)$$

Similarly we find

$$\left\| A^2 \frac{du}{dt} \right\|_{L_2(R_+;H)} \leq \frac{2^{1/3}}{3^{1/2}} \cdot \left\| u \right\|_{W_2^3(R_+;H)} . \qquad (18)$$

Taking into account (17) in (14), we have

$$\left\| A^{1/2} u''(0) \right\|_H^2 \leq 2 \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)} \cdot \left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} \leq$$

$$\leq 2 \left\| u \right\|_{W_2^3(R_+;H)} \cdot \frac{2^{1/3}}{3^{1/2}} \left\| u \right\|_{W_2^3(R_+;H)} = \frac{2^{4/3}}{3^{1/2}} \left\| u \right\|_{W_2^3(R_+;H)}^2 ,$$

or $\left\|A^{1/2}u''(0)\right\| \leq \frac{2^{2/3}}{3^{1/4}}\|u\|_{W_2^3(R_+;H)}$.

Then from (13) we have

$$\|P_0u\|^2_{L_2(R_+;H)} \geq \left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)\|u\|^2_{W_2^3(R;H)}.$$

The lemma is proved.

Theorem 2. *Let the conditions of lemma 2 be fulfilled. Then for any* $u \in \overset{\circ}{W}{}_2^3(R_+;H;K)$ *there hold the inequalities*

$$\left\|A^3u\right\|_{L_2(R_+;H)} \leq C_0(k_0)\|P_0u\|_{W_2^3(R_+;H)}, \tag{19}$$

$$\left\|A^2u'\right\|_{L_2(R_+;H)} \leq C_1(k_0)\|P_0u\|_{W_2^3(R_+;H)}, \tag{20}$$

$$\left\|Au''\right\|_{L_2(R_+;H)} \leq C_2(k_0)\|P_0u\|_{W_2^3(R_+;H)}, \tag{21}$$

where $C_0(k_0)=\left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1/2}$, $C_1(k_0)=\frac{2^{1/3}}{3^{1/2}}\cdot\left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1/2}$, $C_2(k_0)=\frac{2^{1/3}}{3^{1/2}}\cdot\left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1/2}$.

Proof. It follows from lemma 2 that

$$\|u\|^2_{W_2^3(R_+;H)} \leq \left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1}\|P_0u\|^2_{L_2(R_+;H)},$$

$$\left\|A^3u\right\|^2_{L_2(R_+;H)} \leq \left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1}\|P_0u\|^2_{L_2(R_+;H)},$$

or

$$\left\|A^3u\right\|_{L_2(R_+;H)} \leq \left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1/2}\|P_0u\|_{L_2(R_+;H)},$$

i.e. inequality (19) is true.

Prove the remaining inequalities. Let $u \in \overset{\circ}{W}{}_2^3(R_+;H;K)$. From (17) we have

$$\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} \leq \frac{2^{1/3}}{3^{1/2}}\cdot\|u\|_{W_2^3(R_+;H)} \leq$$

$$\leq \frac{2^{1/3}}{3^{1/2}}\cdot\left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1/2}\|P_0u\|_{L_2(R_+;H)}.$$

We proved validity of (21). Similarly, validity of inequality (20) follows from (18)

$$\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)} \leq \frac{2^{1/3}}{3^{1/2}}\cdot\|u\|_{W_2^3(R_+;H)} \leq$$

$$\leq \frac{2^{1/3}}{3^{1/2}}\cdot\left(1-\frac{2^{5/3}}{3^{1/4}}k_0\right)^{-1/2}\|P_0u\|_{L_2(R_+;H)}.$$

The theorem is proved.

Theorem 3. *Let A be a positive-definite self-adjoint operator, conditions 1), 2) be fulfilled, $k_0 < \frac{3^{1/4}}{2^{5/3}}$ and it holds the inequality*

$$\alpha(k_0) = \sum_{j=0}^{2} C_j(k_0)\|B_{3-j}\| < 1.$$

Then problem (1), (2) is regularly solvable.

Proof. By theorem 1, the operator $P_0 : \overset{\circ}{W}{}_2^3(R_+; H; K_0) \to L_2(R_+; H)$ is an isomorphism. Then there exists a bounded inverse operator P_0^{-1}. Write problem (1), (2) in the form of the equation $Pu = P_0u + P_1u = f$, where $f \in L_2(R_+; H)$, $u \in \overset{\circ}{W}{}_2^3(R_+; H; K_0)$. After substituting $P_0u = \upsilon$ we get the equation $\upsilon + P_1P_0^{-1}\upsilon = f$ in $L_2(R_+; H)$. But for any $\upsilon \in L_2(R_+; H)$ by theorem 2,

$$\left\|P_1P_0^{-1}\upsilon\right\|_{L_2(R_+;H)} = \|P_1u\|_{L_2(R_+;H)} = \left\|\sum_{j=0}^{2} A_{3-j}u^{(j)}\right\|_{L_2(R_+;H)} \leq$$

$$\leq \sum_{j=0}^{2}\|B_{3-j}\|\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)} \leq \sum_{j=0}^{2} C_j(k_0)\|B_{3-j}\| = \alpha(k_0) < 1$$

Thus, the operator $E + P_1P_0^{-1}$ is invertible in $L_2(R_+; H)$. Then $\upsilon = \left(E + P_1P_0^{-1}\right)^{-1} f$ and $u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1} f$. Hence it follows that

$$\|u\|_{W_2^3(R_+;H)} \leq const\,\|f\|_{L_2(R_+;H)}\,.$$

The theorem is proved.

Corollary 2. *Let $K_0 = 0$. Then while fulfilling the conditions of theorems 2,3 and*

$$\alpha(0) = \frac{2^{1/3}}{3^{1/2}}\left(\|B_1\| + \|B_2\|\right) + \|B_0\| < 1$$

problem (1), (2) is regularly solvable.

For $K_0 = 0$, the results of the paper (3) and the results of [4,5] follow from theorem 3 if we accept the discontinuous coefficient in the equations for unit.

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