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ON AN INVERSE PROBLEM FOR A REACTION-DIFFUSION TYPE SYSTEM

Abstract

The matters of well-posedness and approximate solution of an inverse problem on definition of time dependent coefficients in the right hand side of equations of a reaction-diffision type system are studied in the paper. A theorem on uniqueness and stability of the solution is proved.

Accept the following denotation: R^n is a real $n-$dimensional Euclidean space, $B \subset R^n$ is a bounded domain with boundary $\partial B \in C^{2+\alpha}$, $\Omega = B \times (0,T]$, $S(B) = \partial B \times [0,T]$, $T > 0$. The spaces $C^l(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0,1,2$, $0 < \alpha < 1$ and the norms in these spaces are determined for example in [1, p. 16], $u = (u_1, ..., u_m)$, $\|u\|_l = \sum\limits_{k=1}^{m} \|u_k\|_{C^l}$.

$u_{kt} = \dfrac{\partial u_k}{\partial t}$, $u_{kx_j} = \dfrac{\partial u_k}{\partial x_j}$, $\dfrac{\partial u_k}{\partial \overrightarrow{v}} = \sum\limits_{j=1}^{n} \dfrac{\partial u_k}{\partial x_j} \cos(\overrightarrow{v},\ x_j)$, $\overrightarrow{v}$ is a unit vector of the inner normal to S at its any point, $\dfrac{\partial}{\partial \overrightarrow{v}}$ means differentiation along $\overrightarrow{v}$, $\Delta u_k = \sum\limits_{i=1}^{n} \dfrac{\partial^2 u_k}{\partial x_i^2}$.

Consider a problem on definition of $\left\{ f_k(t), u_k(x,t),\ k = \overline{1,m} \right\}$ from the conditions

$$u_{kt} - \Delta u_k = f_k(t)\, g_k(x,t,u), \quad (x,t) \in \Omega, \tag{1}$$

$$u_k(x,0) = \varphi_k(x), \quad x \in \overline{B}; \quad \frac{\partial u_k}{\partial \overrightarrow{v}} = \psi_k(x,t), \quad (x,t) \in S(B) \tag{2}$$

$$\int\limits_B u_k(x,t)\, dx = r_k(t),\ t \in [0,T]. \tag{3}$$

here $g_k(x,t,v)$, $\varphi_k(x)$, $\psi_k(x,t)$, $r_k(t)$, $k = \overline{1,m}$ are the given functions.

Such problems, as a rule, are ill-pased in Hadamard sense and were studied in the papers [3-5].

Naturally, if a part of the function $f_k(t)$ is known, the appropriate additional conditions from (3) are unnecessary and are not given.

For the input data of problem (1)-(3) we make the following suppositions:

1^0. $g_k(x,t,v) \in C_{x,t}^{\alpha,\alpha/2}(A)$; $g_k(x,t,v)$ is continuous by Lipschits in variable v, is uniform with respect to (x,t,v) in bounded sets A, i.e.

$$\left| g_k\left(x,t,v^1\right) - g_k\left(x,t,v^2\right) \right| \le \sigma_1 \left| v^1 - v^2 \right|, \quad \left(x,t,v^1\right),\ \left(x,t,v^2\right) \in A;$$

where $A = \overline{B} \times [0,T] \times R^m$.

2^0.$\varphi_k(x) \in C^{1+\alpha}\left(\overline{B}\right)$, $\varphi_k(x,t) \in C^{1+\alpha,\alpha/2}(S(B))$

3^0. $r_k(t) \in C^{1+\alpha}[0,T]$, $t \in [0,T]$;

Definition 1. *The functions* $\{f_k(t),\ u_k(x,t),\ k=\overline{1,m}\}$ *is called the solution of problem (1)-(3) if:*

1) $f_k(t)\in C[0,T]$;

2) $u_k(x,t)\in C^{2,1}\left(\overline{B}\times[0,T]\right)$;

3) relations (1)-(3) are fulfilled for them.

The uniqueness theorem and also the estimation of stability of the solution to inverse problems occupies a cental place in investigation of their well-posedness matters. Here, under the most general assumptions, we prove the uniqueness of the solution to problem (1)-(3) and establish the estimation defining the solution's stability.

Theorem 1. *Let:*

1) Conditions $1^0,2^0,3^0$; *be fulfilled*

2) There exist a solution of problem (1)-(3) belonging to the set

$$K^{\alpha}=\left\{\left(f_k,u_k,\ k=\overline{1,m}\right)\Big|\ f_k(t)\in C^{\alpha}[0,T],u_k(x,t)\in C^{2+\alpha,1+\alpha/2}\left(\overline{B}\times[0,T]\right)\right\}$$

ining the solution's stability.

Then on the set K^{α}, *the solution of problem (1)-(3) is unique and the following stability estimation is true:*

$$\|u-\overline{u}\|_0+\left\|f-\overline{f}\right\|_0\le M\left[\|g-\overline{g}\|_0+\|\varphi-\overline{\varphi}\|_0+\left\|\psi-\overline{\psi}\right\|_0+\|r-\overline{r}\|_1\right], \qquad (4)$$

where $M>0$ *depends on the data of problem (1)-(3) and the set* K^{α}, $\{\overline{f}_k(t),\ \overline{u}_k(x,t),\ \ k=\overline{1,m}\}$ *is a solution of problem (1)-(3) from the set* K^{α} *with data* $\overline{g}_k(\cdot),\overline{\varphi}_k(\cdot),\overline{\psi}_k(\cdot),\overline{r}_k(\cdot)$ *that satisfy conditions* $1^0,2^0,3^0$ *respectively.*

Proof. Denote

$$z_k(x,t)=u_k(x,t)-\overline{u}_k(x,t),\ \lambda_k(t)=f_k(t)-\overline{f}_k(t),$$

$$\delta_{1k}(x,t,v)=g_k(x,t,v)-\overline{g_k}(x,t,v),$$

$$\delta_{2k}=\varphi_k(x)-\overline{\varphi}_k(x),\quad \delta_{3k}(x,t)=\psi_k(x,t)-\overline{\psi}_k(x,t),\quad \delta_{4k}=r_k(t)-\overline{r}_k(t).$$

We can verify that the functions $\{\lambda_k(t),z_k(x,t),\ k=\overline{1,m}\}$ satisfy the relations of the system

$$z_{kt}-\Delta z_k=\lambda_k(t)\,g_k(x,t,u)+F_k(x,t,u),\quad (x,t)\in Q, \qquad (6)$$

$$z_k(x,0)=\delta_{2k}(x),\quad x\in\overline{B};\quad \frac{\partial z_k}{\partial v}=\ \delta_{3k}(x,t),\quad (x,t)\in S, \qquad (7)$$

$$\lambda_k(t)=\left[\int\limits_{\partial B}\overline{\psi}_k(x,t)\,dx-\overline{r}_{kt}(t)\right]\int\limits_{B}[\overline{g}_k(x,t,u)-$$

$$-\overline{g}_k(x,t,\overline{u})]\,dx\backslash\left[\int\limits_{B}g_k(x,t,u)\,dx\int\limits_{B}\overline{g}_k(x,t,\overline{u})\,dx\right]+H_k(t),\quad t\in[0,T], \qquad (8)$$

where

$$F_k(x,t,u)=\overline{f}_k(t)\left[\delta_{1k}(x,t,u)+\overline{g}_k(x,t,u)-\overline{g}_k(x,t,\overline{u})\right],$$

$$H_k(t)=\left\{\left[\delta_{4kt}(t)-\int\limits_{\partial B}\delta_{3k}(x,t)\,dx\right]\int\limits_B\overline{g}_k(x,t,\overline{u})\,dx+\right.$$

$$\left.+\left[\int\limits_{\partial B}\overline{\psi}_k(x,t)\,dx-\overline{r}_{kt}(t)\right]\int\limits_B\delta_{1k}(x,t,u)\,dx\right\}\setminus\left[\int\limits_B g_k(x,t,u)\,dx\int\limits_B\overline{g}_k(x,t,\overline{u})\,dx\right].$$

It follows from the conditions of the theorem that the right hand side of equation (6) satisfies the Hölder condition. So, there exists a classic solution of problem (6)-(7) on definition of $z_k(x,t)$ and may be represented in the form [2, p. 182]:

$$z_k(x,t)=\int\limits_0^t\int\limits_{\partial B}\Gamma_k(x,t;\xi,\tau)\,P_k(\xi,t)\,d\xi d\tau+\int\limits_B\Gamma_k(x,t;\xi,0)\,\delta_{2k}(\xi)\,d\xi-$$

$$-\int\limits_0^t\int\limits_{\partial B}\Gamma_k(x,t;\xi,\tau)\left[\lambda_k(\tau)\,g_k(\xi,t,u)+F_k(\xi,t,u)\right]d\xi d\tau, \qquad (9)$$

where $P_k(x,t)$ is a solution of the integral equation

$$P_k(x,t)=2\int\limits_0^t\int\limits_{\partial B}\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}P_k(\xi,\tau)\,d\xi d\tau+2\int\limits_B\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\delta_{2k}(\xi)\,d\xi-$$

$$-2\int\limits_0^t\int\limits_B\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\left[\lambda_k(\tau)\,g_k(\xi,t,u)+F_k(\xi,t,u)\right]d\xi d\tau-2\delta_{3k}(x,t), \qquad (10)$$

here $\xi=(\xi_1,...,\xi_n)$, $d\xi=d\xi_1...d\xi_n$. $\Gamma_k(\cdot)$ are fundamental solutions of equation (6) for which the following estimations [1, p. 427, 444] are valid:

$$\left|D_x^l\Gamma_k(x,t;\xi,\tau)\right|\le c_1(t-\tau)^{-\frac{n+1}{2}}\exp\left(-c_2\frac{|x-\xi|^2}{t-\xi}\right),$$

$$\left|\int\limits_{R_n}\Gamma_k(x,t;\xi,\tau)\,d\xi\right|\le c_3\ , \qquad (11)$$

$$\left|\int\limits_{R_n}D_x^l\Gamma_k(x,t;\xi,\tau)\,d\xi\right|\le c_4(t-\tau)^{-\frac{l-\alpha}{2}}\ ,\quad l=0,1,2$$

where $c_1,c_2,c_3,c_4>0$ are positive constants. Assume

$$\chi=\|u-\overline{u}\|_0+\left\|f-\overline{f}\right\|_0$$

Estimate the function $z_k(x,t)$, $k=\overline{1,m}$. It follows from (9) that

$$|z_k(x,t)|\le\int\limits_0^t\int\limits_{\partial B}|\Gamma_k(x,t;\xi,\tau)|\,|P_k(\xi,t)|\,d\xi d\tau+\int\limits_B|\Gamma_k(x,t;\xi,\tau)|\,|\delta_{2k}(\xi)|\,d\xi+$$

$$+\int_0^t\int_B \left|\Gamma_k\left(x,t;\xi,\tau\right)\right|\left[\left|\lambda_k\left(\tau\right)g_k\left(\xi,t,u\right)\right|+\left|F_k\left(\xi,\tau,u\right)\right|\right]d\xi d\tau, \tag{12}$$

For $|P_k(\xi,t)|$ from (10) we get

$$|P_k(x,t)|\le 2\int_0^t\int_{\partial B}\left|\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\right||P_k(\xi,\tau)|\,d\xi d\tau+2\int_B\left|\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\right||\delta_{2k}(\xi)|\,d\xi+$$

$$+2\int_0^t\int_{\partial B}\left|\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\right|\left[|\lambda_k(\tau)|\,|g_k(\xi,t,u)|+|F_k(\xi,t,u)|\right]d\xi d\tau-2\,|\delta_{3k}(x,t)|\,, \tag{13}$$

At first we estimate $|P_k(x,t)|$. Considering estimation (11), we get

$$\int_0^t d\tau\int_{\partial B}\left|\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\right|d\xi\le c_4\int_0^t(t-\tau)^{-\frac{1-\alpha}{2}}\,d\tau=c_5t^{(1+\alpha)/2}, \tag{14}$$

$$\int_0^t d\tau\int_{\partial B}\left|\frac{\partial\Gamma_k(x,t;\xi,\tau)}{\partial v}\right|d\xi\le c_5t^{(1+\alpha)/2}, \tag{15}$$

where $c_5>0$ depends on the problem's data.

By the requirements imposed on input data and on the set K^α, the integrand function $|\lambda_k(t)\,g_k(x,t,u)|$ in the third term of the right hand side of (13) satisfies the estimation

$$|\lambda_k(t)\,g_k(x,t,u)|\le c_6\left[\left\|f-\overline{f}\right\|_0\right],\quad (x,t)\in\overline{\Omega} \tag{16}$$

where $c_6>0$ depends on the problem's data.

By the conditions of the theorem, for the integrand function $|F_k(x,t,u)|$ we get

$$|F_k(x,t,u)|\le c_7\left[\|g-\overline{g}\|_0+\|u-\overline{u}\|_0\right],\quad (x,t)\in\overline{\Omega} \tag{17}$$

Considering the estimations $(14),(15),(16),(17)$, from (13) we get

$$|P_k(x,t)|\le c_8\left[\|g-\overline{g}\|_0+\|\varphi-\overline{\varphi}\|_0+\left\|\psi-\overline{\psi}\right\|_0\right]+$$

$$+c_9t^{(1+\alpha)/2}\,\|P\|_0+c_{10}t^{(1+\alpha)/2}\chi,\ (x,t)\in\overline{\Omega}, \tag{18}$$

where $c_7,c_8,c_9,c_{10}>0$ depend on the problem's data and the set K^α.

Inequality (18) is satisfied for all $(x,t)\in\overline{\Omega}$. It should be satisfied also for the maximal values of the left hand side.

Consequently,

$$\|P\|_0\le c_8\left[\|g-\overline{g}\|_0+\|\varphi-\overline{\varphi}\|_0+\left\|\psi-\overline{\psi}\right\|_0\right]+c_9t^{(1+\alpha)/2}\,\|P\|_0+c_{10}t^{(1+\alpha)/2}\chi$$

Let $T_1\quad(0<T_1\le T)$ be such a number that $c_9t^{(1+\alpha)/2}<1$. Then, from the last inequality we get:

$$\|P\|_0\le c_{11}\left[\|g-\overline{g}\|_0+\|\varphi-\overline{\varphi}\|_0+\left\|\psi-\overline{\psi}\right\|_0\right]+c_{12}t^{(1+\alpha)/2}\chi, \tag{19}$$

where $c_{11}, c_{12} > 0$ depend in the problem's data and the set K^α.

Similarly, for $|z_k(x,t)|$ from (12) we get:

$$|z_k(x,t)| \le c_{13}\left[\|g-\overline{g}\|_0 + \|\varphi-\overline{\varphi}\|_0 + \left\|\psi-\overline{\psi}\right\|_0\right] + c_{14}t^{(1+\alpha)/2}\chi, \tag{20}$$

where $c_{13}, c_{14} > 0$ depend on the problem's data and the set K^α.

Now, estimate the function $|\lambda_k(t)|$. It follows from (8) that

$$|\lambda_k(t)| \le \left[\int\limits_{\partial B}\left|\overline{\psi}_k(x,t)\right|dx + |\overline{r}_{kt}(t)|\right]\int\limits_{\partial B}|\overline{g}_k(x,t,u)-\overline{g}_k(x,t,\overline{u})\,dx| \setminus$$

$$\setminus\left[\int\limits_B |g_k(x,t,u)|\,dx\int\limits_B |\overline{g}_k(x,t,\overline{u})|\,dx\right] + |H_k(t)|$$

Considering the theorem's conditions, definition of the set K^α, inequality (20), behaving as in deriving inequalities (19) and (20), from the last inequality we get:

$$|\lambda_k(x,t)| \le c_{15}\left[\|g-\overline{g}\|_0 + \|\varphi-\overline{\varphi}\|_0 + \left\|\psi-\overline{\psi}\right\|_0 + \|r-\overline{r}\|_1\right] +$$

$$+c_{16}t^{(1+\alpha)/2}\chi, \quad t\in[0,T], \tag{21}$$

where $c_{15}, c_{16} > 0$ depend on the problem's data and the set K^α.

Inequalities (20) and (21) are satisfied for any values of $(x,t)\in\overline{\Omega}$. Therefore, they should be satisfied also for maximal values of the left hand sides. Consequently, combining these inequalities, we get

$$\chi \le c_{17}\left[\|g-\overline{g}\|_0 + \|\varphi-\overline{\varphi}\|_0 + \left\|\psi-\overline{\psi}\right\|_0 + \|r-\overline{r}\|_1\right] + c_{18}t^{(1+\alpha)/2}\chi,$$

where $c_{17}, c_{18} > 0$ depend on the problem's data and the set K^α.

Let T_2 $(0 < T_2 \le T)$ be such a number that $c_{18}t^{(1+\alpha)/2} < 1$. Then we get that for $(x,t)\in\overline{B}\times[0,T_3]$, $T_3 = \min(T_1,T_2)$ the stability estimation (4) for the solution of problem (1)-(3) is true.

Uniqueness of the solution of problem (1)-(3) follows from estimation (4) for

$$g_k(x,t,u) = \overline{g}_k(x,t,u), \quad \varphi_k(x) = \overline{\varphi}_k(x), \ \psi_k(x,t) = \overline{\psi}_k(x,t), \ r_k(t) = \overline{r}_k(t)$$

The theorem is proved.

Remark. The similar problem was also considered for the exterior domain $Q = (R^n\setminus B)\times(0,T]$ for which appropriate results were obtained.

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