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n_z-PLANE, n_z-ANALYTIC FUNCTIONS

Abstract

The notion of n_z-plane, n_z-derivative, n_z-analytic functions generalizing the appropriate notion of complex analysis are introduced. Some of their basic properties are reduced.

Necessity to solve some elliptic systems by functional-analytic methods reduced to generalization of classic notion of complex numbers, analytic functions and etc. in the appropriate direction. Apparently, these ideas take its origin from the papers of I.N.Vekua [1], L.Bers [2], A.Douglis [3]. Aterwards, these ideas were developed in the papers of A.V.Bitsadze [4], I.N.Vekua [5], G.N.Nile [6] and in several papers of A.P.Soldatov [7-10]. The authors of the papers [11-14] successfully used the similar method for solving Sobolev type non-classic differential equations. Desire for writing the solutions of such elliptic systems in the form of series or applying the Fourier method to the system of equations containing time derivatives in addition to the elliptic part, compels to study basis properties of special systems that is a generalization of the classic system of exponents in appropriate spaces.

In this paper, the notion of n_z - plane, n_z -monogeneity and etc. and the Cauchy theory is taken for this case. The notion of the system from n_z-exponents is introduced and its basic properties are studied.

1. n_z**- plane**. n_z-derivative. Accept the following standard denotation. N are natural numbers; Z are integers; C is a complex plane; $L(X)$ is an algebra of bounded operators acting from the Banach space X to X; n is a class of Nilpotent operators from $L(X)$.

Take a pair of nilpotent operators $(n_1; n_2) \in n \times n$. To each complex number $z = x + iy \in C$ associate an operator $n_z = zI + xn_1 + iyn_2$, where $I \in L(X)$ is a unit operator. It is obvious that $n_{z_1} + n_{z_2} = n_{z_1+z_2}$, $n_0 = 0 \in L(X)$ is a zero operator. The mapping $n : C \to L(X)$ is not holomorphic.

It follows directly from $n_z = z\left(I + \frac{xn_1+iyn_2}{z}\right)$ that for $z \neq 0$ the operator n_z is invertible. The set $\{n_2 : z \in C\}$ is said to be n_z -plane.

2. Let $f : C \supset D_f \to X$ be some function, where $z_0 \in D_f$ with some vicinity. f is said be n_z - differentiable in Z_0 if there exists a limit in X

$$\lim_{\Delta z \to 0} n_{\Delta z}^{-1} \Delta f, \text{ where } \Delta z = z - z_0,\ \Delta f = f(z) - f(z_0).$$

This limit will be called n_z - derivative of the function f and denote it by $\frac{df}{dn_z}$;

$$\frac{df}{dn_z} = \lim_{\Delta z \to 0} n_{\Delta z}^{-1} \Delta f. \tag{1}$$

It is easy to see that $\frac{d}{dn_z}$ possesses the properties:

$$\frac{d(\lambda f)}{dn_z} = \lambda \frac{df}{dn_z}; \quad \frac{d(f_1+f_2)}{dn_z} = \frac{df_1}{dn_z} + \frac{df_2}{dn_z} \tag{2}$$

3. Let $f : C \supset D_f \to L(X)$ be some mapping and $z_0 \in D_f$. The derivative $\frac{df}{dn_z}$ is similarly determined by formula (1), where naturally, the convergence at this time is understood in $L(X)$. In addition to (2), the $\frac{df}{dn_z}$ possesses the property

$$\frac{d(f_1 f_2)}{dn_z} = \frac{df_1}{dn_z} f_2 + f_1 \frac{df_2}{dn_z}.$$

If limit (1) exists at the point $z = z_0$, the appropriate mapping will be called monogeneous in z_0. The monogeneous mapping in some vicinity of the point $z = z_0$ is said to be analytic in z_0.

Let

$$n_{\Delta z} = \Delta z I + \Delta x n_1 + i \Delta y n_2.$$

Assuming $\Delta y = 0$, we have

$$n_{\Delta z} = \Delta x I + \Delta x n_1$$

$$n_{\Delta z}^{-1} = (\Delta x I + \Delta x n_1)^{-1} = (I + n_1)^{-1} (\Delta x)^{-1}$$

Consequently,

$$\frac{df}{dn_z} = (I + n_1)^{-1} \frac{\partial f}{\partial x}.$$

Similarly, for $\Delta x = 0$ we have:

$$n_{\Delta z} = i \Delta y I + i \Delta y n_2,$$

$$n_{\Delta z}^{-1} = -i (I + n_2)^{-1} (\Delta y)^{-1},$$

$$\frac{df}{dn_z} = -i (I + n_2)^{-1} \frac{\partial f}{\partial y}.$$

Thus,

$$(I + n_1)^{-1} \frac{df}{dx} = -i (I + n_2)^{-1} \frac{\partial f}{\partial y}. \qquad (3)$$

So, if f is an n_z-differentiable mapping, it satisfies relation (3). Conversely, let (3) hold. The mapping f is said to be n_z -differentiable if

$$\Delta f = \Delta x Z + \Delta y B + \overline{\overline{o}} (n_{\Delta z}),$$

where $A; B$ are the mappings independent of Δx, Δy mapping and $\left\| n_{\Delta z}^{-1} \overline{\overline{o}} (n_{\Delta z}) \right\| \to 0$ as Δx, $\Delta y \to 0$. If is n_z - differentiable, then it is clear that $\frac{\partial f}{\partial x} = A$; $\frac{\partial f}{\partial y} = B$. Consequently

$$(I + n_1)^{-1} A = -i (I + n_2)^{-1} B.$$

We have

$$n_{\Delta z} = \Delta x (I + n_1) + i \Delta y (I + n_2).$$

$$(I + n_2) A = -i (I + n_1) B,$$

$$(I + n_2) \Delta f = \Delta x (I + n_2) A + \Delta y (I + n_2) B + (I + n_2) \overline{\overline{o}} (\Delta x; \Delta y) =$$

$$= -i \Delta x (I + n_1) B + \Delta y (I + n_2) B + (I + n_2) \overline{\overline{o}} (\Delta x; \Delta y) =$$

$$= i [\Delta x (I + n_1) + i \Delta y (I + n_2)] B + (I + n_2) \overline{\overline{o}} (\Delta x; \Delta y) =$$

$$= -i n_{\Delta z} + (I + n_2) \overline{\overline{o}} (\Delta x; \Delta y).$$

Thus,

$$n_{\Delta z}^{-1} \Delta f = -(I + n_2)^{-1} B + n_{\Delta z}^{-1} \overline{\overline{o}} (\Delta x; \Delta y).$$

Hence, it directly follows that

$$\frac{df}{dn_2} = -i (I + n_2)^{-1} B = (I + n_1)^{-1} A. \qquad (4)$$

So, the following theorem is true.

Theorem 1. *If f is an n_z - monogeneous mapping in z, then (3) is valid. Conversely, if it is n_z - differentiable in z and (3) holds , then f is n_z - monogeneous mapping in z and (4) is valid.*

Now, let $\Gamma \subset C$ be a pieccewise-smooth curve and $\Gamma \subset D_f$ be a domain of definition of the mapping $f : D_f \to X$ (or $L(X)$). As usual, we divide Γ into n parts by the points $\{z_k\}_0^n \subset \Gamma$ and consider

$$S(\{z_k\}) = \sum_k n_{\Delta z_k}^{-1} f(\xi_k),$$

where $\xi_k \in \overset{\cup}{z_k z_{k+1}}$ is an arch of Γ. The limit (if it exists)

$$\int_\Gamma dn_z f(z) \overset{def}{\equiv} \lim_{\max_k |\Delta z_k| \to 0} S(\{z_k\}), \tag{5}$$

will be called an n_z - integral. Let $T \in L(X)$ be an operator commuting with the n_z- operator. Then it is clear that

$$\int_\Gamma dn_z (Tf(z)) = T \int_\Gamma dn_z f(z).$$

n_z -integral possesses all ordinary properties (additivity with respect to f and Γ).

Assume that Φ is n_z-differentiable and $\frac{d\Phi}{dn_z} = f$, i.e. $d\Phi = f dn_z$. Let Γ be a pieccewise-smooth curve with the ends a a and b. Take partitioning of Γ into parts by the points $\{z_k\}_0^n \subset \Gamma$.

We have

$$\Phi(b) - \Phi(a) = \Phi(z_n) - \Phi(z_0) =$$
$$= [\Phi(z_n) - \Phi(z_{n-1})] + [\Phi(z_{n-1}) - \Phi(z_{n-2})] + ... + [\Phi(z_1) - \Phi(z_0)] =$$
$$= \Delta\Phi(z_{n-1}) + ... + \Delta\Phi(z_0) = \sum_k \left[\Delta x_k A_k + \Delta y_k B_k + \overline{\overline{o}}(n_{\Delta z_k})\right]. \tag{6}$$

Assume that the expression $\left\| n_{\Delta z}^{-1} \overline{\overline{o}}(n_{\Delta z}) \right\|$ uniformly converges to zero on Γ we have

$$\Delta x A + \Delta y B = \Delta x (I + n_z)(I + n_1)^{-1} A + \Delta y (I + n_2)(I + n_2)^{-1} B =$$
$$= (?) = -i [\Delta x (I + n_1) + i\Delta y (I + n_2)] (I + n_2)^{-1} B =$$
$$= -i n_{\Delta z} (I + n_2)^{-1} B = n_{\Delta z} \frac{d\Phi}{dn_z} = n_{\Delta z} f(z).$$

Taking this into account this relation in (6), we get:

$$\Phi(b) - \Phi(a) = \sum_k n_{\Delta z_k} f(z_k) + \sum_k \overline{\overline{o}}(n_{\Delta z_k}).$$

Hence we directly get

$$\int_\Gamma dn_z f(z) = \Phi(b) - \Phi(a). \tag{7}$$

This is an analogy of Newton-Leibnits formula.

And we prove

Statement 1. Let $\Phi : C \supset \Gamma \to L(X)$ be continuosly-differentiable, $\frac{d\Phi}{dn_z} = f(z)$ and $n_{\Delta_z}^{-1} \overline{\overline{o}}(n_{\Delta z}) \rightrightarrows 0$ uniformly on Γ, where

$$\Delta\Phi = \Delta x A + \Delta y B + \overline{\overline{o}}(n_{\Delta z}),$$

then the Newton-Leibnitz n_z - formula (7) is valid. If Γ is closed and Φ is continuous on Γ, then it holds the Cauchy theorem:

$$\int_{\Gamma} dn_z f(z) = 0.$$

This formula follows directly from (7).

We can prove the Cauchy theorem under other assumptions. For that at first we establish analogy of the Green-Ostrogradskiy formula.

4. Let $n_1; n_2 \in L(X)$ be nilpotent operators. To each $(x; y) \in R^2$ we associate $(n_x; n_y) \in L(X) \times L(X)$, where $n_x = xI + xn_1$; $n_y = yI + yn_2$. We call a linear space $\left\{(n_x; n_y) : (x; y) \in R^2\right\}$ with coordinatewise linear operations R_n^2 - plane. $f : R^2 \to L(X)$ (or X) is called n-differentiable in $(x; y)$ if $\exists A; B \in L(X)$ (or X):

$$\Delta f = n_{\Delta x} A + n_{\Delta y} B + \overline{\overline{o}}(\rho), \text{ as } \rho \to 0,$$

where $\rho = \sqrt{\Delta x^2 + \Delta y^2}$. It is easy to see that $\rho \to 0$ is equivalent to $\sqrt{\|n_{\Delta x}\|^2 + \|n_{\Delta y}\|^2} \to 0$. Assume

$$df = dn_x A + dn_y B.$$

Similar to the classic case we define

$$\frac{\partial f}{\partial n_x} = A, \frac{\partial f}{\partial n_y} = B.$$

In the same way we determined the double integrals

$$\int\int_D dn_x dn_y f(x; y); \int\int_D dn_y dn_x f(x; y),$$

as limits of the sums

$$\sum_{i,j} n_{\Delta x_i} n_{\Delta y_j} f(\xi_i; \eta_j); \sum_{i,j} n_{\Delta y_i} n_{\Delta x_j} f(\xi_i; \eta_j).$$

We define in the same way the curvilinear integral

$$\int_L dn_x f(x; y) + dn_y g(x; y),$$

where $f; g : C \to L(X)$ (or X).

The proof of the Green formula of the classic case is suitable in this case as well, i.e. it is valid

Theorem 2. *Let $f;g : D \subset C \to L(X)$ (or X) be continuous in D and have continuous partial derivatives in D. If there exists the integral*

$$I = \iint_D dn_x dn_y \left(\frac{\partial g}{\partial n_x} - \frac{\partial f}{\partial n_y}\right),$$

it is valid Green's n-formula

$$\iint_D dn_x dn_y \left(\frac{\partial g}{\partial n_x} - \frac{\partial f}{\partial n_y}\right) = \oint_L dn_x f + dn_y g,$$

where $L = \partial D$.

Indeed, at first we see that the relations

$$dn_x = dx\,(I + n_1)\,; \quad \frac{\partial f}{\partial n_x} = (I + n_1)^{-1} \frac{\partial f}{\partial x}$$

are valid.

The similar relations are valid also with respect to the variable y. Assume $\widetilde{f}(x;y) = (I + n_1) f(x;y)$, $\widetilde{g}(x;y) = (I + n_2) g(x;y)$. It is easy to see that it holds

$$I = \iint_D \left(\frac{\partial \widetilde{g}}{\partial x} - \frac{\partial \widetilde{f}}{\partial y}\right) dxdy.$$

Take $\forall \vartheta \in (L(X))^*$ (or $\vartheta \in X^*$) and consider

$$J_\vartheta = \vartheta(I) = I = \iint_D \left(\frac{\partial g_\vartheta}{\partial n_x} - \frac{\partial f_\vartheta}{\partial n_y}\right) dxdy, \tag{8}$$

where $g_\vartheta = \vartheta(\widetilde{g})$, $f_\vartheta = \vartheta\left(\widetilde{f}\right)$. For the functions g_ϑ and f_ϑ all the conditions of the Green classic theorem are fulfilled. Therefore, the Green formula

$$J_\vartheta = \oint_L f_\vartheta dx + g_\vartheta dy = \vartheta\left(\oint_L \widetilde{f} dx + \widetilde{g} dy\right) \tag{9}$$

is valid. From (8) and (9) we get:

$$I = \oint_L \widetilde{f} dx + \widetilde{g} dy = \oint_L (I + n_1)\, dxf + (I + n_2)\, dyg = \oint_L dn_x f + dn_y g.$$

The theorem is proved

Now, assume that the expression $dn_x f + dn_y g$ is the exact differential in D, i.e. $\exists F : D \to L(X)$ (or X): $dF = dn_x f + dn_y g$ and so $\frac{\partial F}{\partial n_x} = f$, $\frac{\partial F}{\partial n_y} = g$. If F is continuous on L, it is clear that

$$\iint_D dn_x dn_y \left(\frac{\partial g}{\partial n_x} - \frac{\partial f}{\partial n_y}\right) = 0.$$

Now, let Φ be an n_z - analytic function in D. Consider

$$\oint_L dn_z f = \oint_L (dn_x + i dn_y)\, f, \tag{10}$$

since it is easy to see that $dn_z = dn_x + idn_y$. Applying to (10) the Green $n_{x,y}$-formula, we get

$$\oint_L dn_z f = \int\int_D dn_x dn_y \left(i\frac{\partial f}{\partial n_x} - \frac{\partial g}{\partial n_y} \right).$$

Hence, from (3)we directly get

$$\oint_L dn_z f = 0.$$

Thus, we proved

Theorem 3. *Let f be n_z- differentiable in $\overline{D}$. Then it holds the formula*

$$\oint_L dn_z f = 0.$$

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