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ON BASISITY OF SOME PERTURBED SYSTEM OF EXPONENTS IN L_p

Abstract

A perturbed system of exponents whose phase has asymptotic with piecewise principal part is considered in this paper. Basisity criterion of this system in the Lebesgue space L_p is established.

Consider the following system of exponents

$$\left\{e^{i\lambda_n(t)\,t}\right\}_{n\in\mathbb{Z}} \quad (\mathbb{Z} \text{ is integers set}) \tag{1}$$

where $\lambda_n(t)$ has the asymptotic

$$\lambda_n(t) = (n + \alpha\, sign\; n \cdot sign\, t)\; t + \underline{\underline{0}}\left(\frac{1}{|n|^\gamma}\right) \qquad n \to \infty\,, \tag{2}$$

$\alpha, \gamma \in \mathbb{R}$. Specific case of system (1) is $\left\{e^{i\lambda_n^0(t)}\right\}_{n\in Z}$, where

$$\lambda_n^0(t) \equiv \begin{cases} \lambda_n^+\, t, & 0 < t < \pi \\ \lambda_n^-\, t, & -\pi < t < 0 \end{cases},$$

$\{\lambda_n^\pm\}$ has appropriate asymptotics. The system $\left\{e^{i\lambda_n^0(t)t}\right\}_{n\in\mathbb{Z}}$ is a collection of eigen functions of the first order ordinary discontinuous differential operator $Lu = u'$ on $(-\pi, 0) \bigcup (0, \pi)$, that is understood in V.I. Il'in's sense [1]. Under the domain of definition of the operator L we understand a Cartesian product $W_p^1(-\pi, 0) \times W_p^1(0, \pi)$, $1 < p < +\infty$. By these and other reasons, it is urgent to study basis properties of such type of systems. Many papers beginning with Paley-Wiener basic result [2] have been devoted to these problems. Basicity problems of system (1) in $L_p(-\pi, \pi)$, $1 < p < +\infty$, in the case $\lambda_n(t) = n + \alpha sign\, n$, $n \in \mathbb{Z}$, have been completely studied in the papers [3,4]. Discontinuous but unperturbed case has been considered in the papers [5,6]. In this paper, we study basicity problems of system (1) in $L_p \equiv L_p(-\pi, \pi)$, $1 < p < +\infty$ when asymptotics (2) holds.

Necessary notion and facts. Further, we will need some notion and results from the theory of close basis. We will accept the following denotation.

$B-$ space, is a Banach space; X^* is a space conjugated to X; $f(x)$ is the value of the functional $f \in X^*$ in $x \in X$.

Definition 1. *The system $\{x_n\}_{n\in\mathbb{N}} \subset X$ in $B-$space X is said to be ω -linearly independent if from $\sum\limits_{n=1}^{\infty} a_n x_n = 0$ it follows that $a_n = 0$, $\forall\, n \in \mathbb{N}$.*

The following Lemma is valid.

Lemma 1. *Let X be a B-space with a basis $\{x_n\}_{n\in N} \subset X$, $F : X \to X$ be a Fredholm operator. Then the following properties of the system $\{y_n = Fx_n\}_{n\in N}$ in X are equivalent.*

1. $\{y_n\}_{n\in N}$ *is complete;*
2. $\{y_n\}_{n\in N}$ *is minimal;*
3. $\{y_n\}_{n\in N}$ *is ω-linearly independent ;*
4. $\{y_n\}_{n\in N}$ *is a basis isomorphic to* $\{x_n\}_{n\in N}$.

Accept following definitions.

Definition 2. *The systems $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in the $B-$space, X with the norm $\|\cdot\|$ are called p-close if $\sum\limits_n \|x_n - y_n\|^p < +\infty$.*

Definition 3. *The system $\{x_n\}_{n\in N}$ with conjugated $\{x_n^*\}_{n\in N} \subset X^*$ minimal in the B-space X is said to be p-system if for $\forall\ x \in X :\ \{x_n^*(x)\}_{n\in N} \in l_p$, where l_p is an ordinary space of sequences. In the case of basisity, such system will be called p-basis.*

The following lemma is also valid.

Lemma 2. *Let X be a B-space with q-basis $\{x_n\}_{n\in\mathbb{N}}$ and the system $\{y_n\}_{n\in\mathbb{N}} \subset X$ be p-close to it,where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < +\infty$. Then the expression $Fx = \sum\limits_n x_n^*(x)\, y_n$ generates a Fredholm operator in X.*

Concerning these and other results, we can consider for example the papers [7-10].

2. Basic results

Before to formulate the basic results we cite some arguments that we will need.

So, let X be a $B-$ space with a basis $\{x_n\}_{n\in N}$ and $\{x_n^*\}_{n\in N} \subset X^*$ be a conjugated system. Let $\{y_n\}_{n\in N} \subset X$ be a some what defective system.

Assume, that $\exists\, \chi \in N$ such that for $\forall\, m \ge \chi$ the system $\{x_k\}_1^m \bigcup \{y_k\}_{k\ge m+1}$ forms a basis in X. By $\{\vartheta_k\}_{k\in N} \subset X^*$ we denote a system conjugated to $\{x_k\}_1^\chi \cup \{y_k\}_{k\ge\chi+1}$. It is obvious that the following properties of the system $\{y_k\}_{k\in\mathbb{N}}$ are equivalent:

a) $\{y_k\}_{k\in N}$ is complete ;

b) $\{y_k\}_{k\in N}$ is minimal ;

c) $\{y_k\}_{k\in N}$ forms a basis.

Expand y_k, $k = \overline{1,m}$, by the basis $\{x_k\}_1^m \bigcup \{y_k\}_{k\ge m+1}$:

$$y_k = a_{k1}x_1 + ... + a_{km}x_m + y_k^m, \quad k = \overline{1,m}\,,$$

where y_k^m belongs to the closure of the linear system $\{y_k\}_{k\ge m+1}$and we denote this closure by Y^m.

Consequently, fulfilment of the condition det $(a_{ij})_{i,j=\overline{1,m}} \ne 0$ is equivalent to the properties a)-c).

Thus, the condition

$$\det\left(a_{ij}\right)_{i,j=\overline{1,m}} \neq 0, \tag{3}$$

is fulfilled for $\forall\, m \geq \chi$. By n_0 we denote $\min\{\,\chi : \text{satisfying (3)}\}$. If $n_0 = 1$, it is clear that the system $\{y_k\}_{k\in N}$ forms a basis. For $n_0 > 1$ the defect d of the system $\{y_k\}_{k\in N}$ satisfies $1 \leq d \leq n_0 - 1$. By Y_m we denote a linear span of the system $\{y_k\}_{k=1}^{m}$. It is clear that if the system $\{y_k\}_1^{\infty}$ forms a basis then $\dim Y_m = m$. For $\dim Y_m = m$ it forms a basis iff $Y_m \cap Y^m = 0$.

Now state the basic results. Consider the system (1).

Assume $\mu_n(t) = (n + \alpha\, sign n \cdot sign t)\, t,\ \ n \in Z$.

Let $\delta_n(t) = \lambda_n(t) - \mu_n(t)$.

Using the obvious relation $\left|e^{i\lambda_n(t)} - e^{i\mu_n(t)}\right| = \left|e^{i\delta_n(t)} - 1\right|$, and asymptotics (2), we can establish

$$\left|e^{i\lambda_n(t)} - e^{i\mu_n(t)}\right| = \left|\sum_{k=1}^{\infty} \frac{\delta_n^k}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|\delta_n|^k}{k} \leq c\,|n|^{\gamma}, \qquad \forall n \neq 0, \tag{4}$$

where c is some constant. From the results of the paper [5] we get that the system $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ forms a basis in $L_p(-\pi,\pi)$, $1 < p < +\infty$, for $\forall\alpha \in R$.

By the denotation of this paper we have $\beta(t) = -\alpha\,|t|$, $\alpha(t) = \alpha\,|t|$ and so $\theta(t) = \beta(t) - \alpha(t) = -2\alpha\,|t|$. $\theta(t)$ does not have discontinuity points in $(-\pi,\pi)$. Since $\theta(-\pi) - \theta(\pi) = 0$, then the required one directly follows from theorem 1 of the paper [5].

Then as it follows from the results of the paper [11], the system $\left\{e^{i\mu_n(t)}\right\}_{n\in\mathbb{Z}}$ is isomorphic in L_p to the classic system of exponents$\left\{e^{int}\right\}_{n\in\mathbb{Z}}$. Hence it follows that Hausdorff-Young type theorem is valid for the system $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ i.e. let $\{e_n\}_{n\in Z} \subset L_q$ be a space orthogonal to it $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

Let $f \in L_p$. By $\{f_n\}_{n\in\mathbb{Z}}$ we denote biorthogonal coefficients of the function f by the system $\left\{e^{i\mu_n(t)}\right\}_{n\in\mathbb{Z}}$: $f_n = \int_{-\pi}^{\pi} f(t)\,\overline{e_n(t)}dt$, $\forall n \in \mathbb{Z}$, where $\overline{(\cdot)}$ is a complex conjugation. The following Theorem is valid.

Theorem 1. *The following statements hold:*

1. $1 < p \leq 2$, $f \in L_p$. *Then* $\{f_n\}_{n\in\mathbb{Z}} \in l_q$, *moreover* $\left\|\{f_n\}_{n\in\mathbb{Z}}\right\|_{l_q} \leq M_p\,\|f\|_p$, *where* $\|.\|_{l_q}$ $\left(\|.\|_p\right)$ *is an ordinary norm in* l_q,*(in* $L_p(-\pi,\pi)$*),* M_p *is a constant dependent only on* p.

2. $p > 2$, $\{f_n\}_{n\in\mathbb{Z}} \in l_q$. *Then* $\exists f \in L_p$ *such that* $\{f_n\}_{n\in\mathbb{Z}}$ *are its biorthogonal coefficients by the system* $\left\{e^{i\mu_n(t)}\right\}_{n\in\mathbb{Z}}$, *moreover*

$$\|f\|_p \leq \tilde{M}_p \left\|\{f_n\}_{n\in Z}\right\|_{l_q},$$

$\tilde{M}_p$ *is a constant dependent only on* p.

Consider these different cases.

1) $1 < p \leq 2$, $\gamma > \frac{1}{p}$.

In this case, by Theorem 1, the system $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ forms a q-basis in L_p. It directly follows from estimation (4) that the systems $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ and (1) are p-close in L_p. Then by lemma 2, the expression

$$Ff = \sum_n f_n e^{i\lambda_n(t)}$$

generates a Fredholm operator $F : L_p \to L_p$. It is clear that

$$F[e^{i\mu_n(t)}] = e^{i\lambda_n(t)}, \ \ \forall n \in Z.$$

2) $2 < p < +\infty, \ \ \gamma > \frac{1}{q}$.

In this case, the systems (1) and $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ are q-close in L_p.

Taking into attention Theorem 1, we obtain

$$\left\|\{f_n\}_{n\in Z}\right\|_{l_q} \le \tilde{M}_p \left\|f\right\|_q \le C_p \left\|f\right\|_p, \ \ \forall f \in L_p.$$

Thus, the system $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ forms a p-basis in L_p. Similar to the previous case, we understand that the operator $F : L_p \to L_p$ determined by the expression $Ff = \sum\limits_{n\in Z} f_n e^{i\lambda_n(t)}$, is Fredholm and

$$F[e^{i\mu_n(t)}] = e^{i\lambda_n(t)}, \ \ \forall n \in \mathbb{Z}.$$

As a result, from lemma 1 we establish the following theorem.

Theorem 2. *Let asymptotics (2) hold and* $\gamma > \max\left\{\frac{1}{p}; \frac{1}{q}\right\}$. *Then in* L_p, $1 < p < +\infty$ *the following properties of system (1) are equivalent:*

1. *(1) is complete in* L_p*;*
2. *(1) is minimal in* L_p*;*
3. *(1) forms a basis in* L_p *isomorphic to the classic system of exponents* $\left\{e^{int}\right\}_{n\in Z}$.

Remark. When the principal part of asymptotics (2) has a linear function (i.e. in (2) there is no multiplier sign t) then basisity in L_p of system (1) takes another pattern. In this case, basisity holds not for arbitrary $\alpha \in R$. For basicity, a p dependent inequality type condition is imposed on α. Furthermore, in this case, using N. Levinson's one result [12], the completeness of system (1) in L_p under natural conditions $\lambda_i(t) \not\equiv \lambda_j(t)$, $i \ne j$, and at the same time the basisity of the same system in L_p is easily established. Our case is a more complicated one. In order to demonstrate what has been explained , we consider the following example.

Example. Consider the system of exponents

$$\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}, \tag{5}$$

where $\lambda_n(t) = nt, \ \ \forall n \ne 0$;

$$\lambda_o(t) \equiv \begin{cases} \lambda t, & 0 < t < \pi \\ -\mu t, & -\pi < t < 0 \end{cases},$$

where $\lambda;\mu \neq 0$. Expand the function $e^{i\lambda_0(t)}$ in the basis $\left\{e^{int}\right\}_{n\in Z}$:

$$e^{i\lambda_0(t)} = a_0 + \sum_{n\neq 0} a_n e^{int},$$

where $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda_0(t)}dt$.

From the above mentioned arguments we understand that for basicity of system (5) in L_p it is necessary and sufficient to fulfill the condition $a_0 \neq 0$.
We have

$$2\pi a_0 = \int_0^{\pi} e^{i\lambda t}dt + \int_{-\pi}^{0} e^{-i\mu t}dt = \frac{1}{i\lambda}\left(e^{i\lambda\pi} - 1\right) + \frac{1}{i\mu}\left(e^{i\mu\pi} - 1\right).$$

Assume $\omega(\lambda) = \frac{1}{\lambda}\left(e^{i\lambda\pi} - 1\right)$. Thus, for the basicity of system (5) in L_p it is necessary and sufficient that λ and μ satisfy the condition $\omega(\lambda) + \omega(\mu) \neq 0$. In principle, this is an analogy of Levinson's conditions.

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