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FOLD COMPLETENESS FOR DISCONTINUOUS BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY AND TRANSMISSIONS CONDITIONS

Abstract

In this paper we consider a discontinuous boundary value problem with spectral parameter in the boundary and transmission conditions. We found simply algebric conditions on the coefficients that guarantee the isomorphism, coercive solvability and two-fold completeness of eigen and associated functions of the considered problem

1. Introduction

The division of variables for mathematical physics equations with boundary conditions containing oblique derivative reduces to ordinary differential equations with boundary conditions containing a spectral parameter.

The investigation of fold completeness of eigen and associated functions originated from the work [1]. Later, the results of this work were extended by many mathematicians (a detailed bibliography may be found, for example, in [7] and [8]) on to more recent papers on this topic we should mention the works [4, 5, 7, 8, 9]. It is necessary note that many important results dealing with fold completeness of eigenfunctions and associated functions and its applications are found in the series of S.Y.Yakubov and Y.Yakubov works.

Basically, it has been investigated equations with continuous coefficients standing by the senior derivative.

In this paper, a boundary value problem is studied with a spectral parameter taking part both in the equation and boundary conditions and with discontinuous variable coefficients standing by the senior derivative of the equations. In this case, the transmission conditions at the point of discontinuity naturally appear. Moreover, the considered problem is not pure differential, but in the equation contains the abstract linear operator and also in the boundary and transmission conditions the abstract linear functionals. Therefore, our problem is covers a wide class of boundary value problems.

2. Statement of the problem

In the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$, the equation

$$L(\lambda)u = a(x)u''(x) + (Bu)(x) - \lambda^2 u(x) = 0 \tag{1}$$

is considered with the boundary conditions

$$L_1(\lambda)u=\sum_{k=0}^{1}\lambda^{1-k}\Big(\alpha_{1k}u^{(k)}(-1)+\beta_{1k}u^{(k)}(-0)+\sum_{p=1}^{n_{1k}}\eta_{1pk}u^{(k)}(x_{1pk})+T_{1k}u\Big)=0$$

$$L_2(\lambda)u=\sum_{k=0}^{1}\lambda^{1-k}\Big(\alpha_{2k}u^{(k)}(+0)+\beta_{2k}u^{(k)}(1)+$$

$$+\sum_{p=1}^{n_{1k}}\eta_{2pk}u^{(k)}(x_{2pk})+T_{2k}u\Big)=0 \qquad (2)$$

and with the transmission conditions

$$L_3(\lambda)u=\sum_{k=0}^{1}\lambda^{1-k}\left(\delta_{1k}u^{(k)}(-0)+\gamma_{1k}u^{(k)}(+0)+T_{3k}u\right)=0$$

$$L_4(\lambda)u=\sum_{k=0}^{1}\lambda^{1-k}\left(\delta_{2k}u^{(k)}(-0)+\gamma_{2k}u^{(k)}(+0)+T_{4k}u\right)=0 \qquad (3)$$

where $a(x)\neq 0$, the coefficients α_{ik}, β_{ik}, η_{ipk}, δ_{ik} and $\gamma_{ik}\in\mathbb{C}$ are complex numbers; $-1<x_{1pk}<0$, $0<x_{2pk}<1$ are intermediate points; B is an abstract linear operator and T_{yk} are general linear functionals.

Below, $W_2^m=W_2^m(-1,0)\oplus W_2^m(0,1)$ denotes the space of measurable functions belonging to the Sobolev spaces $W_2^m(-1,0)$ and $W_2^m(0,1)$ in $(-1,0)$ and $(0,1)$, respectively and $C[-1,0]\oplus C[0,1]$denotes the set of the functions $a(x)$, which is defined $[-1,0)\cup(0,1]$ on are continuous on $[-1,0)$ and $(0,1]$ and has a finite limit $a(\pm 0)=\lim_{x\to\pm 0}a(x)$.

3. Auxiliary results

In the paper [3] we established the following proposition for estimation the norms of solution to problem (1)-(3), which we use it for proving mean result in this paper.

Theorem 1. *Assume that the following conditions hold true:*

1. $a(x)\in C[-1,0]\oplus C[0,1]$; $a(-1)=a(x_{1pk})=a(-0)$, $a(+0)=a(x_{2pk})=a(1)$

2. $\left(\alpha_{10}\sqrt[2]{a(-0)}-\alpha_{11}\right)\left(\beta_{20}\sqrt[2]{a(+0)}+\beta_{21}\right)\neq 0$

3. $\left(\delta_{10}\sqrt[2]{a(-0)}-\delta_{11}\right)\left(\gamma_{20}\sqrt[2]{a(+0)}+\gamma_{21}\right)-$
$-\left(\delta_{20}\sqrt[2]{a(-0)}-\delta_{21}\right)\left(\gamma_{10}\sqrt[2]{a(+0)}+\gamma_{11}\right)\neq 0$

4. The linear operator B acts compactly from $W_2^2(-1,0)\oplus W_2^2(0,1)$ to $L_2(-1,0)\oplus L_2(0,1)$.

5. The linear functionals T_{yk} are continuous in $W_2^k(-1,0)\oplus W_2^k(0,1)$.

Then, for any $\varepsilon>0$, there exists $R_\varepsilon>0$ such that under all $\lambda\in G_\varepsilon$ for which $|\lambda|>R_\varepsilon$ the operator $\widetilde{L}(\lambda):u\to(L(\lambda)u,L_1(\lambda)u,...,L_4(\lambda)u)$ from $W_2^2(-1,0)\oplus W_2^2(0,1)$ to $L_2(-1,1)\oplus\mathbb{C}^4$. is an isomorphism under those λ for the solution of the problem

$$L(\lambda)u=f,\quad L_y(\lambda)u=g_y,\quad y=1,2,3,4$$

the following estimation takes place:

$$\sum_{k=0}^{2} |\lambda|^{2-k} \|u\|_{W_2^k} \le C(\varepsilon) \left(\|f\|_{L_2} + \sum_{y=10}^{4} |\lambda|^{1/2} |g_y| \right)$$

where $G_\varepsilon = \left\{ \lambda \in \mathbb{C} | \, \frac{(-\pi+\overline{\omega}+\varepsilon)}{2} < \arg\lambda < \frac{(\pi+\underline{\omega}-\varepsilon)}{2} \right\}, \overline{\omega} = \sup \{\arg a(x)\},$ $\underline{\omega} = \inf \{\arg a(x)\}$.

Here and below $\|u\|_{W_2^k}$ means

$$\|u\|_{W_2^k(-1,0)\oplus W_2^k(0,1)} = \left(\|u\|^2_{W_2^k(-1,0)} + \|u\|^2_{W_2^k(0,1)} \right)^{1/2}.$$

For the consideration introduce the functional L_{yk} $(y = 0, 1, \ k = 0, 1)$ defined by equalities

$$L_y(\lambda) u = \lambda L_{y0} u + L_{y1} u.$$

Theorem 2. *Let* $\alpha_{11}\beta_{21}(\gamma_{11}\delta_{21} - \gamma_{21}\delta_{11}) \ne 0$, *and linear functionals* T_{yK} *be continuous in the space* $W_2^k(-1,0) \oplus W_2^k(0,1)$, $y = 1, ..., 4$, $K = 0, 1$. *Then, the set*

$$\left\{ u | \, u = (u_1, u_2) \in \bigoplus_{K=0}^{1} \left(W_2^{2-K}(-1,0) \oplus W_2^{2-K}(0,1) \right) \right.$$

$$L_{y1}u_1 + L_{y2}u_2, \quad y = 1, ..., 4\}$$

is dense in the space $\bigoplus_{K=0}^{1} \left(W_2^{1-K}(-1,0) \oplus W_2^{1-K}(0,1) \right)$.

Proof: Let $u = (u_1, u_2) \in \bigoplus_{K=0}^{1} \left(W_2^{2-K}(-1,0) \oplus W_2^{2-K}(0,1) \right)$. We construct the functions $v_{2n} \in C^\infty[-1,0] \oplus C^\infty[0,1]$ such that $\lim_{n\to\infty} \|v_{2n} - u_2\|_{L_2} = 0$ and consider the auxiliary problems

$$a(x) u'' - \lambda^2 u(x) = 0, \quad L_{y1}u = -L_{y0}v_{2n}, \quad y = 0, ..., 4 \tag{4}$$

where $\lambda_n = n \max \left\{ 1, \max_{y=1,...,4} |L_{y0}v_{2n}|^2 \right\} e^{i\varphi}, \quad \varphi \in G_\varepsilon$. Then, in virtue of Theorem 1 for the solutions v_{1n} of problem (4) it holds the estimate

$$\|v_{1n}\|_{W_2^1} \le C \sum_{y=1}^{4} |\lambda_n|^{-1/2} |L_{y0}v_{2n}| \le C n^{-1/2}, \quad C = \text{constant}.$$

Consequently, $\lim_{n\to\infty} \|v_{1n}\|_{W_2^1} = 0$.

In the paper [2], we established that the set

$$\{ u \in C^\infty[-1,0] \oplus C^\infty[0,1] \mid L_{y1}u = 0, \ y = 1, ..., 4\}$$

is dense in the space $W_2^1(-1,0) \oplus W_2^1(0,1)$. Therefore, there exist the functions $\omega_{1n} \in C^\infty[-1,0] \oplus C^\infty[0,1]$ such that $L_{y1}\omega_{1n} = 0$, $\lim_{n\to\infty} \|\omega_{1n} - u_{1n}\|_{W_2^1} = 0$.

Now, it is easy to note that for the function $u_{1n} = v_{1n} + \omega_{1n}$ and $u_{2n} = v_{2n}$ it holds $L_{y1}u_{1n} + L_{y0}u_{2n} = 0$, $\lim_{n\to\infty} \|u_{1n} - u_1\|_{W_2^1} = 0$ and $\lim_{n\to\infty} \|u_{2n} - u_2\|_{L_2} = 0$.

This completes the proof.

4. Two-fold completeness of the eigen and associated functions

The main result of this work is the following theorem.

Theorem 3. *Suppose all the conditions of Theorem 1, 2 are valid. Then, the spectrum of the problem (1)-(3) is discrete and the system of root functions of (1)-(3) is 2-fold comlete in the space*

$$\Big\{ u|\, u = (u_1, u_2) \in \underset{K=0}{\overset{1}{\oplus}} \left(W_2^{2-K}(-1,0) \oplus W_2^{2-K}(0,1) \right)$$

$$L_{y1}u_1 + L_{y2}u_2, \quad y = 1, ..., 4\}$$

Proof. In the Hilbert space $L_2(-1,0) \oplus L_2(0,1) = L_2(-1,1)$ we introduce the operator A which is defined by the equalities $D(A) = W_2^2(-1,0) \oplus W_2^2(0,1)$, $Au = a(x)\, u''(x) + (Bu)(x)$. Then, the problem (1)-(3) can be rewritten in the form of a system of the operator pencils, as

$$L(\lambda)\, u = -\lambda^2 u + Au = 0$$

$$L_y(\lambda)\, u = \lambda L_{y1}u_1 + L_{y0}u_2 = 0, \quad y = 1, ..., 4 \tag{5}$$

We shall use the abstract results [7, Theorem 2, 3, 4], in which the sufficiently conditions have been found for multiply completeness of root vectors (i.e. eigen- and associated vectors) for a system of the operator pencils. By virtue of [6, p. 258] the imbeddings $W_2^2(a,b) \subset W_2^1(a,b) \subset L_2(a,b)$ are compact. Consequently, for the Hilbert spaces $H = L_2(-1,0) \oplus L_2(0,1)$, $H^y = C \quad (y = 1,2,3,4)$ and $H_K = W_2^K = W_2^k(-1,0) \oplus W_2^k(0,1)$ $K = 0, 1$ the conditions 1, 3 and 4 of the Theorem 2, 3, 4 in [7] are obvious.

Further, from [6, p. 350] it follows that for the S-numbers of the imbedding operators $J_x = x : H_{K+1} \to H_K$ the following inequalities take place,

$$C_1 n^{-1} \le S_n(J : H_{K+1}, H_K) \le C_2 n^{-1}, \quad n = 1, 2, ...$$

where $C_1 > 0$ and $C_2 > 0$ are constants. Consequently, for any $p > 1$

$$\sum_{n=1}^{\infty} S_n^p(J : H_{K+1}, H_K) < \infty$$

i.e. the condition 4 also holds.

The principal conditions 5 and 6 which are directly connected with the problem, hold by virtue of Theorem 1 and 2, which have respectively been mentioued above. Indeed, condition 5 immediately follows from Theorem 1. Further, from Theorem 2, in particular, it follows that for the operator

$$\widetilde{L}(\lambda)\, u = (L(\lambda)\, u, L_1(\lambda)\, u, L_2(\lambda)\, u, L_3(\lambda)\, u, L_4(\lambda)\, u) : W_2^2(-1,0) \oplus W_2^2(0,1) \to$$

$$L_2(-1,0) \oplus L_2(0,1) \oplus \mathbb{C}^4$$

the estimate $\left\| \widetilde{L}^{-1}(\lambda) \right\| \leq C |\lambda|^{1/2}$, $C = const$ takes place, i.e. the last condition, 6 also holds for the operator pencils (5). Now, for completing the proof, it is enough to apply Theorem 2 and 3 immediately follows the next important results.

Corollary: *If all the conditions of Theorem 1 and 2 are valid, then the eigen and associated functions of problem (1)-(3) is 2-fold complete in the Hilbert space* $[L_2(-1,1)]^2$.

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