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NECESSARY OPRIMALITY CONTIONS IN A CONTROL PROBLEM DESCRIBED BY A SYSTEM OF VOLTERRA TYPE TWO-DIEMNSIONAL DIFFRENCE EQAUTIONS

Abstract

In the paper, an optimal control problem described by a system of Volterra type two-dimensional difference equations is studied. A necessary optimality condition is obtained in the form of Pontryagin's discrete maximum principle. In the case of convexity of controls domain the necessary optimality condition in the form of linearized maximum principle is proved. Analogy of the Euler equation is introduced under the assumption of openness of the controls domain.

Introduction. Various difference equations representing difference analogies of differential, integro-differential, integral equations and also the equations of mathematical physics (see for example [1-6] are oftenly used while modeling many real processes of military matters, production, economy, dynamics of population and etc.

To the present time the optimal control problems described by ordinary differential equations and equations of mathematical physics (see for example [7-14] have been studied enough.

The suggested paper is devoted to investigation of an optimal control problem described by a system of two-dimensional difference equations of Volterra type representing a difference analogy of hyperbolic type integro- difference equation with the Goursat boundary conditions. Different necessary optimality conditions of first order are obtained. Necessary and sufficient optimal condition is proved in one special case. Notice that the optimal control problems described by Volterra type integral equations have been studied in the papers [5-21] and others. Notice that such a problem is studied for the first time.

1. Problem Statement. Assume that the controlled problem is described by the following system of Volterra type difference equations:

$$\begin{gathered} z(t+1,x+1)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x} f(t,x,\tau,s,z(\tau,s),u(\tau,s)), \\ t\in T=\{t_0,t_0+,...,t_1-1\}, \quad x\in X=\{x_0,x_0+,...,x_1-1\} \end{gathered} \tag{1.1}$$

with boundary conditions

$$\begin{gathered} z(t_0,x)=\alpha(x), \quad x\in X\cup x_1, \\ z(t,x_0)=b(t), \quad t\in T\cup t_1, \quad a(x_0)=b(t_0)=a_0. \end{gathered} \tag{1.2}$$

Here, $z(t,x)$ is an n dimensional state vector, $a(x),\ b(t)$ are the given $n-$dimensional discrete vector-functions, $t_0,\ t_1,\ x_0,\ x_1$ are given, the differences $t_1-t_0,\ x_1-x_0$

are natural numbers, $f(t,x,\tau,s,z,u)$ is a given $n-$dimensional vector-function discrete with respect to (t,x,τ,s) and continuous with respect to (z,u) together with $f_z(t,x,\tau,s,z,u)$, $u(t,x)$ is an r dimensional vector of control actions with the values from the given non-empty and bounded set $U\subset R^r$, i.e.

$$u(t,x)\in U\subset R^r,\quad (t,x)\in T\times X. \tag{1.3}$$

The control functions satisfying these restrictions are called admissible controls.

On the solutions of problem (1.1) – (1.2) generated by all possible admissible controls define the functional

$$S(u)=\varphi(z(t_1,x_1)). \tag{1.4}$$

Here $\varphi(z)$ is a given continuously differentiable scalar function.

The problem is to find the minimum of the functional (1.4) under restrictions (1.4)-(1.3).

The admissible control $u(t,x)$ delivering minimum to the functional (1.4) under restrictions (1.1-1.3) is said to be an optimal control, the appropriate process $(u(t,x),\ z(t,x)$ an optimal process.

2. Increment formula of the quality test. Let $u(t,x)$ be a fixed admissible control. Consider an arbitrary admissible control $\overline{u}(t,x)=u(t,x)+\Delta u(t,x)$ and by $z(t,x),\ \overline{z}(t,x)=z(t,x)+\Delta z(t,x)$ denote appropriate solutions of problem (1.1)-(1.2).

It is clear that the increment $\Delta z(t,x)$ of the state vector will be a solution of the problem

$$\begin{gathered}\Delta z(t+1,x+1)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x} f(t,x,\tau,s,\overline{z}(\tau,s),\overline{u}(\tau,s))-\\ -f(t,x,\tau,s,z(\tau,s),u(\tau,s))], \end{gathered}\tag{2.1}$$

$$\begin{gathered}\Delta z(t_0,x)=0,\qquad x\in X\cup x_1,\\ \Delta z(t,x_0)=0,\qquad t\in T\cup t_1.\end{gathered}\tag{2.2}$$

Write the increment formula of the quality test

$$\Delta S(u)=S(\overline{u})-S(u)=\varphi(\overline{z}(t_1,x_1))-\varphi(z(t_1,x_1)). \tag{2.3}$$

Multiplying scalarly the both hand sides of identity (2.1) from the left by the vector-function $\psi(t,x)$ unknown to the present time and summing over t from t_0 to t_1-1 and over x from x_0 to x_1-1, we get

$$\begin{gathered}\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t,x)\Delta z(t+1,x+1)=\\ =\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\psi'(t,x)\left[f(t,x,\tau,s,\overline{z}(\tau,s),\overline{u}(\tau,s))-\right.\right.\\ \left.\left.-f(t,x,\tau,s,z(\tau,s),u(\tau,s))\right]\right].\end{gathered}\tag{2.4}$$

It is easy to prove that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t,x)\Delta z(t+1,x+1)=\psi'(t_1-1,x_1-1)\Delta z(t_1,x_1)-$$

$$-\psi'(t_1-1,x_0-1)\Delta z(t_1,x_0)-\psi'(t_0-1,x_1-1)\Delta z(t_0,x_1)+$$

$$+\psi'(t_0-1,x_0-1)\Delta z(t_0,x_0)+\sum_{x=x_0}^{x_1-1}\psi'(t_1-1,x_1-1)\Delta z(t_1,x)-$$

$$-\sum_{x=x_0}^{x_1-1}\psi'(t_0-1,x-1)\Delta z(t_0,x)+\sum_{t=t_0}^{t_1-1}\psi'(t-1,x_1-1)\Delta z(t,x_1)-$$

$$-\sum_{t=t_0}^{t_1-1}\psi'(t-1,x_0-1)\Delta z(t,x_0)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t-1,x-1)\Delta z(t,x). \quad (2.5)$$

Reduce the discrete analogy of Foubini's two-dimensional theorem.

Lemma 2.1. *Let $L(t,x,\tau,s)$ and $M(t,x,\tau,s)$ be the given $(n\times n)$ matrix functions. Then, the following identity is valid*

$$\sum_{t=t_0}^{m}\sum_{x=x_0}^{\ell}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}L(m,\ell,t,x)M(t,x,\tau,s)\right]=$$

$$=\sum_{t=t_0}^{m}\sum_{x=x_0}^{\ell}\left[\sum_{\tau=t_0}^{m}\sum_{s=x_0}^{\ell}L(m,\ell,t,x)M(\tau,s,t,x)\right].$$

The lemma is proved by the scheme, for example from [22].

Further, using this two-dimensional analogy of the discrete analogy of Foubini's formula ([23-24]), we get

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\psi'(t,x)\left[f(t,x,\tau,s,\overline{z}(\tau,s),\overline{u}(\tau,s))-\right.\right.$$

$$\left.\left.-f(t,x,\tau,s,z(\tau,s),u(\tau,s))\right]\right]=$$

$$=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\left[\sum_{\tau=t}^{t_1-1}\sum_{s=x}^{x_1-1}\psi'(\tau,s)\left[f(\tau,s,t,x,\overline{z}(\tau,s),\overline{u}(\tau,s))-\right.\right.$$

$$\left.\left.-f(\tau,s,t,x,z(\tau,s),u(\tau,s))\right]\right] \quad (2.6)$$

Introducing the Hamilton-Pontryagin's functions in the following way:

$$H(t,x,z(t,x),u(t,x),\psi(t,x))=\sum_{\tau=t}^{t_1-1}\sum_{s=x}^{x_1-1}\psi'(\tau,s)f(\tau,s,t,x,z(\tau,x),u(\tau,s)).$$

and taking into account the identities (2.1), (2.2), (2.4), (2.5), (2.6), the increment formula (2.3) of the quality test (1.4) is written in the form

$$\Delta S(u)=\psi_z'(z(t_1,x_1))\Delta z(t_1,x_1)+0_1(\|\Delta z(t_1,x_1)\|)+$$

$$+\psi'(t_1-1,x_1-1)\Delta z(t_1,x_1)+\sum_{s=x_0}^{x_1-1}\psi'(t_1-1,x-1)\Delta z(t_1,x)+$$

$$+\sum_{t=t_0}^{t_1-1}\psi'(t-1,x_1-1)\Delta z(t,x_1)+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\psi'(t-1,x-1)\Delta z(t_1,x)-$$

$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}[H(t,x,z(t,x),\overline{u}(t,x)\psi(t,x))-H(t,x,z(t,x),u(t,x)\psi(t,x))]-$$

$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}[H_z(t,x,z(t,x),\overline{u}(t,x)\psi(t,x))-H_z(t,x,z(t,x),u(t,x),\psi(t,x)]'\times$$

$$\times\Delta z(t,x)-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}H'_z(t,x,z(t,x),u(t,x),\psi(t,x)\Delta z(t,x)-$$

$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}0_2\left(\|\Delta z(t,x)\|\right). \tag{2.7}$$

Assume that the vector-function $\psi(t,x)$ is a solution of the problem

$$\psi(t-1,x-1)=H_z(t,x,z(t,x),u(t,x),\psi(t,x)), \tag{2.8}$$

$$\begin{array}{ll}\psi(t_1-1,x-1)=0, & x\in X,\\ \psi(t-1,x_1-1)=0, & t\in T,\\ \psi(t_1-1,x_1-1)=-\varphi_z(z(t_1,x_1)).\end{array} \tag{2.9}$$

The problem (2.8)-(2.9) is said to be a conjugate problem (or a system) in problem (1.1)-(1.4). Therewith, the increment formula accepts the form

$$\Delta S(u)=-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}[H(t,x,z(t,x),\overline{u}(t,x)\psi(t,x))-$$

$$-H(t,x,z(t,x),u(t,x)\psi(t,x))]+0_1\left(\|\Delta z(t_1,x_1)\|\right)-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}0_2\left(\|\Delta z(t,x)\|\right)$$

$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}[H_z(t,x,z(t,x),\overline{u}(t,x)\psi(t,x))-$$

$$-H_z(t,x,z(t,x),u(t,x),\psi(t,x)]'\,\Delta z(t,x) \tag{2.10}$$

3. Necessary optimality condition. Suppose that the set

$$f(t,x,\tau,s,z(\tau,s),U)=\{\alpha{:}\alpha=f(t,x,\tau,s,z(\tau,s),\nu),\nu\in U\} \tag{3.1}$$

is convex for all (t,x,τ,s).

Then a special increment of the admissible control $u(t,x)$ may be determined by the formula

$$\Delta u_\varepsilon(t,x) = \nu_\varepsilon(t,x) - u(t,x), \quad (t,x) \in T \times X. \tag{3.2}$$

Here, $\epsilon \in [0,1]$ is an arbitrary number, $\nu_\varepsilon(t,x) \in U$, $(t,x) \in T \times X$ is an arbitrary admissible control such that

$$f(t,x,\tau,s,z(\tau,s),\nu_\varepsilon(\tau,s)) - f(t,x,\tau,s,z(\tau,s),u(\tau,s)) =$$

$$= \varepsilon \left[f(t,x,\tau,s,z(\tau,s),\nu(\tau,s)) - f(t,x,\tau,s,z(\tau,s),u(\tau,s)) \right].$$

Here $\nu(t,x) \in U$, $(t,x) \in T \times X$ is an arbitrary admissible control.

By $\Delta z_\varepsilon(t,x)$ we'll denote a special increment of the state vector $z(t,x)$ responding to increment (3.2) of the control $u(t,x)$.

Passing to the norm, after some transformations from (2.1) we get

$$\|\Delta z(t+1,x+1)\| \le \sum_{\tau=t_0}^{t_1-1} \sum_{s=x_0}^{x_1-1} \| f(t_1,x_1,\tau,s,z(\tau,s),\overline{u}(\tau,s)) -$$

$$- f(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s)) \| + L_1 \sum_{\tau=t_0}^{t} \sum_{s=x_0}^{x} \|\Delta z(\tau,s)\|,$$

where $L_1 = const > 0$.

Applying a discrete analogy of Gronwall-Bellman lemma to this inequality (see [11, 24, 25]), we get

$$\|\Delta z(t,x)\| \le L_2 \sum_{\tau=t_0}^{t_1-1} \sum_{s=x_0}^{x_1-1} \| f(t_1,x_1,\tau,s,z(\tau,s),\overline{u}(\tau,s)) -$$
$$- f(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s)) \|, \tag{3.3}$$

$$(L_2 = const > 0), \quad (t,x) \in (T \cup t_1) \times (X \cup x_1).$$

Inequality (3.3) yields the validity of the estimation

$$\|\Delta z_\varepsilon(t,x)\| \le L_3 \varepsilon, \quad (t,x) \in (T \cup t_1) \times (X \cup x_1). \tag{3.4}$$

The expansion

$$S_\varepsilon(u) = S(u + \Delta u_\varepsilon) - S(u) =$$

$$-\varepsilon \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} \left[H(t,x,z(t,x),\nu(t,x),\psi(t,x)) - H(t,x,z(t,x),u(t,x),\psi(t,x)) \right] + 0(\varepsilon).$$

follows from the increment formula (2.10) allowing for (3.2), (3.4).

This means that the inequality

$$-\varepsilon \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} [H(t,x,z(t,x),\nu(t,x),\psi(t,x)) -$$

$$- H(t,x,z(t,x),u(t,x),\psi(t,x))] + 0(\varepsilon) \ge 0.$$

is fulfilled along the optimal process $(u(t,x), z(t,x)$.

Hence, by arbitrariness of $\varepsilon \in [0,1]$ we have

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}[H(t,x,z(t,x),\nu(t,x),\psi(t,x)) - H(t,x,z(t,x),u(t,x),\psi(t,x))] \leq 0.$$

Formulate the obtained result.

Theorem 3.1. *If the set (3.1) is convex, then for the optimality of the admissible control $u(t,x)$ in problem (1.1)-(1.4) the inequality (3.5) should be fulfilled for all $\nu(t,x) \in U, \quad (t,x) \in T \times X$.*

The inequality (3.5) is an analogy of discrete maximum condition in the considered problem.

The following statement is a direct Corollary of theorem 3.1.

Theorem 3.2. *While fulfilling the conditions of theorem 3.1. for optimality of the admissible control $u(t,x)$ in problem (2.1)-(2.4), the relation*

$$\max_{w\in U} H(\theta,\xi,z(\theta,\xi),w,\psi(\theta,\xi)) = H(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi))$$

should be fulfilled for all $(\theta,\xi) \in T \times X$.

For proving theorem 3.2. it suffices to determine the admissible control $\nu(t,x)$ in (3.5) by the formula

$$\nu(t,x) = \begin{cases} w, & (t,x) \in T \times X, \\ u(t,x), & (t,x) \neq (\theta,\xi). \end{cases}$$

One can show that (see [26]) theorem 3.1. and 3.2. are equivalent. But the statement of theorem 3.2. is easily verified.

4. The maximum principle as necessary and sufficient optimality condition. Choose a class of problems for which the discrete maximum principle is not only a necessary but also sufficient optimality condition.

Let in problem (1.1)-(1.4)

$$f(t,x,\tau,s,z,u) = A(t,x,\tau,s)z + g(t,x,\tau,s,u), \tag{4.1}$$

$$\varphi(z) = c'z. \tag{4.2}$$

Here $A(t,x,\tau,s)$ is a given $(n\times n)$ discrete matrix function, c is a given $n-$dimensional constant vector, $g(t,x,\tau,s,u)$ is a given discrete with respect to (t,x,τ,s) and continuous with respect to u, $n-$dimensional vector-function.

Let by the definition

$$M(t,x,u(t,x),\lambda(t,x) = \sum_{\tau=t}^{t_1-1}\sum_{s=x}^{x_1-1}\lambda'(\tau,s)g(\tau,s,t,x,u(t,x)),$$

where $\lambda(t,x)$ is a solution of the problem

$$\lambda(t-1,x-1)=\sum_{\tau=t}^{t_1-1}\sum_{s=x}^{x_1-1}A'(\tau,s,t,x)\lambda(\tau,s),$$

$$\lambda(t_1-1,x-1)=0,\qquad x\in X\cup x_1,$$

$$\lambda(t-1,x_1-1)=0,\quad t\in T\cup t_1,$$

$$\lambda(t_1-1,x_1-1)=-c.$$

Then, the increment of the quality test $S(u)=c'z(t_1,x_1)$ is written in the form

$$\Delta S(u)=-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\left[M(t,x,\overline{u}(t,x),\lambda(t,x))-M(t,x,u(t,x),\lambda(t,x))\right]. \qquad (4.3)$$

By means of (4.3) we prove

Theorem 4.1. *For optimality of the admissible control $u(t,x)$ in problem (1.1)−(1.4), (4.1), (4.2) it is necessary and sufficient that the relation*

$$\max_{\nu\in U}M(\theta,\xi,\nu,\lambda(\theta,\xi))=M(\theta,\xi,u(\theta,\xi),\lambda(\theta,\xi)) \qquad (4.4)$$

be fulfilled for all $(\theta,\xi)\in T\times X$.

Proof. Necessity. Suppose that $u(t,x)$ is an optimal control. Then, it follows from (4.3) that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}\left[M(t,x,\overline{u}(t,x),\lambda(t,x))-M(t,x,u(t,x),\lambda(t,x))\right]\le 0, \qquad (4.5)$$

for all $\overline{u}(t,x)\in U,\ (t,x)\in T\times X$.

Let

$$\overline{u}(t,x)=\begin{cases}\nu, & (t,x)=(\theta,\xi)\in T\times X,\\ u(t,x), & (t,x)\neq(\theta,\xi).\end{cases} \qquad (4.6)$$

Here, $\nu\in U$ is an arbitrary vector, and $(\theta\ ,\xi)\in T\times X$ is an arbitrary point.

Taking into account (4.6) in (4.5) be arrive at relation (4.4).

Sufficiency. Let relation (4.4) be fulfilled. This means that for any $(\theta\ ,\xi)\in T\times X,\quad \nu(\theta,\xi)\in U$.

$$M(\theta,\xi,\nu(\theta,\xi),\lambda(\theta,\xi))-M(\theta,\xi,u(\theta,\xi),\lambda(\theta,\xi))\le 0.$$

Hence, we get

$$\sum_{\theta=t_0}^{t_1-1}\sum_{\xi=x_0}^{x_1-1}\left[M(\theta,\xi,\nu(\theta,\xi),\lambda(\theta,\xi))-M(\theta,\xi,u(\theta,\xi),\lambda(\theta,\xi))\right]\le 0.$$

Consequently

$$S(\nu)-S(u)=-\sum_{\theta=t_0}^{t_1-1}\sum_{\xi=x_0}^{x_1-1}\left[M(\theta,\xi,\nu(\theta,\xi),\lambda(\theta,\xi))-M(\theta,\xi,u(\theta,\xi),\lambda(\theta,\xi))\right]\ge 0.$$

Thus, we proved that for any admissible control $\nu(t,x)$

$$S(\nu) \geq S(u).$$

This proves sufficiency of optimality condition (4.4)

5. Pontryagin's linearized maximum principle. Assume that in problem (1.1)-(1.4) the set U is convex, $f(t,x,\tau,s,z,u)$ for all fixed (t,x,τ,s) has also continuous derivatives $f_u(t,x,\tau,s,z,u)$, $f_z(t,x,\tau,s,z,u)$.

By analogy with (2.7), the increment of the quality test is written in the form

$$\Delta S(u) = \varphi_z'(z(t_1,x_1))\Delta z(t_1,x_1) + \psi'(t_1-1,x_1-1)\Delta z(t_1,x_1) +$$
$$+\sum_{x=x_0}^{x_1-1} \psi'(t_1-1,x-1)\Delta z(t_1,x) + \sum_{t=t_0}^{t_1-1} \psi'(t-1,x_1-1)\Delta z(t,x_1) +$$
$$+\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} \psi'(t_1-1,x-1)\Delta z(t_1,x) - \sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} [H(t,x,\overline{z}(t,x),\overline{u}(t,x),\psi(t,x)) -$$
$$-H(t,x,z(t,x),u(t,x),\psi(t,x))] + 0_1\left(\|\Delta z(t_1,x_1)\|\right). \tag{5.1}$$

Hence, using the Taylor formula and taking into account that the vector-function $\psi(t,x)$ is a solution of problem (2.8)-(2.9), we get the relation

$$\Delta S(u) = -\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} H_u'(t,x,z(t,x),u(t,x),\psi(t,x))\Delta u(t,x) + 0_1\left(\|\Delta z(t_1,x_1)\|\right) -$$
$$-\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1} 0_3\left(\|\Delta z(t,x)\| + \|\Delta u(t,x)\|\right). \tag{5.2}$$

Further, in (2.1) passing to the norm and using the Lipschitz condition, we get

$$\|\Delta z(t+1,x+1)\| \leq \sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x_1-1} \|\Delta u(\tau,s)\| + L_3 \sum_{\tau=t_0}^{t-1}\sum_{s=x_0}^{x-1} \|\Delta z(\tau,s)\|. \tag{5.3}$$

Here $L_3 = const > 0$.

Applying it to inequality (5.3), we get validity of the estimation

$$\|\Delta z(t,x)\| \leq L_4 \sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x_1-1} \|\Delta u(\tau,s)\|, \quad (t,x) \in (T\cup t_1)\times(X\cup x_1), \tag{5.4}$$

$$L_4 = const > 0.$$

By convexity of the set , U we can define the increment of the admissible control $(u(t,x)$ by the formula

$$\Delta u(t,x;\varepsilon) = \varepsilon\left[\nu(t,x) - u(t,x)\right], \quad (t,x) \in T\times X. \tag{5.5}$$

Here, $\nu(t,x)\in U,\quad (t,x)\in T\times X$ is an arbitrary admissible control, $\varepsilon\in[0,1]$ is an arbitrary number.

By $\Delta z(t,x;\varepsilon)$ we denote a special increment of the state vector $z(t,x)$ corresponding to the special increment (5.5) of the admissible control $u(t,x)$.

Allowing for (5.5), it directly follows from estimation (5.4) that

$$\|\Delta u(t,x;\varepsilon)\|\leq\varepsilon L_5,\qquad (t,x)\in(T\cup t_1)\times(X\cup x_1)\,. \tag{5.6}$$

Allowing for estimation (5.6) and formula (5.5), the following expansion follows from increment formula (5.2).

$$\begin{gathered}\Delta S_\varepsilon(u(t,x))=S(u(t,x)+\Delta u(t,x{:}\varepsilon))-S(u(t,x))=\\ =-\varepsilon\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}H'_u(t,x,z(t,x),u(t,x),\psi(t,x))(\nu(t,x)-u(t,x))+0(\varepsilon).\end{gathered} \tag{5.7}$$

By arbitrariness of $\varepsilon\in[0,1]$, it follows form expansion (5,7) that

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1-1}H'_u(t,x,z(t,x),u(t,x),\psi(t,x))(\nu(t,x)-u(t,x))\leq 0. \tag{5.8}$$

Formulate the following result.

Theorem 5.1. *If the problem (1.1)-(1.4) the set U is convex, and $f(t,x,\tau,s,z,u)$ for all (t,x,τ,s) is continuous with respect to (z,u) together with partial derivatives (z,u), then for optimality of the admissible control $u(t,x)$ the inequality (5.8) should be fulfilled for all $\nu(t,x)\in U,\quad (t,x)\in T\times X$.*

Inequality (5.8) is a discrete analogy of Pontryagin's linearized maximum principle (see [5-7,11, 26]). It can be written in the following equivalent form

$$H'_u(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi))(w-u(\theta,\xi))\leq 0$$

for all $w\in U$ and $(\theta,\xi)\in T\times X$.

Finally, let's consider the case of open control domain.

6. The case of open control domain. Assume that in problem (1.1)-(1.4) the set U is open. Then the special increment of the admissible control may be defined by the formula

$$\Delta u_\mu(t,x)=\mu\delta u(t,z),\qquad (t,x)\in T\times X. \tag{6.1}$$

Here, $\delta u_\mu(t,x)\in R^r,\quad (t,x)\in T\times X$ is an arbitrary r-dimensional bounded vector-function, μ is an arbitrary, sufficiently small number in absolute value.

By $\Delta z_\mu(t,x)$ denote a special increment of the state responding to increment (6.1) of the control $u(t,x)$.

Allowing for (6.1) and (5.4), the validity of the following expansion follows from increment formula (5.2),

$$\Delta S_\mu(u)=S(u+\Delta u_\mu)-S(u)=$$

$$= -\mu \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} H'_u(t,x,u(t,x),\psi(t,x))\delta u(t,x) + 0(\mu).$$

This expansion means that the along the process $(u(t,x), z(t,x))$ the first variant (in the classic sense) of the functional $S(u)$ is of the form

$$\delta^1 S(u{:}\delta u) = \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} H'_u(t,x,u(t,x),\psi(t,x))\delta u(t,x).$$

Hence, as the first variation of the minimized functional along the optimal process equals zero, it follows that

$$H_u(\theta,\xi,z(\theta,\xi),u(\theta,\xi),\psi(\theta,\xi)) = 0 \tag{6.2}$$

for all $(\theta,\xi) \in T \times X$.

Relation (6.2) is the analogy of the Euler equation for the considered problem.

Theorem 6.1. *If the set U is open, then for optimality of the admissible control $u(t,x)$ in problem (1.1)-(1.4), relation (6.2) should be fulfilled for all $(\theta,\xi) \in T \times X$.*

The authors thank to the reviewer for very useful remarks that improved the first variant of the paper.

References

[1]. Kolmanowski V.B. *On asymptotic equivalence of solutions of Volterra difference equations.* // Avtomatika i telemekhanika, 2001, No 4, pp. 47-55 (Russian).

[2]. Kolmanowski V.B., Myshkis A.D., Richard J.P. *Estimate of solution for some Volterra difference equations* // Nonlinear analysis. 2000, vol. 40, pp. 345-363.

[3]. Gayshun I.V. *Discrete time systems.* Minsk, IM NAS Belarus, 2001, 425 p. (Russian).

[4]. Dymkov M.P. *Extremal problems in multiparametric control systems.* // The author's thesis for Doctor's degree. Minsk, 1997, 32 p. (Russian).

[5]. Ashepkov L.T. *Optimization of discontinuous systems.* Nauka, 1988 (Russian).

[6]. Propoy A.I. *Elements of theory of optimal discrete processes.* M. Nauka, 1973.

[7]. Gabasov R., Kirillova F.M. *Maximum principle in optimal control theory.* Mn. Nauka I tekhnika, 1974 (Russian).

[8]. Gabasov R., Kirillova F.M. *Singular optimal controls.* M., Nauka, 1973, 256 p. (Russian).

[9]. Pontryagin L.S. and others. *Mathematical theory of optimal processes.* M. Nauka, 1969.

[10]. Egorov A.I. *Optimal control of heat and diffusion processes.* M. Nauka, 1978 (Russian).

[11]. Vasil'ev F.P. *Solution methods of extremal problems.* M. Nauka, 1981 (Russian).

[12]. Vasil'ev F.P., Srochko V.A., Terletskiy V.A. *Optimization methods and their applications.* Part II. Nauka, 1990 (Russian).

[13]. Butkowskiy A.A. *Theory of optimal control of distributed parameters systems.* M. Nauka, 1965 (Russian).

[14]. Lourie K.A. *Optimal control of mathematical physics equations.* M. Nauka, 1975 (Russian).

[15]. Angell T.C. *On the optimal control of systems governed by nonlinear integral equations* // J. Optimiz. Theory and Applics. 1976, 19, pp. 63-79.

[16]. Burnap C. and Kazemi M.A. *Optimal control of a system governed by nonlinear Volterra integral equations with delay* // IMAY. Math. Control Info. 1999, 19, pp. 1-17.

[17]. Carlson D.A. *An elementary proof of the maximum principle for optimal control problems governed by a Volterra integral equation* // J. Optimiz. Theory and Applics. 1987, 54, pp. 43-61.

[18]. Jafarov E.E., Mansimov K.B. *Necessary optimality conditions in a variable structure control described by a system of the two-dimensional Volterra type integral equations.* // Transactions of NAS Azerbaijan. Series of Physical-technical and mathematical sciences. 2007, No 7, pp. 191-208.

[19]. Abdullayev A.A., Mansimov K.B. *Necessary second order optimality conditions for the process described by a system of nonlinear integral equations of Volterra type.* // Avtomatika i telemekhanika. 2000, No 2, pp. 3-11.

[20]. Jafarov E.E., Mansimov K.B. *On optimality of Quasi singular controls in a variable structure control problem.* // Proceedings of Institute of Mathematics and mechanics. 2007, vol. XXXV, pp. 159-172.

[21]. Elnagar G.N., Kazemi M.A., Kim Hoonjoo. *Necessary and sufficient optimality conditions for control systems described by integral equations with delay* // J. Koran Math. Soc. 2000, 37, No 4, pp. 625-643.

[22]. Zouyosifain M., Leela S. *Stability results for difference equations of Volterra type* // Apll. Math. Comput. 1990, vol. 36, part I, pp. 51-61.

[23]. Sung. Kyu Choi, Yoon Hoe Goo, Nam Jip Koo. *Asymptotic behavior of nonlinear Volterra difference systems* // Bull. Korean Math. Soc. 2007, vol. 44, No 1, pp. 177-184.

[24]. Salem S.H., Raslan K.R. *Some new discrete inequalities and their applications* // Journal of inequalitites in Pure and Applied Mathematics. 2004, vol. 5, No 1, pp. 1-9.

[25]. Salem S.H. *On some system of two discrete inequalities of Gronwall Type* // Journ. Math. and Appl. 1997, vol. 208, pp. 553-566.

[26]. Mansimov K.B. *Discrete systems.* Baku, publishing house of the Baku University, 2002,116 p.

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Received February 12, 2010; Revised May 17, 2010;