Manaf Dzh. MANAFOV

ON FLOQUET SOLUTIONS FOR PERIODIC "WEIGHT" EQUATION WITH DISTRIBUTIONS

Abstract

In the paper, stability or instability of the solution to periodic "weight" equation with distributions is investigated by means of the Floquet theory.

The problems on study of Sturm-Liouville operator and its multi-dimensional analogies $-\Delta + q(x)$ with short interaction potential (of δ-function type) have appeared in physical literature. Mathematical investigations of appropriate physical models were initiated at the beginning of sixties in the papers [3,10]. This theme intensively has developed in the last two decades. There is a monograph [1] where one can be acquainted with details of Berezin-Minlos-Faddeev theory in its contemporary state and other new directions arising from this theory. In the same place, one can find wide bibliography.

Another approach for studying the Sturm-Liouville operators with non-classic potentials being the derivatives of the bounded variation (charge) functions was undertaken in the papers [2,6,7] and others. In this direction, spectral properties of such a class of higher order operators were studied in the papers [8,9].

It was proved in [11] that the Sturm-Liouville operator may be correctly determined for essentially wider class of potentials being the first order singular distributions.

In this paper we'll state the Floquet theory for the equation

$$l_q^\rho[y] \equiv -\frac{1}{\rho(x)}\frac{d}{dx}(\rho(x)\frac{dy}{dx}) + q(x)y = \lambda y, \quad -\infty < x < \infty, \tag{1}$$

that clarifies the structure of the space of solutions of this equation for each complex value of the parameter λ. Notice that the "weight" function $\rho(x) = 1 + \alpha \sum\limits_{n=-\infty}^{\infty} \delta(x - Nn)$ and the coefficient $q(x)$ is a real valued periodic continuous function with a period equal N; $\delta(x)$ is a Dirac function; $\alpha \neq 0$ and $N \geq 1$ are real and natural numbers, respectively. Spectral analysis for equation (1) in the case $\alpha \equiv 0$ was stated in detail in [5,12]. For generalization for the case $\alpha \neq 0$ as we'll see we have to overcome many difficulties.

Definition. *For the given real value of the parameter λ, equation (1) is said to be* ***stable*** *if all its solutions are bounded on the axis $(-\infty, \infty)$,* ***instable*** *if all its solutions are not bounded on the axis $(-\infty, \infty)$,* ***conditionally stable*** *if it has at least one non-trivial solution bounded on the whole of the axis $(-\infty, \infty)$.*

According to this definition, if for the given value if λ, equation (1) is stable, then it will be conditionally stable as well. The vice versa, generally speaking, is not true.

Here, the approach is based on the idea of approximation of the generalized weight with smooth weights.

Consider the differential expression

$$l_q^\rho[y] \equiv -\frac{1}{\rho_\varepsilon(x)}\frac{d}{dx}(\rho_\varepsilon(x)\frac{dy}{dx}) + q(x)y$$

Here, the density of the function

$$\rho_\varepsilon(x) = 1 + \frac{\alpha}{\varepsilon}\sum_{n=-\infty}^{\infty} X_\varepsilon(x - Nn)$$

is determined by means of the characteristic function

$$X_\varepsilon(x) = \begin{cases} 1, & \text{for } \ x \in [0, \varepsilon] \\ 0, & \text{for } \ x \notin [0, \varepsilon], \end{cases} \qquad \varepsilon << N$$

Notice that the density of the function $\rho_\varepsilon(x)$ is chosen so that as $\varepsilon \to 0^+$ it approaches the function $\rho(x)$ (see [4]). Therefore, the approximation function is of the form:

$$l_q^{\rho_\varepsilon}[y] = \lambda y, \qquad -\infty < x < \infty. \tag{2}$$

Agree that the solution of equation (2) is any function $y(x,\lambda)$ determined on $(-\infty,\infty)$ for which the following conditions are fulfilled:

1. $y(x) \in C^2(Nn,\ Nn+\varepsilon) \cap C^2(Nn+\varepsilon,\ N(n+1))$ for $n \in Z = = \{..., -1, 0, 1, ...\}$;
2. $-y''(x) + q(x)y(x) = \lambda y(x)$ for $x \in (Nn,\ Nn+\varepsilon) \cup (Nn+\varepsilon,\ N(n+1))$ $n \in Z$;
3. $y(Nn)^+) = y((Nn)^-)$, $(1+\frac{\alpha}{\varepsilon})y'((Nn)^+) = y'((Nn)^-)$ for $n \in Z$;
4. $y((Nn+\varepsilon)^+) = y((Nn+\varepsilon)^-)$, $y'((Nn+\varepsilon)^+) = (1+\frac{\alpha}{\varepsilon})y'((Nn+\varepsilon)^-)$ for $n \in Z$.

These conditions guarantee that $y(x)$ and $\rho_\varepsilon(x)y'(x)$ are continuous functions at the points Nn and $Nn+\varepsilon(n \in Z)$.

If $y(x)$ is a solution of equation (1), it follows from periodicity of the functions $\rho(x)$ and $q(x)$ that $y(x+N)$ will be also a solution of this equation. However, generally speaking, $y(x) \not\equiv y(x+N)$. We'll show that there always exists a non-zero number $p = p(\lambda)$ and a non-trivial solution $\psi(x,\lambda)$ of equation (2), such that

$$\psi(0,\lambda) = p\psi(N,\lambda),$$

$$(1+\frac{\alpha}{\varepsilon})\psi'(0,\lambda) = p\psi'(N,\lambda),$$

$$\begin{array}{c} \psi(\varepsilon',\lambda) = \psi(\varepsilon^-,\lambda) \\ \psi'(\varepsilon',\lambda) = (1+\frac{\alpha}{\varepsilon})\psi'(\varepsilon^-,\lambda) \end{array}, \tag{3}$$

To this end, we consider a fundamental system of solutions $\theta(x,\lambda)$, $\varphi(x,\lambda)$ of the equation $-y''+q(x)y = \lambda y$ that will be determined by means of the initial conditions

$$\theta(0,\lambda) = \varphi'(0,\lambda) = 1, \quad \theta'(0,\lambda) = \varphi(0,\lambda) = 0. \tag{5}$$

The general solution of equation (2) will be of the form:

$$\psi(x,\lambda) = \begin{cases} c_1\theta(x,\lambda) + c_2\varphi(x,\lambda), & \text{for } \ 0 < x < \varepsilon \\ c_3\theta(x,\lambda) + c_4\varphi(x,\lambda), & \text{for } \ 0 < x < N \end{cases} \tag{6}$$

Since $\theta(x+N,\ \lambda)$ and $\varphi(x+N,\lambda)$ are also the solutions of equation (4), then

$$\begin{array}{c} \theta(x+N,\lambda) = a_{11}\theta(x,\lambda) + a_{12}\varphi(x,\lambda), \\ \varphi(x+N,\lambda) = a_{11}\theta(x,\lambda) + a_{12}\varphi(x,\lambda) \end{array} \tag{7}$$

and by (5)

$$\begin{array}{ll} a_{11}=\theta(N,\lambda) & a_{12}=\theta'(N,\lambda) \\ a_{21}=\varphi(N,\lambda) & a_{22}=\varphi'(N,\lambda) \end{array}, \qquad (8)$$

Substituting (6) in (3) and using (7), for definition of the constants c_i, $i = 1,2,3,4$ in (6) we get the homogeneous linear system of equations whose non-trivial solvability conditions is the relation

$$\begin{vmatrix} 1 & 0 & -p\theta(N,\lambda) & -p\varphi(N,\lambda) \\ 0 & (1+\dfrac{\alpha}{\varepsilon}) & -p\theta(N,\lambda) & -p\varphi'(N,\lambda) \\ \theta(\varepsilon,\lambda) & \varphi(\varepsilon,\lambda) & -\theta(\varepsilon,\lambda) & -\varphi(\varepsilon,\lambda) \\ \left(1+\dfrac{\alpha}{\varepsilon}\right)\theta'(\varepsilon,\lambda) & \left(1+\dfrac{\alpha}{\varepsilon}\right)\varphi'(\varepsilon,\lambda) & -\theta(\varepsilon,\lambda) & -\varphi'(\varepsilon,\lambda) \end{vmatrix} = 0 \qquad (9)$$

By (5) we'll have the identity

$$\theta(x,\lambda)\varphi'(x,y)-\theta'(x,y)\varphi(x,y)=1 \qquad (10)$$

According to (8) and (10), as $\varepsilon\to 0^+$ equation (9) is scheduled in the form

$$p^2-[\theta(N,\lambda)+\varphi'(N,\lambda)-\alpha\lambda\varphi(N,\lambda)]p+1=1 \qquad (11)$$

Since, this equation has always the root p, and obviously its roots are non-zero, the reduced reasonings prove the existence of the non-trivial solution $\psi(x,N)$ of equation (1) possessing the property $\psi(x,\lambda)=p\psi(x+N,\lambda)$.

Introducing the function of parameter λ

$$F(\lambda)=\frac{1}{2}\left[\theta(N,\lambda)+\varphi'(N,\lambda)-\alpha\lambda\varphi(N,\lambda)\right],$$

we rewrite equation (4) in the form

$$p^2-2F(\lambda)p+1=0 \qquad (12)$$

The roots of this equation are determined by the formula:

$$p_{1,2}=F(\lambda)\pm\sqrt{F^2(\lambda)-1} \qquad (13)$$

The function $F(\lambda)$ is said to be a **discriminant**, the functions $p_1(\lambda)$, $p_2(\lambda)$ the multiplicators of equation (1).

It follows from (13) that

$$p_1\cdot p_2=1 \qquad (14)$$

If for the given λ the function $F^2(\lambda)-1$ is non-zero, equation (12) has two different roots p_1, p_2 and consequently, there exist two solutions $\psi_1(x,\lambda)$, $\psi_2(x,\lambda)$ of equations (1) such that

$$\psi_1(x,\lambda)=p_1\psi_1(x+N,\lambda), \quad \psi_2(x,\lambda)=p_2\psi_2(x+N,\lambda).$$

It is easy to see that $\psi_1(x,\lambda)$ and $\psi_2(x,\lambda)$ will be linear independent. Since $p_1\neq 0$ $p_2\neq 0$, there exist the numbers $\mu_1=\mu_2(\lambda)$ and $\mu_2=\mu_2(\lambda)$ such that

$$e^{N\mu_1}=p_1, \quad e^{N\mu_2}=p_2$$

The quantities μ_1 and μ_2 are called characteristic exponents of equation (1).

Assume

$$Y_1(x,\lambda) = e^{-\mu_1 x}\psi_1(x,\lambda), \quad Y_2(x,\lambda) = e^{-\mu_2 x}\psi_2(x,\lambda).$$

Obviously, the functions $Y_1(x,\lambda)$ and $Y_2(x,\lambda)$ are periodic with respect to x with period N.

Thus, for $F^2(\lambda) - 1 \neq 0$ the general solution of equation (1) is represented in the Floquet's form:

$$y(x,\lambda) = c_1 e^{\mu_1 x} Y_1(x,\lambda) + c_2 e^{\mu_2 x} Y_2(x,\lambda).$$

If $\lambda \in (-\infty, \infty)$, by the real number α and the function $q(x)$, the solutions $\theta(x,\lambda)$ and $\varphi(x,\lambda)$, consequently the function $F(\lambda)$ as well will by real.

Consider in detail the possible cases.

1) $F(\lambda) > 1$. According to (14) there exists such a non-zero real number μ that $p_1 \neq p_2 \quad p_1 > 0, \quad p_2 > 0$

$$p_1 = e^{N\mu}, \quad p_2 = e^{-N\mu}.$$

Consequently, in this case, the general solution of equation (1) is represented in the form

$$y(x,\lambda) = c_1 e^{\mu_1 x} Y_1(x,\lambda) + c_2 e^{-\mu_2 x} Y_2(x,\lambda),$$

wherein μ is a non-zero real number.

2) Let $F(\lambda) < -1$. In this case, again $p_1 \neq p_2$, however $p_1 < 0, \quad p_2 < 0$. Therefore, the general solution of equation (1) will be represented in the form

$$y(x,\lambda) = c_1 e^{\left(\mu + i\frac{\pi}{N}\right)} Y_1(x,\lambda) + c_2 e^{-\left(\mu + i\frac{\pi}{N}\right)x} Y_2(x,\lambda),$$

where μ is a non-zero real number.

3) Let $-1 < F(\lambda) < 1$. In this case, p_1 and p_2 are different, non-real and complex by conjugated. Therefore, it follows from (14) that $|p_1| = |p_2| = 1$. Consequently, the general solution of equation (1) is represented in the form

$$y(x,\lambda) = c_1 e^{i\beta x} Y_1(x,\lambda) + c_2 e^{-i\beta x} Y_2(x,\lambda),$$

where μ is a non-zero real number.

4) Let $F^2(\lambda) - 1 = 0$. In this case

$$p_1 = p_2 \overset{def}{=} p = \begin{cases} 1, & \text{for } F(\lambda) = 1 \\ -1, & \text{for } F(\lambda) = -1 \end{cases} \tag{15}$$

and condition (1) will have at least one non-trivial solution $\psi_1(x,\lambda)$ possessing the property

$$\psi_1(x,\lambda) = p\psi_1(x+N,\lambda). \tag{16}$$

Denote by $C(x,\lambda)$ the solution of equation (1) linearly independent with $\psi_1(x,\lambda)$. Since $C(x+N,\lambda)$ also satisfies equation (1), then

$$C(x+N,\lambda) = d_1\psi_1(x,\lambda) + d_2 C(x,\lambda). \tag{17}$$

From (16) and (17) we have

$$W\left[\psi_1(x+N,\lambda), C(x+N,\lambda)\right] = p^{-1} d_2 W\left[\psi_1(x,\lambda), C(x,\lambda)\right].$$

Since the wron skian of two solutions of equation (1) is independent of x, from the last relation we get $d_2 = p$. Consequently, (17) accepts the form

$$C(x+N,\lambda) = d_1\psi_1(x,\lambda) + pC(x,\lambda) \tag{18}$$

Consider the following two possible cases.

a) Let $d_1 = 0$. Then from (18) we have

$$C(x+N,\lambda) = pC(x,\lambda) \tag{19}$$

Taking into consideration (15), (16) and (19) we get that in the considered case all the solutions of equation (1) are periodic for $F(\lambda) = 1$ and antiperiodic for $F(\lambda) = -1$. We can show that the equality $d_1 = 0$ holds iff

$$\theta'(N,\lambda) = \alpha\lambda \ \text{ and } \ \lambda = (N,\lambda) = 0$$

b) Let $d_1 \neq 0$ i.e. even if one of the numbers $\theta'(N,\lambda) - \alpha\lambda$ and $\varphi(N,\lambda)$ is non-zero. By (15) we have $p = e^{\mu N}$, where

$$\mu = \begin{cases} 0, & \text{for } \ F(\lambda) = 1 \\ i\frac{\pi}{N}, & \text{for } \ F\lambda) = -1 \end{cases} \tag{20}$$

Assume

$$Y_1(x,\lambda) = e^{-\mu x}\psi_1(x,\lambda)$$
$$Y_2(x,\lambda) = e^{-\mu x}C(x,\lambda) - \frac{d_1}{Np}xY_1(x,\lambda)$$

It is easy to verify that

$$Y_1(x+N,\lambda) = Y_1(x,\lambda), \quad Y_2(x+N,\lambda) = Y_2(x,\lambda)$$

Thus, in this case, the fundamental system of solutions of equations (1) is of the form:

$$\psi_1(x,\lambda) = e^{\mu x}Y_1(x,\lambda)$$
$$C(x,\lambda) = e^{\mu x}\left[\frac{d_1}{Np}xY_1(x,\lambda) + Y_2(x,\lambda)\right]$$

where μ is determined by equality (20).

5) Now, let's consider the case when λ is a non-real number. In this case, two cases are possible:

a) $F(\lambda)$ is real. In this case, one of the cases 1)-4) mentioned above will hold.

b) $F(\lambda)$ is non-real. In this case, by (13) the multiplicators p_1 and p_2 will be different. Furthermore, p_1 and p_2 may not equal unit by modulus. Indeed, if $p_1 = e^{i\gamma}$, $\gamma \in (-\infty,\infty)$ then by (14) $p_2 = e^{-i\gamma}$ and $2F(\lambda) = p_1 + p_2 = 2Cos\gamma$, i.e. $F(\lambda)$ is real and this contradicts our assumption. Consequently, there exists such a complex number μ satisfying the condition $\operatorname{Re}\mu \neq 0$ that

$$p_1 = e^{N\mu}, \quad p_1 = e^{-N\mu}$$

and equation (1) will have two linearly independent solutions in the form

$$\psi_1(x,\lambda) = e^{\mu x}Y_1(x,\lambda), \quad \psi_2(x,\lambda) = e^{-\mu x}Y_2(x,\lambda),$$

where $Y_1(x,\lambda)$ and $Y_2(x,\lambda)$ are periodic with respect to x functions with period equal N.

It follows the following theorem.

Theorem. *For the given λ equation (1) is unstable if $|F(\lambda)| > 1$ and is stable if $|F(\lambda)| < 1$. But if $|F(\lambda)| = 1$, then in the case $\theta'(N,\lambda) - \alpha\lambda = \varphi(N,\lambda) = 0$ equation (1) is stable, in the contrary case it will be conditionally stable.*

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Manaf Dzh. Manafov
Department of Mathematics, Faculty of Arts and Sciences,
University of Adiyaman, 02040-Adiyaman / TURKEY
E-mail: mmanafov@adiyaman. edu.tr; manafov 57@gmail.com

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