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SPECTRAL ANALYSIS OF ONE-DIMENSIONAL BIHARMONIC OPERATOR WITH δ'' POTENTIAL

Abstract

In the given paper, the self-adjoint operator A corresponding to the differential operator $\frac{d^4}{dx^4}+\beta\delta''(x)$ is determined in the space $L_2(R)$. Explicit representation of the resolvent of the operator A is found. It is shown that $\sigma_{ese}(A)=\sigma_{ac}(A)=[0,+\infty)$. The negative eigen value of the operator A and corresponding normed eigen function are found.

In the given paper, in the space $L_2(R)$ we find a self adjoint operator corresponding to the formal differential expression

$$\frac{d^4}{dx^4}+\beta\delta''(x)\,, \tag{1}$$

where $\delta(x)$ is Dirac's function, $\delta''(x)$ is its generalized second order derivative, $\beta\in R=(-\infty,+\infty)$ is fixed.

It is known that while determining the operator corresponding to differential expression (1), there arise difficulties related with strong singularity of the distribution $\delta''(x)$. Since $\delta''(x)$ belongs to the Sobolev space $W_2^{-\frac{5}{2}-\varepsilon}(R)\,(\varepsilon>0)$, the known methods (see. for example [1], [2]) are not applicable for determining the operator (1). The methods stated in these papers are not suitable for determining the operator $\frac{d^4}{dx^4}+q(x)$ with the generalized potential $q(x)\in W_2^{-2}(R)$.

In this paper the way for the definition of the operator (1) based on the formula of the product of $\delta''(x)$ by piecewise differentiable functions $f(x)$, for which first and second classic derivatives have first order discontinuous at the point $x=0$.

This formula is of the form ([3]):

$$\delta''(x)\cdot f(x)=\frac{f''(+0)+f''(-0)}{2}\cdot\delta(x)-\left[f'(+0)+f'(-0)\right]\delta'(x)+f(0)\,\delta''(x)\,. \tag{2}$$

Formula (2) allows to give sense to formal operator (1) as a self-adjoint operator in the space $L_2(R)$.

Let $D(A)$ be a set of functions $f\in W_2^4(R\backslash\{0\})\cap W_2^1(R)$ satisfying the boundary conditions:

$$f'(-0)-f'(+0)=\beta f(0)\,, \tag{3}$$

$$f''(+0)-f''(-0)=\beta\left[f'(+0)+f'(-0)\right], \tag{4}$$

$$f'''(-0)-f'''(+0)=\frac{\beta}{2}\left[f''+(0)+f''(-0)\right]. \tag{5}$$

In the space $L_2(R)$ determine the operator A:

$$Af=\frac{d^4f}{dx^4}+\beta\delta''(x)\cdot f,\,f\in D(A)\,,$$

where the derivative $\frac{d^4 f}{dx^4}$–is understood in the sense of distributions, and the product $\delta''(x)\cdot f$ is determined by formula (2).

By boundary conditions (3)-(5), the operator A is a closed symmetric operator in the space $L_2(R)$.

In this paper, the self-adjointness of the operator A is proved. Furthermore, the resolvent $R_z(A)$ is found and the structure of the spectrum of the operator A is researched.

Theorem 1. *The operator A is a self-adjoint operator in the space $L_2(R)$. The resolvent $R_z(A)$ is an integral operator in $L_2(R)$ an the integral trace formula $G(x,y;z)$ for $z=-\lambda^4\ (\lambda>0)\,,-\lambda^4\in\rho(A)$ has the representation*

$$G\left(x,y;-\lambda^4\right)=\frac{1}{2\lambda^3}e^{-\frac{\lambda}{\sqrt{2}}|x-y|}\sin\left(\frac{\lambda}{\sqrt{2}}|x-y|+\frac{\pi}{4}\right)-$$

$$-\frac{\beta}{\sqrt{2}\lambda^3}e^{-\frac{\lambda}{\sqrt{2}}(|x|+|y|)}\left\{\frac{1}{\sqrt{2}\lambda+\beta}\sin\frac{\lambda}{\sqrt{2}}x\cdot\sin\frac{\lambda}{\sqrt{2}}y-\frac{1}{\left(2\sqrt{2}\lambda-\beta\right)^2+\beta^2}\times\right.$$

$$\left.\times\left[\left(\beta-2\sqrt{2}\lambda\right)\cos\frac{\lambda}{\sqrt{2}}(|x|+|y|)+\beta\sin\frac{\lambda}{\sqrt{2}}(|x|+|y|)\right]\right\}. \qquad (6)$$

Proof. By closeness and symmetry of A, for proving the self-adjointness of the operator A is suffices to show that its resolvent set contains even if one real number ([4], Corollary of theorem X.I).

In the space $L_2(R)\,,$ solve the equation

$$Af+\lambda^4=g \qquad (g\in L_2(R)\,,\quad \lambda>0)\,.$$

By formula (2), we can write this equation as follows

$$\frac{d^4 f}{dx^4}+\frac{\beta}{2}\left[f''(+0)+f''(-0)\right]\delta(x)-$$

$$-\beta\left[f'(+0)+f'(-0)\right]\delta'(x)+\beta f(0)\,\delta''(x)+\lambda^4 f=g. \qquad (7)$$

Apply the Fourier transformation F to equation (7). Then, taking into account

$$F\left[\frac{d^4 f}{dx^4}\right]=\xi^4 F[f]\,,\quad F[\delta(x)]=1,\quad F\left[\delta'(x)\right]=-i\xi,\quad F\left[\delta''(x)\right]=-\xi^2,$$

we get

$$F[f]=\frac{1}{\xi^4+\lambda^4}F[g]-\frac{\beta}{2}\left[f''(+0)+f''(-0)\right]\cdot\frac{1}{\xi^4+\lambda^4}-$$

$$-\beta i\left[f'(+0)+f'(-0)\right]\frac{\xi}{\xi^4+\lambda^4}+\beta f(0)\cdot\frac{\xi^2}{\xi^4+\lambda^4}.$$

Now, apply the Fourier inverse transformation F^{-1} and use the known formulae

$$F^{-1}\left[\frac{1}{\xi^4+\lambda^4}\right]=\frac{1}{2\lambda^3}e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\lambda}{\sqrt{2}}|x|+\frac{\pi}{4}\right),$$

$$F^{-1}\left[\frac{\xi}{\xi^4+\lambda^4}\right]=\left(-\frac{i}{2\lambda^2}\right)e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\frac{\lambda}{\sqrt{2}}x,$$

$$F^{-1}\left[\frac{\xi^2}{\xi^4+\lambda^4}\right]=\frac{1}{2\sqrt{2}\lambda}e^{-\frac{\lambda}{\sqrt{2}}|x|}\left(\cos\frac{\lambda}{\sqrt{2}}|x|-\sin\frac{\lambda}{\sqrt{2}}|x|\right)=$$

$$=\frac{1}{2\lambda}e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\pi}{4}-\frac{\lambda}{\sqrt{2}}|x|\right),$$

$$F^{-1}\left[\frac{1}{\xi^4+\lambda^4}F[g]\right]=F^{-1}\left[\frac{1}{\xi^4+\lambda^4}\right]*g=G_0*g,$$

where

$$G_0=G_0\left(x,y;-\lambda^4\right)=\frac{1}{2\lambda^3}e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\lambda}{\sqrt{2}}|x|+\frac{\pi}{4}\right).$$

As a result we get

$$f(x)=G_0*g-\frac{\beta}{4\lambda^3}\left[f''(+0)+f''(-0)\right]e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\lambda}{\sqrt{2}}|x|+\frac{\pi}{4}\right)-$$

$$-\frac{\beta}{2\lambda^2}\left[f'(+0)+f'(-0)\right]e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\frac{\lambda}{\sqrt{2}}|x|+\frac{\beta}{2\lambda}f(0)\,e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\pi}{4}-\frac{\lambda}{\sqrt{2}}|x|\right).\quad(8)$$

Find the quantities $f(0)$, $f'(+0)+f'(-0)$ and $f''(+0)+f''(-0)$. Set $x=0$ in (8). Then

$$f(0)=\frac{2\sqrt{2}\lambda}{2\sqrt{2}\lambda-\beta}(G_0*g)(0)-\frac{\beta}{2\lambda^2\left(2\sqrt{2}\lambda-\beta\right)}\left[f''(+0)+f''(-0)\right].\qquad(9)$$

Calculate the derivative $f'(x)$ for $x\neq 0$ by formula (8) and in the obtained equality pass to limit as $x\to+0$ and $x\to-0$:

$$f'(+0)=(G_0*g)'_x(0)-\frac{\beta}{2\sqrt{2}\lambda}\left[f'(+0)+f'(-0)\right]-\frac{\beta}{2}f(0),$$

$$f'(-0)=(G_0*g)'_x(0)-\frac{\beta}{2\sqrt{2}\lambda}\left[f'(+0)+f'(-0)\right]+\frac{\beta}{2}f(0),$$

Summing these equalities, we get:

$$f'(+0)+f'(-0)=\frac{2\sqrt{2}\lambda}{\sqrt{2}\lambda+\beta}(G_0*g)'_x(0).\qquad(10)$$

Similarly,we calculate the second derivative $f''(x)$ for $x\neq 0$ and pass to limit as $x\to+0$ and $x\to-0$. Then we get the equalities

$$f''(+0)=(G_0*g)''_x(0)+\frac{\beta}{4\sqrt{2}\lambda}\left[f''(+0)+f''(-0)\right]+\frac{\beta}{2}\left[f'(+0)+f'(-0)\right]+\frac{\beta\lambda}{2\sqrt{2}\lambda}f(0),$$

$$f''(-0)=(G_0*g)''_x(0)+\frac{\beta}{4\sqrt{2}\lambda}\left[f''(+0)+f''(-0)\right]-$$

$$-\frac{\beta}{2}\left[f'(+0)+f'(-0)\right]+\frac{\beta\lambda}{2\sqrt{2}\lambda}f(0).$$

Putting together these equalities, we get

$$f''(+0)+f''(-0)=\frac{4\sqrt{2}\lambda}{\sqrt{2}\lambda-\beta}\left(G_0*g\right)''_x(0)+\frac{2\beta\lambda^2}{2\sqrt{2}\lambda-\beta}f(0).$$

Taking into account (9) in the last equality, after some transformations we find

$$f''(+0)+f'(-0)=\frac{4\sqrt{2}\lambda\left(2\sqrt{2}\lambda-\beta\right)}{\left(2\sqrt{2}\lambda-\beta\right)^2+\beta^2}\left(G_0*g\right)''_x(0)+$$

$$+\frac{4\sqrt{2}\beta\lambda^3}{\left(2\sqrt{2}\lambda-\beta\right)^2+\beta^2}\left(G_0*g\right)(0). \tag{11}$$

Find $f(0)$ by formulae (9)

$$f(0)=\frac{2\sqrt{2}\lambda\left(2\sqrt{2}\lambda-\beta\right)}{\left(2\sqrt{2}\lambda-\beta\right)^2+\beta^2}\left(G_0*g\right)(0)-\frac{2\sqrt{2}\beta}{\lambda\left[\left(2\sqrt{2}\lambda-\beta\right)^2+\beta^2\right]}\left(G_0*g\right)''_x(0). \tag{12}$$

Then, we have

$$\left(G_0*g\right)(0)=\frac{1}{2\lambda^3}\int_R e^{-\frac{\lambda}{\sqrt{2}}|y|}\sin\left(\frac{\lambda}{\sqrt{2}}|y|+\frac{\pi}{4}\right)g(y)\,dy,$$

$$\left(G_0*g\right)'_x(0)=\frac{1}{2\lambda^2}\int_R e^{-\frac{\lambda}{\sqrt{2}}|y|}\sin\frac{\lambda}{\sqrt{2}}y\cdot g(y)\,dy,$$

$$\left(G_0*g\right)''_x(0)=\frac{1}{2\lambda}\int_R e^{-\frac{\lambda}{\sqrt{2}}|y|}\sin\left(\frac{\lambda}{\sqrt{2}}|y|-\frac{\pi}{4}\right)\cdot g(y)\,dy.$$

Considering these expressions and relations (10), (11) and (12) in (8), after simple transformations we get

$$f(x)=\int_R G\left(x,y;-\lambda^4\right)g(y)\,dy, \tag{13}$$

where the integral trace formula $G\left(x,y;-\lambda^4\right)$ has the representation (6).

It follows from representation (6) that the operator B determined by the equality

$$Bf=\int_R G\left(x,y;-\lambda^4\right)f(y)\,dy,\quad f\in L_2(R)$$

is a bounded operator in the space $L_2(R)$ if $\lambda>0$ for $\beta\ge 0$ and $\lambda>0$, $\lambda\ne-\frac{\beta}{\sqrt{2}}$ for $\beta<0$. Consequently, for such values of λ, there exists a resolvent $R_{-\lambda^4}(A)=\left(A+\lambda^4 I\right)^{-1}=B$. Thus, the resolvent set $\rho(A)$ of the operator A contains real numbers and therefore the operator A is self adjoint in the space $L_2(R)$.

Continuing $G\left(x,y;-\lambda^4\right)$ analytically in λ on a complex plane, we get $R_z(A)$, $z\in\rho(A)$ that is an integral operator and $\rho(A)$ is of the from:

$$\rho(A)=C\backslash[0,+\infty),\quad \text{if}\quad \beta\ge 0;$$

$$\rho(A) = C \backslash [0,+\infty) \cup \left\{-\frac{\beta^4}{4}\right\}, \quad \text{if} \quad \beta < 0.$$

Theorem 1 is proved.

Classification of the points of the spectrum of the operator A is described by the following theorem.

Theorem 2. *The essential spectrum of the operator A coincides with continuous part of its spectrum, moreover*

$$\sigma_{ess}(A) = \sigma_{ac}(A) = [0,+\infty). \tag{14}$$

If $\beta < 0$, the operator A has only one negative prime eigen value $\lambda_0 = -\frac{\beta^4}{4}$. The corresponding normed eigen function is of the form:

$$f(x) = \sqrt{2|\beta|}e^{\frac{\beta}{2}|x|}\sin\frac{\beta}{2}x. \tag{15}$$

In the case $\beta \geq 0$, the operator A has no eigen values.

Proof. Relations (14) are proved by means of the standard method that is usually used while investigating such problems. Namely, the Weyl theorem on essential spectrum ([5], theorem XIII, 14) and a theorem on preservation of absolutely continuous parts of the spectra of perturbed and non-perturbed operators ([6], ch. X, theorem 42) are used. Application of these theorems leads to equalities (14).

Find the negative eigen values of the operator A in the case $\beta < 0$. Let $-\lambda^4\,(\lambda > 0)$ be a negative eigen value of the operator A, and $f(x)$ be a an appropriate eigen function. Then $Af + \lambda^4 f = 0$. Assuming $g(x) = 0$ in (8), we get

$$f(x) = -\frac{\beta}{4\lambda^3}\left[f''(+0) + f''(-0)\right]e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\lambda}{\sqrt{2}}|x| + \frac{\pi}{4}\right) -$$

$$-\frac{\beta}{4\lambda^2}\left[f'(+0) + f'(-0)\right]e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\frac{\lambda}{\sqrt{2}}x + \frac{\beta}{2\lambda}f(0)\,e^{-\frac{\lambda}{\sqrt{2}}|x|}\sin\left(\frac{\pi}{4} - \frac{\lambda}{\sqrt{2}}|x|\right). \tag{16}$$

For brevity we denote

$$c_1 = f'(+0) + f'(-0),\, c_2 = f''(+0) + f''(-0),\, c_3 = f(0).$$

It is obvious that $f(x) \neq 0$ iff $c_1^2 + c_2^2 + c_3^2 \neq 0$. From representation (16) for determining the quantities c_1, c_2 and c_3 we get the system of equations:

$$\begin{cases} \beta c_2 + 2\lambda^2\left(2\sqrt{2}\lambda - \beta\right)c_3 = 0, \\ \left(\sqrt{2}\lambda + \beta\right)c_1 = 0, \\ \left(2\sqrt{2}\lambda - \beta\right)c_2 - 2\beta\lambda^2 c_3 = 0. \end{cases} \tag{17}$$

The system of equations (17) has a non-zero solution iff the determinant of this system equals zero:

$$\Delta = 2\lambda^2\left(\sqrt{2}\lambda + \beta\right)\left[\beta^2 + \left(2\sqrt{2}\lambda - \beta\right)^2\right] = 0.$$

Hence we find $\lambda = -\frac{\beta}{\sqrt{2}}$. For this values of λ, from system (17) we have $c_2 = c_3 = 0$. Consequently, the system of equations (17) has a non-zero solution only for $\lambda = -\frac{\beta}{\sqrt{2}}$.

Therefore, $\lambda_0 = -\frac{\beta^4}{4}$ is a negative eigen value of the operator A and appropriate eigen function is of the from:

$$f(x) = ce^{\frac{\beta}{2}|x|}\sin\frac{\beta}{2}x,$$

where $c \neq 0$ is an arbitrary constant.

Find the normed eigen function. Choose the constant c from the condition $\|f\|_{L_2} = 1$, i.e.

$$2c^2\int_0^{+\infty} e^{\beta x}\sin^2\frac{\beta}{2}xdx = 1.$$

The non-singular integral in the last equality is easily calculated:

$$\int_0^{+\infty} e^{\beta x}\sin^2\frac{\beta}{2}xdx = -\frac{1}{4\beta}.$$

Therefore $c^2 = 2|\beta|$. Choosing $c = \sqrt{2|\beta|}$, we get that the normed eigen function is of the from (15).

It is directly verified that when $\beta \geq 0$, the operator A has no eigen values. Theorem 2 is proved.

The basic results of the paper were announced by the author in [7].

References

[1]. Savchuk A. M., Shkalikov A. A. *The Sturm-Liouville operators with potentials-distributions.* Trudy Moskow. Matem. Obsh. 2003, vol. 64, pp. 159-212 (Russian).

[2]. Albeverio S., Gesztesy F., Hoegh-Krohn R, Holden H. *Solvable models in quantum mechanics.* M., Mir, 1991, 568 p. (Russian).

[3]. Heydarov A. H. *On products of distributions by discontinuous functions.* Vestnic Bakinskoqo Univ. ser. fiz. mat. Nauk, 2008, pp. 13-18 (Russian).

[4]. Reed M., Saimon B. *Methods of modern mathematical physics.* Vol. 2, M., Mir, 1978, 395 p. (Russian).

[5]. Reed M., Saimon B. *Methods of modern mathematical physics.* Vol. 4, M., Mir, 1982, 428 p. (Russian).

[6]. Kato T. *Theory of perturbations of linear operators* M., Mir, 1972, 740p. (Russian).

[7]. Heydarov A.H. *On spectrum of one dimensional biharmonic operator with δ''-potential.* Abstracts of the XII International Conference on mathematics and mechanics devoted to 70-th birthday of corr. Number of NASA prof. B.A. Iskenderov. Baku, 2006, p. 66.

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