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ON FINITENESS OF THE MOMENTS OF INTERSECTION TIME OF ONE-SIDED NONLINEAR BOUNDARY BY A RANDOM WALK

Abstract

Necessary and sufficient condition for finiteness of the first passage moment of one - sided nonlinear boundary by a random walk and the value of the random walk at the first passage moment is found.

Let ξ, ξ_1, ξ_2 be independent identical random variables determined on some probability space $(\Omega, \mathfrak{F}, P)$.

In the paper we consider the family of the first passage moments

$$\tau_a = \inf\{n \geq 1 : S_n > f_a(n)\}$$

of the sums $S_n = \xi_1 + ... + \xi_n$, $n \geq 1$ for a positive nonlinear boundary $f_a(t)$, $t > 0$, $a > 0$. The goal of the paper is to study the problem on finiteness of the moments $E\tau_a^r$ and $S_{\tau_a}^r$ for $r \geq 1$. As it was noted in the papers [3], [12], the study of the finiteness problem of the first and higher moments of the first passage of boundaries often is of interest in applied fields of the theory of random processes. In the case $f_a(t) \equiv a$ and $f_a = at^\gamma$, $\gamma \in (0,1)$ such problems were studied in $[1], [2], [3]$. The problems on finiteness and asymptotic behavior as $a \to \infty$ of mean $E\tau_a$ and variance $D\tau_a$ were investigated in the papers [3] – [8].

In the case $f_a(t) \equiv f(t)$ the problem on finiteness of the mean ES_τ was studied in $[4], [6]$. For the first passage time for the level of first order autoregressional sequence $(AR(1))$ this problem was studied in [5].

The following theorems are the basic results of the present paper.

Theorem 1. *Let for each a the function $f_a(t)$ be continuous and $f_a(t) = 0(t)$ as $t \to \infty$.*

Then the implications

$$E\left(\xi^-\right)^r \Longleftrightarrow E(\tau_a)^r$$

where $\xi^- = \max(0, -\xi)$ are valid for each $a > 0$ and $r \geq 1$.

Theorem 2. *Let the conditions of theorem 1 be fulfilled for the nonlinear boundary $f_a(t)$. Then the implication*

$$E|\xi|^r < \infty \Longrightarrow ES_{\tau_a}^r < \infty$$

is valid for all $a > 0$ and $r \geq 1$.

Corollary 1. *Let $0 < E\left(\xi_1^-\right) < \infty$. Then τ_a is an eigen random variable, i.e.*

$$P(\tau_a < \infty) = 1.$$

Corollary 2. *Let $f_a(t) = at^\beta$, $0 \leq \beta < 1$. Then the implications $E(S_{\tau_a})^r < \infty \Longleftrightarrow E\left(\xi_1^+\right)^r < \infty$ and $E\left(\xi_1^-\right)^{\beta r} < \infty$, where $\xi^+ = \max(0, \xi)$, are valid.*

2. Auxiliary facts. To prove the basic results we'll need some facts from the theory of linear boundary value problems for random walks.

Assume $t_x = \inf\{n \geq 1 : S_n > x\}$ is the first passage of the random walk S_n for the level $x \geq 0$.

Lemma 1. *The implications*

$$E\left(\xi^-\right)^r < \infty \Longleftrightarrow Et_x^r < \infty \text{ and } E\left(\xi^+\right) < \infty \Longleftrightarrow ES_{t_x}^r < \infty$$

are valid.

Lemma 1 was proved in the paper [2] (see also [3]).

By $\{N(t), t \geq 0\}$ we denote a family of stopping times relative to σ-algebras $\mathfrak{F}_n = \sigma(\xi_1 ... \xi_n)$, $n \geq 1$ and $\mathfrak{F}_0 = \{\varnothing, \Omega\}$.

Lemma 2. *If* $E|\xi_1|^r < \infty$, $r \geq 1$ *and* $EN(t) < \infty$, *then* $E\left|\xi_{N(t)}\right| \leq \left(E|\xi_1|^r\right)^{1/r} \leq \left(EN(t)\, E|\xi_1|^r\right)^{1/r}$.

Statement of Lemma 2 was proved in [3].

3. Proof of theorem 1. At first assume that $E\left(\xi_1^-\right)^r < \infty$. By the made assumptions for the boundary $f_a(t)$ it is easy to understand that for each $a > 0$ there exist the numbers $0 < \theta < \mu = E\xi_1^-$ and $c > 0$ such that for sufficiently large t

$$f_a(t) \leq c + \theta\, t, \tag{1}$$

where $c = c(a)$ and $\theta = \theta(a)$.

Assume $\nu_a = \inf\{n \geq 1 : S_n > c + \theta n\}$ and $S_n' = S_n - \theta n = \sum_{k=1}^{n}(\xi_k - \theta) = \sum_{k=1}^{n}\xi_k^1$, $\xi_k' = \xi_k - \theta$.

It is clear that $\nu_a = \inf\{n \geq 1 : S_n' > c + \theta\}$ and $\left(\xi_k'\right)^- \leq \xi_k^- + \theta$.

By means of the Minkovskiy inequality it is easy to show that $E\left(\left(\xi_k'\right)^-\right)^r < \infty$. Therefore, by lemma 1

$$E\nu_a^r < \infty. \tag{2}$$

Then (2) yields $E(\tau_a)^r < \infty$, since $\tau_a \leq \nu_a$.

Now, let $E(\tau_a)^r < \infty$. Consider the first appearance moment

$$\tau_+ = \inf\{n \geq 1 : S_n > 0\}$$

of the positive sum S_n.

It is clear that $\tau_+ \leq \tau_a$, since $f_a(n) > 0$ for $a > 0$ and $n \geq 1$. Therefore, for $r \geq 1$ we have

$$E\tau_+^r \leq E\tau_a^r < \infty.$$

Then lemma 1 yields $E\left(\xi_1^-\right)^r < \infty$.

Proof of theorem 2. Statement of this theorem for $r = 1$ follows from theorem 1 and from the Wald identity $ES_{\tau_a} = \mu E\tau_a < \infty$. Let $r > 1$. By definition of the quantity τ_a

$$S_{\tau_a} > f_a(\tau_a) \text{ and } S_{\tau_a - 1} \leq f_a(\tau_a - 1).$$

Hence, taking into account $S_{\tau_a} = S_{\tau_a - 1} + \xi_{\tau_a}$ we get

$$f_a(\tau_a) < S_{\tau_a} \leq f_a(\tau_a - 1) + \xi_{\tau_a}. \quad (3)$$

By (1) from the right hand side of (3) we have

$$S_{\tau_a} \leq c + \theta(\tau_a - 1) + \xi_{\tau_a}.$$

In view of $\xi_{\tau_a} \leq \xi^+_{\tau_a}$ we have

$$S_{\tau_a} \leq c + \theta\tau_a + \xi^+_{\tau_a}. \quad (4)$$

By means of the Minkovskiy inequality and lemma 2, from (4) we find

$$\left(E(S_{\tau_a})^r\right)^{1/r} \leq c + \theta\left(E\tau_a^r\right)^{1/r} + E\left(\tau_a E\left(\xi_1^+\right)^r\right)^{1/2}. \quad (5)$$

It follows from the condition of the proved theorem that $E\left(\xi_1^-\right)^r < \infty$ and $E\left(\xi_1^+\right)^r < \infty$. Then by theorem 1, it follows from (5) that $E(S_{\tau_a})^r < \infty$.

Statement of Corollary 1, directly follows from theorem 1, since by condition of corollary 1 $E\tau_a < \infty$ for all $a > 0$.

Statement of Corollary 2 follows from bilateral inequality $a(\tau_a)^\beta < S_{\tau_a} \leq a(\tau_a)^\beta < \xi_{\tau_a}$, since by the monotonicity of the boundary $f_a(t) = at^\beta$ the quantity E_{τ_a} is positive and $S_{\tau_a} \geq \xi_1^+$.

3. For applying theorems 1 and 2 consider the following model from the theory of branching processes ([9], [13]). Let at the initial moment at the points $x \in (0, l)$, $0 < l < \infty$ there be n particles of type T_1. These particles are subjected to the reaction diffusion are reproduced (produce the particles of type T_2) and absorbed on the boundary (at the points $x = 0$ and $x = l$), moreover, the parts of the particles of final type T_2 are not reproduced, don't perish and are not absorbed on the boundary (at the points $x = 0$ and $x = l$). Thus, final parts are the end reaction products.

Let S_n be the number of particles of type T_2 and ξ_k be the number of particles of type T_2 obtained from the k-th particle (at any numeration) of type T_2 available at initial moment at the point $x \in (0, l)$.

It is clear that $S_n = \xi_1 + ... + \xi_n$, where $\xi_1 ... \xi_n$ are independent identical random variables.

Consider the first passage time

$$T_a = \inf\{n \geq 1 : S_n > a\ n^\gamma\}, \ a > 0, \ 0 \leq \gamma < 1.$$

The first passage time T_a often arises in many applied problems of the theory of random processes and mathematical statistics ([1], [9], [10], [11]).

Theorem 3. *For $r \geq 1$ it holds*

$$ET_a^r < \infty \text{ and } ES_{T_a}^r = \infty.$$

Proof. It is proved in the paper ([14], remark 1, p. 381) that the distribution of the random variable ξ_k belongs to the domain of attraction of the stable law with the parameters $\alpha = \dfrac{1}{2}$ and $\beta = 1$. Therefore, distribution of the random variable ξ_1 is entirely concentrated on the right semi-axis $(0, \infty)$ ([13]). This means that $P(\xi_1 \geq 0) = 1$. Then it follows from theorem 1 that $ET_a^r < \infty$, and from Corally 2 we have $ES_{T_a}^r = \infty$ for all $r \geq 1$, since $E\left(\xi_1^+\right)^r = \infty$ $(see\ [13])$.

Remark. Statement of theorem 3 for $r = 1$ and $\gamma = 0$ follows from the Wald identity and theorem III. 3.1 of the paper [3].

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Received September 10, 2009; Revised December 02, 2009.