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A PRIORI MORREY ESTIMATES FOR POISSON EQUATION

Abstract

Let Ω a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and let u be a solution of the classical Poisson problem in Ω; i.e.,

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L_{p,\lambda}(\Omega)$, $1 \le p < \infty$ and $0 \le \lambda < n$.

The main goal of this paper is to prove the following Morrey a priori estimate

$$\|u\|_{W^2_{p,\lambda}(\Omega)} \le C\|f\|_{L_{p,\lambda}(\Omega)}.$$

1. Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}^n_+$ we denote $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$, $D^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ and

$$W^k_p(\Omega) = \{v \in L_p(\Omega) : D^\alpha v \in L_p(\Omega) \ \forall \ |\alpha| \le k\}.$$

For $1 \le p < \infty$ and $0 \le \lambda < n$, $L_{p,\lambda}(\Omega)$ is the space of measurable functions f defined in Ω such that

$$\|f\|_{L_{p,\lambda}(\Omega)} = \sup_I \left(\frac{1}{|I|^\lambda} \int\limits_I |f(x)|^p dx \right)^{1/p} < \infty$$

and $W^k_{p,\lambda}(\Omega)$ is the space of functions such that

$$\|f\|_{W^k_{p,\lambda}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha f\|^p_{L_{p,\lambda}(\Omega)} \right)^{1/p} < \infty.$$

Let Γ be the standard fundamental solution of the Laplacian operator, namely,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \log |x|^{-1}, & n = 2, \\ \dfrac{1}{n(n-2)\omega_n} |x|^{2-n}, & n \ge 3, \end{cases}$$

with ω_n the area of the unit sphere in $\mathbb{R}^n$.

Given a function $f \in C_0^\infty(\mathbb{R}^n)$, it is a classic result that the potential u given by

$$u(x) = \int \Gamma(x - y) f(y) dy$$

is a solution of $-\triangle u = f$ in $\mathbb{R}^n$ and satisfies the estimate

$$\|u\|_{W_p^2(\mathbb{R}^n)} \le C\|f\|_{L_p(\mathbb{R}^n)} \tag{1.1}$$

for $1 < p < \infty$. Indeed, this estimate is a consequence of the Calderón-Zygmund theory of singular integrals (see for example [12]).

It is known that many results on Morrey estimates for maximal functions and singular integral operators have been obtained. In particular, generalizations of (1.1) to Morrey norms are known (see for example [13]).

On the other hand, a priori estimates like (1.1) for solutions of the Dirichlet problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

on smooth bounded domains Ω are also well known (see for example the classic paper by Agmon, Douglis and Nirenberg [2] where a priori estimates for general elliptic problems are proved).

Therefore, it is a natural question whether Morrey a priori estimates are valid also for the solution of the Dirichlet problem (1.2). In this paper we give a positive answer to this question, namely, we prove that

$$\|u\|_{W_{p,\lambda}^2(\Omega)} \le C\|f\|_{L_{p,\lambda}(\Omega)},$$

for $1 < p < \infty$ and $0 < \lambda < n$, where the constant C depends only on Ω and on the λ.

2. Preliminaries on Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [8], [9], introduced in 1938 by C. Morrey [10].

Let Ω a bounded domain in $\mathbb{R}^n$ and $d = diam\Omega$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r.

Definition 2.1. *Let $1 \le p < \infty$ and $0 \le \lambda < n$. We denote by $L_{p,\lambda}(\Omega)$ the Morrey space as the set of locally integrable functions $f(x)$, $x \in \Omega$ with the finite norm*

$$\|f\|_{L_{p,\lambda}(\Omega)} = \sup_{x\in\Omega,\, 0<r\le d} \left(r^{-\lambda} \int_{B(x,r)\cup\Omega} |f(x)|^p dx \right)^{1/p}.$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

Let T be a singular integral Calderon-Zygmund operator, briefly a Calderon-Zygmund operator, i. e., a linear operator bounded from $L_2(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $Tf \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy \qquad \text{a. e.} \quad \text{on} \;\; \mathrm{supp} f.$$

Here $K(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|K(x,y)| \le c_1|x-y|^{-n}$$

for all $x, y \in \mathbb{R}^n$, $x \ne y$, and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le c_1 \left(\frac{|x-x'|}{|x-y|}\right)^{\varepsilon} |x-y|^{-n},$$

whenever $2|x-x'| \le |x-y|$. Such operators were introduced in [5].

The operators M and T play an important role in real and harmonic analysis and applications (see, for example [13] and [14]).

F. Chiarenza and M. Frasca [4] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows:

Theorem 2.1. *Let $1 \le p < \infty$ and $0 \le \lambda < n$. Then for $p > 1$ the operator M is bounded in $L_{p,\lambda}(\mathbb{R}^n)$ and for $p = 1$ M is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$.*

G.D.Fazio and M.A.Ragusa [7] studied the boundedness of the Calderón-Zygmund singular integral operators in Morrey spaces, and their results imply the following statement for Calderón-Zygmund operators T.

Theorem 2.2. *Let $1 \le p < \infty$, $0 < \lambda < n$. Then for $1 < p < \infty$ Calderón-Zygmund singular integral operator T is bounded in $L_{p,\lambda}(\mathbb{R}^n)$ and for $p = 1$ T is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$.*

Note that in the case of the classical Calderón-Zygmund singular integral operators Theorem 2.2 was proved by J. Peetre [11]. If $\lambda = 0$, the statement of Theorem 2.2 reduces to the aforementioned result for $L_p(\mathbb{R}^n)$.

3. A priori Morrey estimates

We consider the Dirichlet problem (1.2) in bounded domains Ω. From now on we will assume that $\partial\Omega$ is of class C^2. The solution of this problem is given by

$$u(x) = \int_{\Omega} G(x,y) f(y) dy, \tag{3.1}$$

where $G(x,y)$ is the Green function, which can be written as

$$G(x,y) = \Gamma(x-y) + h(x,y) \tag{3.2}$$

with $h(x,y)$ satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} \triangle_x h(x,y) = 0 & x \in \Omega, \\ h(x,y) = -\Gamma(x-y) & x \in \partial\Omega. \end{cases}$$

If $P(y,Q)$ is the Poisson kernel, $h(x,y)$ is given by

$$h(x,y) = -\frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \frac{1}{|x-Q|^{n-2}} P(y,Q) dS(Q), \tag{3.3}$$

where dS denotes the surface measure on $\partial\Omega$.

In what follows the letter C will denote a generic constant, not necessarily the same at each occurrence. It is known that the Green function satisfies the following estimates (see [15]),

$$G(x,y)\le\begin{cases} C\log|x-y|, & \text{if} \quad n=2,\\ C|x-y|^{2-n}, & \text{if} \quad n\ge 3,\end{cases} \tag{3.4}$$

and

$$|D_{x_i}G(x,y)|\le C|x-y|^{1-n}. \tag{3.5}$$

Therefore

$$D_{x_i}u(x)=\int_\Omega D_{x_i}G(x,y)f(y)dy. \tag{3.6}$$

To obtain the second derivatives of u from the representation (3.1) we will use the following lemma. We denote with $d(x)$ the distance to the boundary, namely, $d(x)=\inf_{Q\in\partial\Omega}|x-Q|$.

Lemma 3.1. *Given $\alpha\in\mathbb{Z}^n$ ($|\alpha|>0$ if $n=2$) there exists a constant C depending only on n and α such that*

$$|D^\alpha h(x,y)|\le Cd(x)^{2-n-|\alpha|}. \tag{3.7}$$

Proof. To simplify notation we assume that $n\ge 3$ (but the argument applies also in the case $n=2$). Using (3.3) and $P(y,Q)\ge 0$, $\forall Q\in\partial\Omega$, we have

$$|D^\alpha h(x,y)=\left|\frac{1}{(n-2)\omega_n}\int_{\partial\Omega}D^\alpha|x-Q|^{2-n}P(y,Q)dS(Q)\right|\le$$

$$\le C\int_{\partial\Omega}|x-Q|^{2-n-|\alpha|}P(y,Q)dS(Q)$$

and then (3.7) follows by using that $\int_{\partial\Omega}P(y,Q)\,dS(Q)=1$.

It follows from this lemma that for each $x\in\Omega$, $D_{x_ix_j}h(x,y)$ is bounded uniformly in a neighborhood of x and so

$$D_{x_ix_j}\int_\Omega h(x,y)f(y)dy=\int_\Omega D_{x_ix_j}h(x,y)f(y)dy. \tag{3.8}$$

On the other hand, since $|D_{x_j}\Gamma(x)|\le C|x|^{1-n}$ we have

$$D_{x_j}\int_\Omega \Gamma(x-y)f(y)dy=\int_\Omega D_{x_j}\Gamma(x-y)f(y)dy.$$

However, $D_{x_ix_j}\Gamma$ is not an integrable function and we cannot interchange the order between second derivatives and integration. A known standard argument shows that

$$D_{x_i}\int_\Omega D_{x_j}\Gamma(x-y)f(y)dy=Kf(x)+c(x)f(x), \tag{3.9}$$

where c is a bounded function and

$$Kf(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}D_{x_ix_j}\Gamma(x-y)f(y)dy. \tag{3.10}$$

Here and in what follows we consider f defined in $\mathbb{R}^n$ extending the original f by zero.

The operator K is a Calderón-Zygmund singular integral operator. Indeed, since $D_{x_j}\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and it is a homogeneous function of degree $1-n$, it follows that $D_{x_ix_j}\Gamma(x-y)$ is homogeneous of degree $-n$ and has vanishing average on the unit sphere (see Lemma 11.1 in [[1], page 152]). Then, it follows from the general theory given in [3] that K is a bounded operator in L_p for $1<p<\infty$.

Moreover, the maximal operator

$$\widetilde{K}f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} D_{x_ix_j}\Gamma(x-y)f(y)dy\right|$$

is also bounded in L_p for $1<p<\infty$.

We will need the following estimate for the Green function. This estimate has been proved by A. Dall'Acqua and G. Sweers in [6], however they assume that the domain is more regular than C_2.

Lemma 3.2. *Let Ω be a bounded C^2 domain and $G(x,y)$ be the Green function of problem (1.2) in Ω. There exists a constant C depending only on n and Ω such that for $(x,y)\in\Omega\times\Omega$*

$$|D_{x_ix_j}G(x,y)| \le C\frac{d(x)}{|x-y|^{n+1}}. \tag{3.11}$$

Lemma 3.3. *There exists a constant C depending only on n and Ω such that, for any $x\in\Omega$,*

$$|u(x)| + |D_{x_i}u(x)| \le CMf(x),$$

$$|D_{x_ix_j}u(x)| \le C\{\widetilde{K}f(x) + Mf(x) + |f(x)|\},$$

where $Mf(x)$ is the usual Hardy-Littlewood maximal function of f.

Proof. Calling δ the diameter of Ω and using (3.5) and (3.6), we have

$$|D_{x_i}u(x)| \le C\int_{|x-y|\le\delta}\frac{|f(y)|}{|x-y|^{n-1}}dy = \sum_{k=0}^{\infty}\int_{\{2^{-(k+1)}\delta\le|x-y|\le 2^{-k}\delta\}}\frac{|f(y)|}{|x-y|^{n-1}}dy$$

and then, it follows easily that

$$|D_{x_i}u(x)| \le CMf(x). \tag{3.12}$$

(see Lemma 2.8.3 in ([16], page 85) for details).

Analogously we obtain

$$|u(x)| \le CMf(x)$$

using now (3.1) and (3.4).

Therefore, the most interesting and difficult part of the lemma is the estimate of the second derivatives. Using the representation given by (3.1) and (3.2), (3.8), (3.9) and (3.10), we have

$$D_{x_ix_j}u(x) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|\le d(x)} D_{x_ix_j}\Gamma(x-y)f(y)dy +$$

$$+\int_{|x-y|>d(x)} D_{x_ix_j}\Gamma(x-y)f(y)dy + c(x)f(x)+$$

$$+\int_{|x-y|\le d(x)} D_{x_ix_j}h(x,y)f(y)dy + \int_{|x-y|>d(x)} D_{x_ix_j}h(x,y)f(y)dy$$

and then

$$D_{x_ix_j}u(x) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|\le d(x)} D_{x_ix_j}\Gamma(x-y)f(y)dy + c(x)f(x)+$$

$$+\int\limits_{|x-y|\le d(x)} D_{x_ix_j}h(x,y)f(y)dy + \int\limits_{|x-y|>d(x)} D_{x_ix_j}G(x,y)f(y)dy =: I_1+I_2+I_3+I_4. \quad (3.13)$$

Now we have

$$I_1 = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|} D_{x_ix_j}\Gamma(x-y)f(y)dy - \int_{|x-y|>d(x)} D_{x_ix_j}\Gamma(x-y)f(y)dy$$

but,

$$\left|\int_{|x-y|>d(x)} D_{x_ix_j}\Gamma(x-y)f(y)dy\right| \le \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} D_{x_ix_j}\Gamma(x-y)f(y)dy\right| = \widetilde{K}f(x)$$

and therefore

$$|I_1| \le |Kf(x)| + \widetilde{K}f(x) \le 2\widetilde{K}f(x).$$

Since c is a bounded function, we have $|I_2| \le C|f(x)|$. Therefore, it only remains to estimate the last two terms in (3.13).

By (3.7) we have

$$|I_3| = \frac{C}{d(x)^n}\int_{|x-y|\le d(x)} |f(y)|dy \le CMf(x).$$

Finally, from (3.11) we obtain

$$|I_4| \le C\int_{|x-y|>d(x)} \frac{d(x)}{|x-y|^{n+1}} f(y)dy$$

and therefore, by the same arguments used to prove (3.12) we conclude that

$$|I_4| \le CMf(x)$$

and the lemma is proved.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain. If $1 < p < \infty$ and $0 \le \lambda < n$, $f \in L_{p,\lambda}(\Omega)$ and u is the solution of problem (1.2), then there exists a constant C depending only on n, λ and Ω such that*

$$\|u\|_{W^2_{p,\lambda}(\Omega)} \le C\|f\|_{L_{p,\lambda}(\Omega)}.$$

Proof. By Theorems 2.1 and 2.2 implies that the operators M and $\widetilde{K}$ are bounded in $L_{p,\lambda}$. Therefore (3.3) follows immediately from Lemma 3.3.

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