Bilal T. BILALOV, Sabina R. SADIQOVA

ON PERTURBATIONS OF BANACH SPACES BASES

Abstract

One method for establishing the basicity of the perturbed system from the basis in Banach spaces is cited. The obtained results are applied to the system of exponents.

The Paley-Wiener type theorem [1] on basicity of close in this or other sense systems in the Banach spaces play an important part in the theory of bases and spectral theory of linear operators. Generally speaking, such kind theorems are equivalent to small perturbations in the terms of closeness of the systems in Banach spaces. Using such statements, earlier numerical results concering the basicity in the spaces L_p of the perturbed systems of exponents, $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ (Z is a set of integers) where $\{\lambda_n\} \subset C$ (C is a complex plane) is a sequence of complex numbers, were obtained. This direction was called "Fourier inharmonic series" and R. Young's mongraph [2] was devoted to it. Some kinds were given in the papers [3-5]. If $\{\lambda_n\}$ has much perturbation, for example, its asymptotics contains a principal part of the form $n + asignn$, $n \in Z$, application of such methods generally speaking, are impossible. In these cases, another study methods are attractive. One of these methods is the method of boundary value problems of the theory of analytic functions. Apparently, it takes its origin from the A.V. Bitsadze's paper [6]. S.M. Ponomarev [7], E.I. Moiseev [8; 9] and others successfully used this method. The very general case was considered in the papers [10; 11].

In this paper, we try to generalize these results for the abstract case.

1. Necessary notion and facts. Accept the following standard denotation.

B - space is a Banach space; H - space is a Hilbert space;

$\|\cdot\|_X$ is the norm in X (if there are no misunderstandings, sometimes we'll omit the index);

$L(X;Y)$ is a $B-$space of bounded operators acting from X to Y;

$L(X) \equiv L(X;X)$;

X^* is a space conjugated to X; T^* is an operator conjugated to T;

δ_{nk} is a Kronecker symbol.

Let X be some B - space. If there exist the subspaces (closed) $X_k \subset X$, $k = 1, 2$, such that $X_1 \cap X_2 = \varnothing$ and $X = X_1 + X_2$, they say that X is a direct sum of the subspaces $X_1; X_2$ and write $X = X_1 \dot{+} X_2$. Let $X = X^+ \dot{+} X^-$ and the system $\{x_n^+\}_{n\in N}$ $\left(\{x_n^-\}_{n\in N}\right)$ form a basis in X^+ (X^-). Then the double system $\{x_n^+; x_n^-\}_{n\in N}$ forms a basis in X in the following sense.

Definition. *The system* $\{x_n^+; x_n^-\}_{n\in N}$ *is said to be a basis in* X *, if* $\forall x \in X$, $\exists! \left\{\lambda_n^\pm\right\}_{n\in N} \subset C$*:*

$$x = \sum_{n=1}^{\infty} \lambda_n^+ x_n^+ + \sum_{n=1}^{\infty} \lambda_n^- x_n^- .$$

So, let the expansion

$$X = X^+ \dotplus X^- \tag{1}$$

hold.

Denote by P^+ and P^- the propectors generated by expansion (1).

Take $\forall x \in X$. We have

$$x = x^+ + x^-, \quad x^\pm \in X^\pm.$$

Expand $x^\pm$ by the basis $\{x_k^\pm\}_{k\in N}$:

$$x^\pm = \sum_{k=1}^{\infty} \lambda_k^\pm x_k^+.$$

Thus,

$$x = \sum_{k=1}^{\infty} \lambda_k^+ x_k^+ + \sum_{k=1}^{\infty} \lambda_k^- x_k^-.$$

This means that $\forall x \in X$ has an expansion by the system $\{x_k^+; x_k^-\}_{k\in N}$. Show the uniqueness of such an expansion. Let there be the expansion

$$x = \sum_{k=1}^{\infty} a_k^+ x_k^+ + \sum_{k=1}^{\infty} a_k^- x_k^-.$$

Consequently,

$$0 = \sum_{k=1}^{\infty} \left(\lambda_k^+ - a_k^+\right) x_k^+ + \sum_{k=1}^{\infty} \left(\lambda_k^- - a_k^-\right) x_k^-.$$

Since $X^\pm$ are closed, it follows from $X^+ \cap X^- = \{0\}$ that $\sum\limits_{k=1}^{\infty} \left(\lambda_k^\pm - a_k^\pm\right) x_k^\pm = 0$ $\Longrightarrow \lambda_k^\pm = a_k^\pm$, $\forall k \in N$. As a result $\{x_k^+; x_k^-\}_{k\in N}$ forms a basis in X. Let $\{f_k^\pm\}_{k\in N} \subset (X^\pm)^*$ be a system conjugated to $\{x_k^\pm\}_{k\in N} \subset X^\pm$. Then the system $\{(P^+)^* f_k^+;\ (P^-)^* f_k^-\}_{k\in N}$, is the system conjugated to $\{x_k^+; x_k^-\}_{k\in N}$ where $P^\pm$ are the projectors generated by expansion (1). Indeed, we have

$$\left\lfloor (P^+)^* f_k^+ \right\rfloor \left(x_n^+\right) = (\text{definition of the conjugated operator}) =$$

$$= f_k^+ \left(P^+ x_n^+\right) = f_k^+ \left(x_n^+\right) = \delta_{nk};$$

$$\left\lfloor (P^+)^* f_k^+ \right\rfloor \left(x_n^-\right) = f_k^+ \left(P^+ x_n^-\right) = f_k^+ (0) = 0.$$

Similarly, it is established

$$\left\lfloor (P^-)^* f_k^- \right\rfloor \left(x_n^+\right) = 0,$$

$$\left\lfloor (P^-)^* f_k^- \right\rfloor \left(x_n^+\right) = \delta_{nk}.$$

Generally speaking, it directly follows from these relations that if the systems $\{x_k^+\}_{k\in N}$, $\{x_k^-\}_{k\in N}$ are minimal in X^+ and X^-, respectively, the system $\{x_k^+; x_k^-\}_{k\in N}$ is minimal in X. The similar reasoning is valid for the completeness as well. Thus, it is valid

Statement 1. Let X be a B- space and the expansion (1) by the subspaces $X^{\pm}$ hold. If the systems $\{x_k^+\}_{k\in N}$, $\{x_k^-\}_{k\in N}$ are complete, minimal, form bases in X^+ and X^-, the system $\{x_k^+; x_k^-\}_{k\in N}$ possesses the corresponding properties in X.

In the case of H-space, the same statement is true for the Riesz basicity. Really, let X be a $H-$ space, expansion (1) hold and $\{x_k^{\pm}\}_{k\in N} \subset X^{\pm}$ form a Riesz basis in $X^{\pm}$. Consequently, there exist the automorphisms $T^{\pm} \in L(X^{\pm}) : T^{\pm}x^{\pm} = e_n^{\pm}$, $\forall n \in N$, where $\{e_n^{\pm}\}_{n\in N} \subset X^{\pm}$ are the bases orthonormed in $X^{\pm}$. Identify each $x \in X$ with the element $x = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$, where $x = x^+ + x^-$, $x^{\pm} \in X^{\pm}$. We identify $\forall x^{\pm} \in X^{\pm}$ with $\begin{pmatrix} x^+ \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x^- \end{pmatrix}$, respectively. Consider the matrix operator $T \in L(X)$

$$T = \begin{pmatrix} T^+ & O \\ O & T^- \end{pmatrix},$$

where O is a zero operator. It is easy to show that T is an automorphism in X. Furthermore,

$$Tx_n^{\pm} = e_n^{\pm}, \quad \forall n \in N.$$

Hence, it directly follows that the system $\{x_n^+; x_n^-\}_{n\in N}$ forms a Riesz basis in X. Let $x = x^+ + x^-$, $x^{\pm} \in X^{\pm}$. Consequently, $Tx = T^+x^+ + T^-x^-$. Having accepted $y = Tx$, we get $y = T^+x^+ + T^-x^-$. Thus, for $\forall y \in X$ the equation

$$T^+x^+ + T^-x^- = y, \tag{2}$$

has the solution $x^{\pm} \in X^{\pm}$. It is obvious that such a solution is unique.

Now, vice versa, let for $\forall y \in X$ equation (2) have a unique solution

$(x^+; x^-) \in X^+ \times X^-$, where $T^{\pm} \in L(X)$ are some operators. Expand the element $x^{\pm}$ by the basis $\{x_k^{\pm}\}_{k\in N} : x^{\pm} = \sum_{k=1}^{\infty} \lambda_k^{\pm} x_k^{\pm}$. Consequently,

$$y = \sum_{k=1}^{\infty} \lambda_k^+ T^+ x_k^+ + \sum_{k=1}^{\infty} \lambda_k^- T^- x_k^-.$$

Thus, $\forall y \in X$ is expanded by the system $\{T^+x_k^+; T^-x_k^-\}_{k\in N}$ in X. It is obvious that if T^+ and T^- are automorphisms in $L(X)$, then the expansion is unique and as a result the system $\{T^+x_k^+; T^-x_k^-\}_{k\in N}$ forms a basis in X. In the case when X is a H-space, assume that $\{x_k^{\pm}\}_{k\in N}$ forms a Riesz basis in $X^{\pm}$. Let $\{e_k^{\pm}\}_{k\in N}$ be an orthonormed basis in $X^{\pm}$ and $R^{\pm}e_k^{\pm} = x_k^{\pm}$, $\forall k \in N$. Assume

$$R = \begin{pmatrix} R^+ & O \\ O & R^- \end{pmatrix}.$$

Consider

$$T = \begin{pmatrix} T^+ & O \\ O & T^- \end{pmatrix}.$$

It is obvious that T is an automorphisim in X (this follows from the Banach theorem). And what is more, $TR\, e_k^{\pm} = T^{\pm}x_k^{\pm}$, $\forall k \in N$ and therefore $\{T^+x_k^+; T^-x_k^-\}_{k\in N}$ forms a Riesz basis in X. So, it is valid

Theorem 1. *Let X be a H-space, the direct sum (1) hold, $\{x_n^{\pm}\}_{n\in N} \subset X^{\pm}$ be Riesz bases in $X^{\pm}$, respectively. If $T^{\pm}$ are automorphisms in X and for $\forall y \in X$ equation (2) has a unique solution $(x^+;x^-) \in X^+ \times X^-$ the system $\left\{T^+x_k^+;T^-x_k^-\right\}_{k\in N}$ forms a Riesz basis in X.*

The isomorphism between X and $X^+ \times X^-$ generated by expansion (1) generated by expansion (1) denote by $P : X \leftrightarrow X^+ \times X^- \equiv \overline{X}$.

Consider equation (2). Accept

$$\Delta T^{\pm} = T^{\pm} - I^{\pm}.$$

Thus, we can write equation (2) in the form

$$x^+ + \Delta T^+ x^+ + x^- + \Delta T^- x^- = y.$$

In the matrix form we get

$$\left(P^{-1} + \Delta T\right)\overline{x} = y, \tag{3}$$

where

$$\Delta T = \begin{pmatrix} \Delta T^+ & O \\ O & \Delta T^- \end{pmatrix},$$

$$P = \begin{pmatrix} I^+ & O \\ O & I^- \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix},$$

$I^{\pm}$ is a unit operator in $X^{\pm}$.

Let $\|\cdot\|$ be a norm in X. Accept that $\|\cdot\|_1$ is a norm in $X^+ \times X^-$; such that

$$\left\|P^{-1}\overline{x}\right\| \sim \|\overline{x}\|_1, \quad \forall \overline{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix},$$

i.e. $\exists m, M > 0$:

$$m\left\|P^{-1}\overline{x}\right\| \le \|\overline{x}\|_1 \le M\|P\overline{x}\|. \tag{4}$$

Let $\overline{x} \in X^+ \times X^-$ be an arbitrary element.

Let $x = P^{-1}\overline{x}$, i.e. $\overline{x} = Px$. We have

$$\|\Delta T\overline{x}\| = \left\|\Delta T^+x^+ + \Delta T^-x^-\right\| \le \left\|\Delta T^+\right\|\left\|x^+\right\| + \left\|\Delta T^-\right\|\left\|x^-\right\|.$$

It is known that $\exists m > 0$:

$$\left\|x^{\pm}\right\| \le m\|x\|, \quad \forall x \in X. \tag{5}$$

The $\inf\{m :$ satisfying (5)$\}$ denote by $\theta(X^+;X^-)$ and call the direct norm of expansion (1). Consequently,

$$\left\|x^{\pm}\right\| \le \theta\left(X^+;X^-\right)\|x\|, \quad \forall x \in X.$$

Thus,

$$\|\Delta T\overline{x}\| \le \theta\left(X^+;X^-\right)\left(\left\|\Delta T^+\right\| + \left\|\Delta T^-\right\|\right)\|x\|,$$

$$\|\Delta TPx\| \le \theta\left(X^+;X^-\right)\left(\left\|\Delta T^+\right\| + \left\|\Delta T^-\right\|\right)\|x\| \Rightarrow$$
$$\Rightarrow \|\Delta TP\| \le \theta\left(X^+;X^-\right)\left(\left\|\Delta T^+\right\| + \left\|\Delta T^-\right\|\right).$$

Represent (3) in the form

$$\left(I + \Delta TP\right)P^{-1}\overline{x} = y,$$

i.e.

$$\left(I + \Delta TP\right)x = y. \tag{6}$$

It is clear that equations (6) and (2)are equivalent.

If $\theta\left(X^+;X^-\right)\left(\left\|\Delta T^+\right\| + \left\|\Delta T^-\right\|\right) < 1$, then $\left(I + \Delta TP\right)$ is invertible and as a result equation (2) has a unique solution. Take $\forall y \in X$. Let $x^\pm \in X^\pm$ be a solution of (2). Expand $x^\pm$ by the basis $\left\{x_k^\pm\right\}_{k\in N} \subset X^\pm$.

$$x^\pm = \sum_{n=1}^{\infty} a_n^+ x_n^+,$$

i.e.

$$y = \sum_{n=1}^{\infty} a_n^+ T^+ x_n^+ + \sum_{n=1}^{\infty} a_n^- T^- x_n^-.$$

Thus, an arbitrary element $y \in X$ is expanded by the system $\left\{T^+x_n^+;T^-x_k^-\right\}_{n,k\in N}$. Show that such an expansion is unique, i.e., let the expansion

$$y = \sum_{n=1}^{\infty} b_n^+ T^+ x_n^+ + \sum_{n=1}^{\infty} b_n^- T^- x_n^-.$$

hold. Consequently,

$$0 = \sum_{n=1}^{\infty} c_n^+ T^+ x_n^+ + \sum_{n=1}^{\infty} c_n^- T^- x_n^-,$$

where $c_n^\pm = a_n^\pm - b_n^\pm$.

As the $T^\pm$ is an isomorphism, hence it follows that the series $\sum\limits_{n=1}^{\infty} c_n^\pm x_n^\pm$ converges in $X^\pm$. As a result

$$0 = T^+y^+ + T^-y^-,$$

where $y^\pm = \sum\limits_{n=1}^{\infty} c_n^\pm x_n^\pm$. It is clear that $y^\pm \in X^\pm$. Then from the uniqueness of the solution we get $y^\pm = 0$ and as a result $c_n^\pm = 0$, $\forall n \in N \Rightarrow a_n^\pm = b_n^\pm$, $\forall n \in N$. This proves the basicity of the system $\left\{T^+x_n^+;T^-x_k^-\right\}_{n,k\in N}$ in X. Construct a system biorthogonal to it. Let $\left\{\vartheta_k^\pm\right\}_{k\in N}$ be a system biorthogonal to $\left\{x_k^\pm\right\}_{k\in N}$ i.e. $\vartheta_k^\pm\left(x_k^\pm\right) = \delta_{nk}$, $\forall n,k \in N$. Accept

$$\omega_k^\pm = \left[\left(T^\pm\right)^{-1}\right]^*\left(\vartheta_k^\pm\right), \quad \forall k \in N.$$

We have

$$\overline{X} = X^+ \times X^-,$$
$$\left(\overline{X}\right)^* = \left(X^+\right)^* \times \left(X^-\right)^*.$$

We identify the elements $x_n^+\,(x_n^-)$ with the elements

$$\overline{x}_n^+ \equiv \begin{pmatrix} x_n^+ \\ 0 \end{pmatrix} \left(\overline{x}_n^- \equiv \begin{pmatrix} 0 \\ x_n^- \end{pmatrix}\right), \quad \forall n \in N.$$

In the similar way we can assume that

$$\overline{\vartheta}_k^+ \equiv \begin{pmatrix} \vartheta_k^+ \\ 0 \end{pmatrix} \left(\overline{\vartheta}_k^- \equiv \begin{pmatrix} 0 \\ \vartheta_k^- \end{pmatrix}\right), \quad \forall k \in N.$$

We have

$$\omega_k^\pm\left[T^\pm x_n^\pm\right] = \left\lfloor \left(T^\pm\right)^* \omega_k^\pm \right\rfloor \left(x_n^\pm\right) = \vartheta_k^\pm\left(x_n^\pm\right) = \delta_{nk}, \quad \forall n,k \in N.$$

Thus, the system $\left\{\omega_k^\pm\right\}_{k\in N}$ is biorthogonal to $\{T^+x_n^+\}_{n\in N}$. Let

$$T = \begin{pmatrix} T^+ & O \\ O & T^- \end{pmatrix}.$$

$$T^{\pm 1} \in L\left(\overline{X}\right).$$

We have

$$T\overline{x}_n^+ = \begin{pmatrix} T^+ & x_n^+ \\ & 0 \end{pmatrix},$$

$$T\overline{x}_n^- = \begin{pmatrix} T^+ & x_n^- \\ & 0 \end{pmatrix}, \quad \forall n \in N.$$

Let

$$\overline{\omega}_k^+ \equiv \begin{pmatrix} \omega_k^+ \\ 0 \end{pmatrix}, \quad \overline{\omega}_k^- \equiv \begin{pmatrix} 0 \\ \omega_k^- \end{pmatrix}.$$

It is clear that $\left\{\omega_k^\pm\right\}_{k\in N} \subset \overline{X}^*$ and $\overline{\omega}_k^\pm\left(T\overline{x}_n^\pm\right) = \delta_{nk}$,

$$\overline{\omega}_k^+\left(T\overline{x}_n^-\right) = 0, \quad \overline{\omega}_x^-\left(T\overline{x}_n^+\right) = 0, \quad \forall n,k \in N.$$

Consequently, the system $\left\{\overline{\omega}_k^+;\overline{\omega}_k^-\right\}_{k\in N}$ is biorthogonal to $\{T\overline{x}_n^+;T\overline{x}_n^-\}_{n\in N}$. Then, we can pass from $\overline{X}$ to X. So, it is proved.

Theorem 2. *Let the B -space X have the expansion $X = X^+\dot{+}X^-$, $\{x_n^\pm\}_{n\in N} \subset X^\pm$ form a basis in X and $\theta\left(X^+;X^-\right)$ be a direct norm of the expansion (1). If $\|I - T^+\| + \|I - T^-\| < \frac{1}{\theta(X^+;X^-)}$, the system*

$$\left\{T^+x_n^+;T^-x_n^-\right\}_{n\in N}$$

forms a basis in X.

Apply the obtained result to the concrete case.

Example. As X we take the space L_p, $1 < p < +\infty$, $X^\pm = H_p^\pm$, where $H_p^\pm$ are ordinary Hardy classes of functions an analytic interior and exterier, respectively. Take the system $\left\{e^{int}\right\}_{n\in Z}$.

Let $T^\pm f = A^\pm(t)\cdot f(t)$ be a multipication operator. Let $\theta\left(H_p^+;H_p^-\right)$ be a direct norm of the expansion

$$L_p = H_p^+ \dotplus H_p^-.$$

From the theorem we directly get

Corollary 1. *Let the inequality*

$$\left\|1-A^{+}(t)\right\|_{\infty}+\left\|1-A^{-}(t)\right\|_{\infty}<\frac{1}{\theta\left(H_{p}^{+};H_{p}^{-}\right)}$$

be fulfilled. Then the system

$$\left\{A^{+}(t)\,e^{int};A^{-}(t)\,e^{ikt}\right\}_{n\geq 0;k\geq 1} \tag{7}$$

forms a basis isomorphic to $\left\{e^{int}\right\}_{n\in Z}$ *in* L_p, *where* $\|\cdot\|_{\infty}$*is an ordinary norm in* $L_{\infty}(-\pi,\pi)$. *In particular, for* $p=2$ *it is clear that* $\theta\left(H_2^{+};H_2^{-}\right)=1$ *and as a result we get*

Corollary 2. *Let the inequality*

$$\left\|1-A^{+}(t)\right\|_{\infty}+\left\|1-A^{-}(t)\right\|_{\infty}<\frac{1}{\sqrt{2}}.$$

be fulfilled. Then system (7) forms a Riesz basis in $L_2(-\pi,\pi)$.

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Bilal T. Bilalov, Sabina R. Sadiqova
Institute of Mathematics and Mechanics NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (+99412) 439 47 20 (off).

Received September 02, 2009; Revised November 27, 2009.