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COERCIVE SOLVABILITY OF QUADRATIC OPERATOR AND DIFFERENTIAL PENCILS BY SPECTRAL PARAMETER

Abstract

In the paper, coercive solvability of a second order differential operator is studied, two fold completeness of the root vectors of corresponding quadrature operator pencil is proved.

Introduction. In the paper [1] published in 1951 M.V. Keldysh introduced an important notion of n-fold completeness of the root vectors and proved a fundamental theorem on n-fold completeness for a polynomial operator pencil with a principal part generated by a self-adjoint operator. In the sequel, these results were completed in the papers [2] – [5] (detailed references is for example in [5]).

In this paper, coercive solvability by a spectral parameter and two fold completeness of the root vectors is proved for a class of quadratic operator pencils of not Keldysh type.

Sometimes it is succeeded to replace the conditions of coerciveness or coercivenesss with defect of operator pencils on rays by the conditions on operator coefficients of the pencil. In the present paper, we show that for a class of quadratic operator pencils of not Keldysh type the coerciveness condition on the rays may be replaced by the conditions on operator coefficients.

In the Banach space E consider the operator pencil:

$$L(\lambda) = \lambda^n I + \lambda^{n+1} A_1 + ... + A_n, \tag{0.1}$$

where A_n , $k = 1 \div n$ are, generally speaking, unbounded operators. Obviously, $D(L(\lambda)) = \bigcap\limits_{k=1}^{n} D(A_k)$ for $\lambda \neq 0$.

The number λ_0 is said to be eigen value of the pencil $L(\lambda)$ if the equation $L(\lambda_0) u = 0$ has a non-trival solution $u \in D(L(\lambda_0))$. The vector $u_0 \neq 0$ satisfying the equation $L(\lambda_0) u_0 = 0$ is called an eigen vector of the pencil $L(\lambda)$ corresponding to the eigen value λ_0.

The vectors $u_1, ..., u_k$ connected with the eigen vector u_0 by the relations

$$L(\lambda_0) u_p + \frac{1}{1!} L'(\lambda_0) u_{p-1} + ... + \frac{1}{p} L^{(p)}(\lambda) u_0 = 0, \ p = 0 \div k$$

are said to be adjoint vectors to the eigen vector u_0 of the pencil $L(\lambda)$.

The eigen and adjoint vectors of a pencil are united under the general name of the root vectors of a pencil.

The point μ of a complex plane is called a regular point of the pencil $L(\lambda)$ if the operator $L(\mu)$ has a bounded inverse $L^{-1}(m)$ determined on the space.

Completion of regular points set in a complex plane is said to be spectrum of the pencil $L(\lambda)$.

A spectrum of the pencil $L(\lambda)$ is said to be discrete if

a) all the points of λ not coinciding with eigen values of the pencil $L(\lambda)$ are regular points of the pencil $L(\lambda)$;

b) eigen values are isolated and have finite algebraic multiplicitics;

c) infinity is a unique limit point of eigen values set of the pencil $L(\lambda)$.

Below, $L^{-1}(\lambda)$ is written only in regular points of the pencil.

In the Banach space E consider the Cauchy problem:

$$L(D_t)u = u^{(n)}(t) + A_1 u^{(n-1)}(t) + ... + A_n u(t) = 0, \tag{0.2}$$

$$u^{(k)}_{(0)} = \vartheta_{k+1},\ k = 0 \div (n-1), \tag{0.3}$$

where ϑ_{k+1} are the given vectors from E.

The function $u(t)$ of the form

$$u(t)\, e^{\lambda_0 t}\left(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + ... + \frac{t}{1!}u_{k-1} + u_k\right) \tag{0.4}$$

is a solution of equation (0.2) iff $u_0, u_1, .., u_k$ is a chain of the root vectors corresponding to the eigen value λ_0 of pencil (1). The solution of the form (0.4) is called an elementary solution of equation (0.2).

Natural desire to approximate the solution of problem (0.2),(0.3) by linear combinations of elementary solutions of (0.4) leads to the fact that the vector $(\vartheta_1, ..., \vartheta_n)$ should be approximated by linear combinations of vectors of the form

$$\left(u(0), u'(0), ..., u^{(n-1)}(0)\right). \tag{0.5}$$

The system of vectors (0.5) is called a Keldysh system of the pencil (0.1).

Let E_k, $k = 1 + n$ be Banach spaces continuously imbedded into E.

The system of the root vectors of the pencil $L(\lambda)$ is called n- fold complete in $E_1 \times ... \times E_n$ if the Keldush system (0.5) of the pencil $L(\lambda)$ is complete in the space $E_1 \times ... \times E_n$.

The lower bound of the numbers η satisfying the estimation

$$\sum_{k=0}^{n} |\lambda|^{n-k-\eta} \left\|L^{-1}(\lambda)\right\|_{B(E,E_k)} \leq C,\ \lambda \in G(a,\varphi)$$

$|\lambda| \to \infty$, where $G(a,\varphi)$ is an angle of complex plane centered at a and with angle φ is called a defect of coerciveness of the pencil $L(\lambda)$ at the angle $G(a,\varphi)$.

Now, cite a theorem from [7, p. 430, II].

Theorem A. *Let*

1. There exist Hilbert spaces H_k, $k = 0 \div n$, for which compact imbeddings

$$H_n \subset H_{n-1} \subset ... \subset H_0 = H \text{ hold}$$

2. $\overline{H}_k\big|_{H_{k-1}} = H_{k-1}$, $k = 1 \div n$

3. For some $p > 0$, $J \in \sigma_p (H_k, H_{k-1})$, $k = 1 \div n$

4. The operators A_k, $k = 1 \div n$ from H_k to H be bounded.

5. There exist the rays $l_k (a)$ with angles between neighboring rays no more than π/p and integer m such that

$$\left\|L^{-1} (\lambda)\right\|_{B(H,H_{n-1})} \le C |\lambda|^m , \lambda \in l_k (a) , \ |\lambda| \to \infty.$$

Then the spectrum of the pencil $L (\lambda) = \lambda^n I + \lambda^{n-1} A_1 + ... + A_n$ is discrete and the system of the root vectors of the pencil $L (\lambda)$ is n-fold complete in the space $H_n \times H_{n-1} \times ... \times H_1$.

1. Coercive solvability by spectral parameter of quadratic operator pencils

Consider a quadratic operator bundle

$$L (\lambda) = \lambda^2 I + \lambda (A + A_1) + (B + B_1) , \tag{1.1}$$

with unbounded operators A, A_1, B, B_1.

Theorem 1.1. *Let*

1^0. *The operator A in the Banach space E be invertible the operator B in E be closed;*

2^0. *There exist the numbers $\alpha \in (0, 1]$, $\beta \in (0, 1]$ such that*

$$\|R (\lambda, -A)\| \le c |\lambda|^{-\alpha} , \lambda \in S, \ |\lambda| \to \infty,$$

$$\left\|R \left(\lambda, -BA^{-1}\right)\right\| \le c |\lambda|^{-\beta} , \lambda \in S, \ |\lambda| \to \infty,$$

where S is a set of complex plane diverging to infinity;

3^0. $\alpha + \beta > 1$, $D (B) \supset D \left(A^2\right)$ *and for any* $\varepsilon < 0$

$$\|Bu\| \le \varepsilon \left\|A^2 u\right\|^{\alpha+\beta-1} \|u\|^{2-\alpha-\beta} + c (\varepsilon) \|u\| , \ u \in D \left(A^2\right) ;$$

4^0. *A_1 be an operator in E, $D (A_1) \supset D (A)$ and for any $\varepsilon < 0$*

$$\|A_1 u\| \le \varepsilon \|Au\|^{\alpha+\beta-1} \|u\|^{2-\alpha-\beta} + c (\varepsilon) \|u\| , \ u \in D (A) ;$$

5^0. *B_1 be an operator in E, $D (B_1) \supset D (B)$ and for any*

$$\|B_1 u\| \le \varepsilon \|Bu\|^{\frac{\alpha+\beta}{2}} \|u\|^{\frac{2-\alpha-\beta}{2}} + c (\varepsilon) \|u\| , \ u \in D (B) .$$

Then for all $\lambda \in S$, $|\lambda| \to \infty$ the operator

$$L (\lambda) = \lambda^2 I + \lambda (A + A_1) + (B + B_1) ,$$

in E is invertible and it holds the estimation

$$|\lambda|^{\alpha+\beta}\left\|L^{-1}(\lambda)\right\|+|\lambda|^{\alpha+\beta-1}\left\|AL^{-1}(\lambda)\right\|+|\lambda|^{\alpha+\beta-2}\left\|BL^{-1}(\lambda)\right\|\le C.$$

$$\lambda\in S,\quad |\lambda|\to\infty.$$

Proof. Show that for a pencil

$$L_0(\lambda)=\left(\lambda I+BA^{-1}\right)(\lambda I+A)=\lambda^2 I+\lambda\left(A+BA^{-1}\right)+B \tag{1.2}$$

it holds the estimation

$$|\lambda|^{\alpha+\beta}\ \left\|L_0^{-1}(\lambda)\right\|+|\lambda|^{\alpha+\beta-1}\left\|AL_0^{-1}(\lambda)\right\|+$$
$$+|\lambda|^{\alpha+\beta-2}\left\|BL_0^{-1}(\lambda)\right\|\le C,\ \lambda\in S,\quad |\lambda|\to\infty. \tag{1.3}$$

By condition 2^0 and representation

$$L_0^{-1}(\lambda)=(\lambda I+A)^{-1}\left(\lambda I+BA^{-1}\right)^{-1}$$

the first two summands in (1.3) are easily estimated.

Since from (1.2) we have

$$BL_0^{-1}(\lambda)=I-\lambda^2L_0^{-1}(\lambda)-\lambda\left(A+BA^{-1}\right)L_0^{-1}(\lambda)$$

for $\lambda\in S,\quad |\lambda|\to\infty$

$$|\lambda|^{\alpha+\beta-2}\left\|BL_0^{-1}(\lambda)\right\|\le|\lambda|^{\alpha+\beta-2}+|\lambda|^{\alpha+\beta}\left\|L_0^{-1}(\lambda)\right\|+$$
$$+|\lambda|^{\alpha+\beta-1}\left(\left\|AL_0^{-1}(\lambda)\right\|+\left\|BA^2\right\|\cdot\left\|AL_0^{-1}(\lambda)\right\|\right)\le C.$$

So, estimation (1.3) is proved.

Since the perturbation operator pencil is of the form

$$L_1(\lambda)=L(\lambda)-L_0(\lambda)=\lambda\left(A_1-BA^{-1}\right)+B_1, \tag{1.4}$$

then by conditions $3^0,4^0,5^0$ and estimation (1.3) for $\lambda\in S,\quad |\lambda|\to\infty$ and any $\varepsilon>0$ we have

$$|\lambda|\cdot\left\|\left(A_1-BA^{-1}\right)L_0^{-1}(\lambda)\right\|+\left\|BL_0^{-1}(\lambda)\right\|\le|\lambda|\left(\left\|A_1L_0^{-1}(\lambda)\right\|+\right.$$
$$\left.+\left\|BA^{-1}L_0^{-1}(\lambda)\right\|\right)+\varepsilon\left\|BL_0^{-1}(\lambda)\right\|^{\frac{\alpha+\beta}{2}}\left\|L_0^{-1}(\lambda)\right\|^{\frac{2-\alpha-\beta}{2}}+c(\varepsilon)\left\|L_0^{-1}(\lambda)\right\|\le$$
$$\le|\lambda|\,\varepsilon\left\|AL_0^{-1}(\lambda)\right\|^{\alpha+\beta-1}\left\|L_0^{-1}(\lambda)\right\|^{2-\alpha-\beta}+\varepsilon\left\|AL_0^{-1}(\lambda)\right\|^{\alpha+\beta-1}\times$$
$$\times\left\|A^{-1}L_0^{-1}(\lambda)\right\|^{2-\alpha-\beta}+c(\varepsilon)\left\|A^{-1}L_0^{-1}(\lambda)\right\|+c(\varepsilon)+c(\varepsilon)\left\|L_0^{1}(\lambda)\right\|\le$$
$$\le C(\varepsilon)+C(\varepsilon)\,|\lambda|^{1-\alpha-\beta}.$$

So, for $\lambda\in S,\quad |\lambda|\to\infty$

$$\left\|L_1(\lambda)L_0^{-1}(\lambda)\right\|\le\left\|\left[\lambda\left(A_1-BA^{-1}\right)+B_1\right]L_0^{-1}(\lambda)\right\|\le q<1. \tag{1.5}$$

Then it follows from (1.4) that for $\lambda \in S, \quad |\lambda| \to \infty$ the operator $L(\lambda)$ in E is invertible and

$$L^{-1}(\lambda) = L_0^{-1}(\lambda)\left[I + L_1(\lambda) L_0^{-1}(\lambda)\right]^{-1}.$$

Hence, by (1.3) and (1.5) we have

$$|\lambda|^{\alpha+\beta}\left\|L^{-1}(\lambda)\right\| + |\lambda|^{\alpha+\beta-1}\left\|AL^{-1}(\lambda)\right\| + |\lambda|^{\alpha+\beta-2}\left\|L^{-1}(\lambda)\right\| \le C.$$

for $\lambda \in S, \quad |\lambda| \to \infty$. The theorem is proved.

2. Two-fold completeness of root vectors of an unbounded quadratic pencil

Let the operator A act in the Hilbert space H with domain of definition $D(A)$. Convert $D(A)$ into the Hilbert space with the norm:

$$\|u\|_{H(A)} = \left(\|Au\|^2 + \|u\|^2\right)^{\frac{1}{2}},$$

where $\|\cdot\|$ is a norm in the space H.

Theorem 2.1. *Let*

1^0. *Operator A in the Hilbert space H be invertible and have a dense domain of definition; the operator B and H be closed;*

2^0. $D(A) \supset D(B), \overline{H(B)}\Big|_{H(A)} = H(A)$ *for some* $p > 0$, $J \in \sigma_p(H(A), H)$, $J \in \sigma_p(H(A), H(A))$

3^0. *there exist the rays $\ell_k(a)$ with angles between the neighbering rays no more than π/p and the numbers $\alpha \in (0,1]$, $\beta \in (0,1]$ such that*

$$\|R(\lambda, -A)\| \le C|\lambda|^{-\alpha}, \ \lambda \in \ell_k(a), \quad |\lambda| \to \infty,$$

$$\left\|R\left(\lambda, -BA^{-1}\right)\right\| \le C|\lambda|^{-\beta}, \ \lambda \in \ell_k(a), \quad |\lambda| \to \infty;$$

4^0. $\alpha + \beta > 1$, $D(B) \supset D\left(A^2\right)$ *and for any* $\varepsilon > 0$

$$\|Bu\| \le \varepsilon \left\|A^2u\right\|^{\alpha+\beta-1} \|u\|^{-2-\alpha-\beta} + c(\varepsilon)\|u\|, \ u \in D\left(A^2\right);$$

5^0. *A_1 be an operator in H: $D(A_1) \supset D(A)$ and for any $\varepsilon > 0$*

$$\|A_1u\| \le \varepsilon \|Au\|^{\alpha+\beta-1} \|u\|^{-2-\alpha-\beta} + c(\varepsilon)\|u\|, \ u \in D(A)$$

6^0. *B_1 be an operator in H,$D(B_1) \supset D(B)$ and for any $\varepsilon > 0$*

$$\|B_1u\| \le \varepsilon \|Bu\|^{\frac{\alpha+\beta}{2}} \|u\|^{\frac{-2-\alpha-\beta}{2}} + c(\varepsilon)\|u\|, \ u \in D(B).$$

Then the spectrum of the pencil (1.1) is discrete and the system of the root vectors of the pencil (1.1) is two fold complete in the space $H(B) \times H(A)$.

Proof. Apply theorem A to the pencil (1.1).

By $1^0, 2^0$, for $H_2 = H(B)$, $H_1 = H(A)$, $H_0 = H$, conditions 1-3 of theorem A are fulfilled.

By theorem 1.1. conditions 4 and 5 of theorem A are also fulfilled.

There by, all the conditions of theorem A are fulfilled and consequently the system of root vectors of the pencil (1.1) is two fold complete in the space $H(B)\times H(A)$.

3. Coercive problems for in principle weightless ordinary differential equations

The problems both for ordinary differential equations and weightless partial differential equatons have not been almost investigated. Here we investigate regular problems for a class of in principle weightless ordinary differential equations with quadratic spectral parameter.

3.1. Coerciveness of the problem. Consider a problem with a quadratic spectral parameter

$$L(\lambda)u=\lambda^2u(x)+\lambda\left(au^{(2m)}(x)+A_1u\right)+\left(bu^{2p}_{(x)}+B_1u\right)=f(x), \tag{3.1}$$

for $p\le m$ with conditions

$$L_\nu u=\alpha_\nu u^{(m_\nu)}(0)+\beta_\nu u^{(m_\nu)}(1)+\sum_{k=1}^{N_\nu}\delta_{\nu k}u^{(m_\nu)}(x_{\nu k})+T_\nu u=0, \tag{3.2}$$

$\nu=1\div 2m$, and for $m<p<2m$ additionally

$$L_{2m+s}u=L_{\nu_s}u^{(2m)}=0,\ s=1\div(2p-2m), \tag{3.3}$$

where $m\ge 1,\ \chi_{\nu k}\in(0,1),\ 1\le\nu_s\le 2m.$

Theorem 3.1. *Let*

1. $a\ne 0,\ b\ne 0,\ m\ge 1,\ m_\nu\le 2m-1;$

2. $p\le m$

$$\theta_1=\begin{vmatrix}\alpha_1\omega_1^{m_1}...\alpha_1\omega_m^{m_1}\ \beta_1\omega_{m_1-1}^{m_1}...\ \beta_1\omega_{2m}^{m_1}\\ ....\ ......\ ....\ ...\ ...\ .....\\ \alpha_{2m}\omega_1^{m_{2m}}...\alpha_{2m}\omega_m^{m_{2m}}\ \beta_2\omega_{m+1}^{m_2}...\ \beta_2\omega_{2m}^{m_{2m}}\end{vmatrix}\ne 0, \tag{3.4}$$

where $\omega_1=1,\ \omega_2=e^{i\frac{\pi}{m}},...,\omega_{2m}=e^{i\frac{\pi}{m}(2m-1)}$.

3. *for* $m<p<2m$, *in addition* $\theta_1\ne 0$, $m_{\nu_s}\le 2p-2m+1$ *and*

$$\theta_2=\begin{vmatrix}\alpha_{\nu_1}s_1^{m_{\nu_1}}...\beta_{\nu_1}s_{p-m+1}^{m_{\nu_1}}...\\ ....\ ......\ ....\ ...\ ...\ .....\\ \alpha_{\nu_{2p-2m}}s_1^{m_{\nu_{2p-2m}}}...\beta_{\nu_{2p-2m}}s_{p-m+1}^{m_{\nu_{2p-2m}}}\end{vmatrix}\ne 0, \tag{3.5}$$

where $s_1=1,\ s_2=e^{i\frac{\pi}{p-m}},...,s_{2p-2m}=e^{i\frac{\pi}{p-m}(2p-2m-1)}$.

4. *The operator* A_1 *from* $W_q^{2m}(0,1)$ *in* $L_q(0,1)$ *is compact, where* $q\in(1,\infty)$.

5. The operator B_1 from $W_q^{2p}(0,1)$ in $L_q(0,1)$ is compact;

6. *for some* $\eta \in [1,\infty)$ *the functionals* T_ν *in* $W_\eta^{m_\nu}(0,1)$ *are continuous.*

Then for any $\varepsilon > 0$ *there exists* $R**$ *such that for all complex numbers* λ *for which* $|\lambda| > R$ *and* $\lambda \in G(0, m\pi - \pi + \arg a + \varepsilon, m\pi + \pi + \arg a - \varepsilon)$ *for* $p \le m$;

$$\lambda \in G(0, m\pi - \pi + \arg a + \varepsilon, m\pi + \pi + \arg a - \varepsilon) \cap$$

$$\cap G(0, (p-m)\pi - \pi + \arg b - \arg a + \varepsilon, (p-m)\pi + \pi + \arg b - \arg a - \varepsilon)$$

for $m < p < 2m$, *problem (3.1)-(3.3) has a unique solution* $u \in W_q^{2m}(0,1) \cap W_q^{2q}(0,1)$ *and for these* λ *for the solution of problem (3.1)-(3.3) it holds the estimation*

$$|\lambda|^2 \|u\|_{L_q(0,1)} + |\lambda| \cdot \|u\|_{W_q^{2m}(0,1)} + \|u\|_{W_q^{2p}(0,1)} \le c\|f\|_{L_q(0,1)}.$$

Proof. In order to reduce the problem (3.1)-(3.3) to the operator pencil to which the results of 1 of theorem 1.1 are applicable, in $L_q(0,1)$ we introduce the operators A, B by the equalities

$$D(A) = W_q^{2m}((0,1);\ L_\nu u = 0, \nu = 1 \div 2m),$$

$$Au = au^{(2m)}(x) + \omega u(x),\ \omega \in \mathbb{C},$$

$$D(B) = W_q^{2m}((0,1);\ L_\nu u = 0, \nu = 1 \div 2p).$$

Then problem (3.1)-(3.3) is reduced in the space $L_q(0,1)$ to the equation

$$L(\lambda)u = \lambda^2 u + \lambda(A + A_1)u + (B + B_1)u = f. \tag{3.6}$$

From conditions 1.2 and 6 by theorem 2 [8] it follows that the operator A for some $\omega \in \mathbb{C}$ is invertible i.e. it satisfies condition 1 of theorem 1.1. Theorem 2 [8] also yields that for

$$\lambda \in S_1 = G(0, m\pi - \pi + \arg a + \varepsilon, m\pi + \pi + \arg a - \varepsilon),\ |\lambda| \to \infty,$$

$$\|R(\lambda, -A)\| \le c|\lambda|^{-1}, \tag{3.7}$$

i.e. the operator A satisfies condition 2^0 of theorem 1.1 in the set S_1.

For $p \le m$ the operator BA^{-1} in $L_q(0,1)$ is bounded. So, for all $|\lambda| \to \infty$

$$\left\|R\left(\lambda, -BA^{-1}\right)\right\| \le c|\lambda|^{-1},$$

i.e. the operator BA^{-1} satisfies condition 2^0 of theorem 1.1 in the set $S_2 = \mathbb{C}$.

Now, consider the case $m < p < 2m$. Since for $2m \le k$, $u \in W_q^{k-2m}(0,1)$ it holds the estimation

$$D^k A^{-1}u = a^{-1}D^{k-2m}\left[\left(aD^{2m} + \omega\right) - \omega\right]A^{-1}u = a^{-1}D^{k-2m}u - \omega D^{k-2m}A^{-1}u,$$

then for $u \in D\left(BA^{-1}\right) \subset W_q^{2p-2m}(0,1)$ we have

$$BA^{-1}u = ba^{-1}D^{2p-2m}u - b\omega D^{2p-2m}A^{-1}u, \tag{3.8}$$

and for $s = \div (2p-2m)$

$$L_{2m+s}A^{-1}u = a^{-1}\left(\alpha_{p_s}u^{(m_{p_s})}(0) + \beta_{\nu_s}u^{(m_{\nu_s})}(1) + \sum_{k=1}^{m_{\nu_s}} \delta_{\nu_s k}u^{(m_{\nu_k})}(x_{\nu_k})\right) -$$

$$-\omega\left(\alpha_{\nu_s}\left[D^{m_{\nu_s}}A^{-1}u\right]_{x=0} + \beta_{\nu_s}\left[D^{m_{\nu_s}}A^{-1}u\right]_{x=1} + \sum_{k=1}^{m_{\nu_s}} \delta_{\nu_s k}\left[D^{m_{\nu_s}}A^{-1}u\right]_{x=x_{\nu_k}}\right) +$$

$$+T_{\nu_s}A^{-1}u.$$

Obviously, the functional $aL_{2m+s}A^{-1}$ is of the form

$$\widetilde{L}_s u = aL_{2m+s}A^{-1}u = \alpha_{\nu_s}u^{(m_{\nu_s})}(0) + \beta_{\nu_s}u^{(m_{\nu_s})}(1) +$$

$$+\sum_{k=1}^{N_{\nu_s}} \delta_{\nu_s k}u^{(m_{\nu_s})}(x_{\nu_s k}) + \widetilde{T}_s u,$$

where the functional

$$\widetilde{T}_s u = -a\omega\left(\alpha_{\nu_s}\left[D^{m_{\nu_s}}A^{-1}u\right]_{x=0} + \beta_{\nu_s}\left[D^{m_{\nu_s}}A^{-1}u\right]_{x=1} +\right.$$

$$\left.+\sum_{k=1}^{N_{\nu_s}} \delta_{\nu_s k}\left[D^{m_{\nu_s}}A^{-1}u\right]_{x=x_{\nu_s k}}\right) + aT_{\nu_s}A^{-1}u$$

is continuous in $L_q(0,1)$. So, the operator BA^{-1} is defined by the equalities

$$D\left(BA^{-1}\right) = W_q^{2p-2m}\left((0,1), \widetilde{L}_s u = 0,\ s = 1 \div (2p-2m)\right)$$

in (3.8). By conditions 1.3 and 6, to the operator BA^{-1} we apply the norm 2 [8], whence it follows that for

$$\lambda \in S_2 =$$

$$= G\left(0, (p-m)\pi - \pi + \arg b - \arg a + \varepsilon, (p-m)\pi + \pi + \arg b - \arg a - \varepsilon\right),$$

$$|\lambda| \to \infty$$

$$\left\|R\left(\lambda, -BA^{-1}\right)\right\| \leq C|\lambda|^{-1}, \tag{3.9}$$

i.e. the operator BA^{-1} satisfies condition 2^0 of theorem 1.1 in the set S_2.

Show that condition 3^0 of theorem 1.1 holds i.e. $D(B) \supset D\left(A^2\right)$ and for $\varepsilon > 0$

$$\|Bu\| \leq \varepsilon\left\|A^2u\right\| + C(\varepsilon)\|u\|,\ u \in D\left(A^2\right).$$

In fact, it follows from $u \in D\left(A^2\right)$ that $u \in D(A)$, $Au \in D\left(A^2\right)$. Consequently, $L_\nu u = 0$, $L_\nu Au = aL_\nu u^{(2m)} + \omega L_\nu u = 0,\ \nu = 1 \div 2m$. So, the function $u(x)$ satisfies both the conditions (2.2) and (3.3), i.e. $u \in D(B)$.

On the other hand, by the known estimation [9, p. 145]

$$\left\|u^{(k)}\right\|_{L_\infty(0,1)} \le \varepsilon \left\|u^{(n)}\right\|_{L_q(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)}, **$$

for $u \in D\left(A^2\right)$ we have

$$\|Bu\|_{L_q(0,1)} \le C\left\|u^{(2p)}\right\|_{L_q(0,1)} \le \varepsilon \left\|u^{(4m)}\right\|_{L_q(0,1)} +$$

$$+C(\varepsilon)\|u\|_{L_q(0,1)} \le \varepsilon \left\|A^2 u\right\|_{L_q(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)},$$

Q.E.D.

It follows from condition 4 that

$$\|A_1 u\|_{L_q(0,1)} \le \varepsilon \|u\|_{W_q^{2m}(0,1)} \le C(\varepsilon)\|u\|_{L_q(0,1)} \le$$

$$\le \varepsilon \|Au\|_{L_q(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)},$$

i.e. the operator A_1 satisfies condition 4^0 of theorem 1.1. In the same way, it follows from condition S that

$$\|B_1 u\|_{L_q(0,1)} \le \varepsilon \|Bu\|_{L_q(0,1)} \le C(\varepsilon)\|u\|_{L_q(0,1)}, u \in D(B).$$

So, all the conditions of theorem 1.1 are verified for problem (3.6), whence the statement of theorem 3.1 follows.

3.2. Two-fold completeness of root functions of the problem

For $m < p$ consider the homogeneous equation

$$L(\lambda)0 = \lambda^2 u(x) + \lambda\left(au^{(2m)}(x) + A_1 u\right) +$$

$$+\left(bu^{(2p)}(x) + B_1 u\right) = 0, \tag{3.10}$$

with conditions

$$L_\nu u = 0,\ \nu = 1 \div 2p, \tag{3.11}$$

where L_ν are determined by equalities $(3.2), (3.3)$.

Theorem 3.2. *Let*

1. $a \neq 0$, $b \neq 0$, $m \ge 1$, $m < p < 2m$, $m_\nu \le 2m-1$, $m_{\nu_s} \le 2p-2m$.

2. The determinants of (3.4) *and* (3.5) *be not zero;*

3. The operator A_1 *from* $W_2^{2m}(0,1)$ *in* $L_2(0,1)$ *be compact;*

4. The operator B_1 *from* $W_2^{2p}(0,1)$ *in* $L_2(0,1)$ *be compact;*

5. For some $\eta \in [1,\infty)$ *the functionals* T_ν *in* $W_2^{m_\nu}(0,1)$ *be continuous.*

Then the spectrum of problem $(3.10)-(3.11)$ *is discrete and the system of root functions of problem* $(3.10)-(3.11)$ *is two-fold complete in*

$$W_2^{2p}((0,1), L_\nu u = 0,\ \ \nu = 1 \div 2p) \times W_2^{2m}((0,1), L_\nu u = 0,\ \ \nu = 1 \div 2m).$$

Proof. Introducing in $L_2(0,1)$ the operators A and B by the equalities

$$D(A) = W_2^{2m}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2m)$$

$$Au = au^{(2m)}(x) + \omega u(x), \omega \in \mathbb{C},$$

$$D(B) = W_2^{2p}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2p)$$

$$Bu = bu^{(2p)}(x),$$

we reduce problem (3.10) – (3.11) in $L_2(0,1)$ to the equation

$$L(\lambda)u = \lambda^2 u + \lambda(A + A_1)u + (B + B_1)u = 0. \tag{3.12}$$

In proving theorem 3.1 for problem (3.12), all the conditions of theorem 3.2 except $\overline{D(A)} = L_2(0,1)$ and conditions 2 are verified. The equality

$$\overline{W_2^{2p}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2p)} = L_2(0,1)$$

follows from [12].

The relation

$$D(A) = W_2^{2m}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2m) \supset W_2^{2p}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2p)$$

follows from $m < p$. The equality

$$\overline{H(B)}|_{H(A)} = \overline{W_2^{2p}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2p)}|_{W_2^{2m}(0,1)} =$$

$$= W_2^{2m}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2m) = H(A)$$

then $D(A^2) \subset D(B)$. In fact since $\left.\overline{H(A^2)}\right|_{H(A)} = H(A)$, then $\left.\overline{H(B)}\right|_{H(A)} = H(A)$. From [10, p. 437/14] the equivalence relation

$$S_j\left(J, W_2^{S_1}(0,1), W_2^{S_0}(0,1)\right) \sim j^{-(S_1 - S_0)} S_1 > S_0, \tag{3.13}$$

it follows that for some

$$\rho > 0 \quad J \in \sigma_\rho(H(A), H), J \in \sigma_\rho(H(B), H(A)).$$

Really, it follows from (3.13)

$$S_j\left(J, W_2^{2m}(0,1), L_2(0,1)\right) \sim j^{-2m}$$

that

$$S_j\left(J, W_2^{2p}(0,1), W_2^{2m}(0,1)\right) \sim j^{-(2p-2m)}.$$

On the other hand, $H(A) = W_2^{2m}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2m)$ and $H(B) = W_2^{2p}((0,1), L_\nu u = 0, \quad \nu = 1 \div 2p)$ are the sub-spaces of $W_2^{2m}(0,1)$ and $W_2^{2p}(0,1)$, respectively. Then, for $\rho > \max\left\{\frac{1}{2m}, \frac{1}{2p-2m}\right\}$, $J \in \sigma_\rho(H(A), H)$ and

$J \in \sigma_\rho(H(B), H(A))$. So, condition 2^0 of theorem 2.1 is verified. Since the estimation (3.7) and (3.9) are fulfilled expect two arbitrary small angles

$$G(0, m\pi + \pi + \arg a - \varepsilon, m\pi + \pi + \arg a + \varepsilon)$$

and

$$G(0, (p-m)\pi + \pi + \arg b - \arg a - \varepsilon, (p-m)\pi + \pi + \arg b - \arg a + \varepsilon)$$

everywhere in a complex plane, then condition 3^0 of theorem 2.1 is fulfilled in the complete form.

So, theorem 2.1 is applicable to problem (3.12), whence the statement of theorem 3.2 follows.

References

[1]. Keldysh M.V. *On eigen values and eigen functions of some classes of not self adjoint equations.* Dokl. An SSSR, 1951, vol. 27, No 1, pp. 11-14 (Russian).

[2]. Allahverdiev Y.E. *On completeness of a system of eigen and adjoint elements of not self adjoint operators close to normal ones.* DAN SSSR, 1957, vol. 115, No 2, pp. 207-210 (Russian).

[3]. Palant Yu.A. *On a test for completeness of the system of eigen and adjoint vectors of polynomial pencil of operators.* DAN SSSR, 1961, vol. 141, No 3, pp. 558-560 (Russian).

[4]. Yakubov S.Ya., Mamedov K.S. *On multiple completeness of the system of eigen and adjoint elements of a polynomial operator pencil and in multiple expansions in this system.* Funk. analiz. i. ego prilozhenia. 1975, vol. 9, No 1, pp. 91-93 (Russian).

[5]. Radzievskii G.V. *A problem on completeness of root vectors in spectral theory of operator functions.* Uspekhii matem. Nauk, 1982, vol. 37, 2, pp. 81-143 (Russian).

[6]. Dimfod N., Shwartz Y.G. *Linear operator.* spectral theory M.Mir, 1966 (Russian).

[7]. Yakubov S.Ya. *Multiple completeness of root vectors of unbounded polynomial operator pencil.* Revue Roumaine de Math. pures appl. 1986, vol. XXXI, No 5, pp. 423-138 (Russian).

[8]. Yakubov S.Ya. *Nonlocal problem for a class of correct by Petrovsky equations.* Mat. Sbornik.1982, 118 (160) , No 2, (6), pp. 252-261 (Russian).

[9]. Besov O.V., Il'in V.P., Nikolskiy S.M. *Integral representations of functions and imbedding theorem.* M.Nauka, 1975 (Russian).

[10]. Tribel Kh. *Interpolation theory Functional spaces. Differential operators.* M. Mir, 1980 (Russian).

[11]. Yakubov S.Ya. *Multiple completeness by M.V Keldysh of root functions of elliptic boundary value problems with polynomial spectral parameter.* Izv. AN SSSR, 1986, vol. 50, No 2, pp. 425-432 (Russian).

[12]. Yakubov S.Ya. *Operator pencils and differential operator equations.* Dep in VINITI, No 6295-84, 1984 (Russian).

[13]. *Some problems of theory of non-linear analysis.* Issue II, 1990. (Trudy IMM, AN Azerb Rep) (Russian).

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