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LIMIT THEOREMS FOR THE FIRST PASSAGE TIME FOR THE LEVEL BY NONLINEAR FUNCTION OF MARKOV CHAIN

Abstract

In the paper, strong law of large numbers and integral limit theorem for the first passage time of high level by the Markov chain dependent function are proved.

1. Introduction. Let on some probability space (Ω, F, P) a Markov chain $X = (X_n, n \geq 0)$ be given with on a real straight line $R = (-\infty, \infty)$ with transitional probability

$$P_n(X, B) = P(X_{n+1} \in B / X_n = x),$$

where $x \in R$ and $B \in \beta(R)$ is σ-algebra of Borel sets in R.

Homogeneity of transitional probability in time is not assumed.

Let $\Delta(x),\ x \in R$ be some Borel function in R.

Assume

$$T_n = n\Delta\left(\frac{X_n}{n}\right),\ n \geq 1,\ T_0 = 0 \tag{1}$$

and consider the first passage time $\tau_c = \inf\{n \geq 1 :\ T_n \geq c\}$ of a level $c \geq 0$ by the process T_n, where we everywhere suppose that $\inf\{\varnothing\} = \infty$.

In the case when $\Delta(x) \equiv x$ and a Markov chain X is made by the sums of independent identical random variables, in references there are many results on distribution of τ_a and overshoot $\chi_c = T_{\tau_c} - c$. Statement of results in this direction may be found in [1]-[4] and in many other papers.

In the papers [2], [4], [5], integral and local limit theorems for τ_c are studied for a nonlinear case $\Delta(x) \neq x$ and ordinary random walk. Note that in these papers, the theory of limit theorems for the sum of independent random variables is on the basis of the used method.

Development of theory of limit theorems for Markov's general chains (see [4]) allows to promote theory of boundary value problems for random walks ([6], [7], [8]).

In the given paper, we cite generalizations of results of the paper [8] on limit theorems for the first passage time of Markov chain beyond the level.

2. Conditions and formulation of basic results.

By $\xi_n(u)$ we denote a jump of the chain X from the state u at time n, whose distribution is given by the equality

$$P(u + \xi_n(u) \in B) = P_n(u, B),\ B \in \beta(R),\ u \in R.$$

For the Markov chain we'll assume that it is a chain with a shift asymptotically homogeneous in time and in the space, i.e. $E\xi_n(u)$ converges as $n, u \to \infty$ to some number $\mu \in R$, and the existence of $E\xi_n(u)$ is not assumed for all values of n and u (see [9]).

For the function $\Delta(x),\ x \in R$ we'll assume that it accepts positive values and is twice differentiable on $R = (-\infty, \infty)$, moreover $\Delta(\mu) > 0$ and $\Delta'(\mu) \neq 0$.

Under the made assumptions, we can write

$$T_n = n\Delta(X_n/n) = Z_n + \varepsilon_n,\ n \geq 1, \tag{2}$$

where

$$Z_n = n\Delta(\mu) + n\Delta'(\mu)\left(\frac{X_n}{n} - \mu\right) \quad \text{and} \quad \varepsilon_n = \frac{n}{2}\Delta''(v_n)\left(\frac{X_n}{n} - \mu\right)^2$$

and ν_n is an intermediate point between $\dfrac{X_n}{n}$ and μ.

We have

$$Z_n = n(\Delta(\mu) - \mu\Delta'(\mu)) + \Delta'(\mu)X_n$$

and

$$Z_n - Z_{n-1} = \Delta(\mu) - \mu\Delta'(\mu) + \Delta'(\mu)(X_n - X_{n-1}).$$

Hence it is seen that the sequence $Z_n, n \geq 1$ $(Z = 0)$ in (2) is a Markov chain with a shift $\Delta(\mu) > 0$ asymptotically homogeneous in time and in space.

The following theorems are valid.

Theorem 1. *Let the above mentioned conditions for the function $\Delta(x)$ and the Markov chain be fulfilled, and it hold a convergence $X_n \to \infty$ almost sure as $n \to \infty$. Assume that for some spatial level U and time N, a family of random variables $\{|\xi_n(u)|,\ n \geq N,\ u \geq U\}$ possesses integrable majorant, i.e. there exists a random variable ξ with finite mean value ($E|\xi| < \infty$), such that $P(|\xi_n(u) \leq \xi|) = 1$ for any $n \geq N,\ u \geq U$.*

Then

1) $P(\tau_c < \infty) = 1$ for all $c \geq 0$,

2) $\tau_c \to \infty$ almost sure as $c \to \infty$.

3) $\dfrac{\tau_c}{c} \to \dfrac{1}{\Delta(\mu)}$ almost sure as $c \to \infty$.

Theorem 2. *Let all the above mentioned conditions be fulfilled for the function $\Delta(x)$ and the Markov chain X with shift $\mu > 0$, it hold convergence $X_n \to \infty$ almost sure as $n \to \infty$. Assume that for some time N and spatial level U a family of squares of jumps $\left\{\left|\xi_n^2(u)\right|,\ n \geq N,\ u \geq U\right\}$ is uniformly integrable with respect to n and u, and as $n, u \to \infty$ there hold the relations*

$$E\xi_n(u) = \mu + o\left(1/\sqrt{u} + 1\sqrt{u}\right),$$

and

$$D\xi_n(u) \to \sigma^2 > 0.$$

Furthermore, let for any $\varepsilon > 0$

$$\lim_{\delta \to 0} \sup_{n \geq 1} P\left(\max_{1 \leq k \leq n\delta}\left|X_{n+1}^* - X_n^*\right| \geq \varepsilon\right) = 0,$$

where

$$X_n^* = \frac{X_n - n\mu}{\sigma\sqrt{n}}.$$

Then

$$\lim_{c\to\infty} P\left(\frac{\tau_c - c/\Delta(\mu)}{\Delta'(\mu)\sigma\sqrt{c}} \le (\Delta(\nu))^{-\frac{3}{2}}\, x\right) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} d_y.$$

Remark 1. Condition (2) means that the sequence $X_n^* = \dfrac{X_n - n\mu}{\sigma\sqrt{n}}$, $n \ge 1$ is uniformly continuous in probability (see [2]).

Remark 2. Note that the ordinary process of summation $X_n = \xi_1 + ... + \xi_n$ of independent identical random variables ξ_k, $k \ge 1$ with positive mean value and random walk with delay in zero $X_{n+1} = \max\left(0, X_n + \xi_{n+1}\right)$ make a Markov chain that satisfy the conditions of theorems 1 and 2 (see [9]).

3. Proof of basic results. The following results formulated as lemmas play an important role in proving integral theorems for boundary functionals connected with τ_c.

Lemma 1 (Anscombe theorem). *Let t_c, $c > 0$ be a family of non-negative integer random variables such that $\dfrac{t_c}{c} \xrightarrow{p} a > 0$ as $c \to \infty$, and let a sequence of random variables Y_n, $n \ge 1$ be uniformly continuous in probability. Then*

$$Y_{t_c} - Y_{[ca]} \xrightarrow{p} 0 \quad \text{as} \quad c \to \infty.$$

Moreover, if Y_n weakly converges to random variable Y, then Y_{t_c} also weakly converges to Y as $c \to \infty$.

The proof of Anscombe theorem may be found for example, in the work [2] and [3].

Lemma 2. *If sequence of randopm variables Y_n' and Y_n'', $n \ge 1$ are uniformly continuous in probability, the sequence of their sum $Y_n' + Y_n''$, $n \ge 1$ is also uniformly continuous in probability. In addition, if Y_n' and Y_n'', $n \ge 1$ are stochastically bounded and the function $K(x, y)$ is continuous on R^2, the sequence $k(Y_n', Y_n'')$, $n \ge 1$ is uniformly continuous in probability.*

The proof of this lemma is given in [2].

Proof of theorem 1. In the conditions of theorem 1, the strong law of large for Markov chain numbers from the paper [9] is fulfilled. By this law, for the considered Markov chain, it holds almost sure convergence

$$\frac{X_n}{n} \to \mu \quad \text{as} \quad n \to \infty.$$

Then, there hold almost sure convergences

$$\frac{Z_n}{n} \to \Delta(\mu) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \frac{\varepsilon_n}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, by equality (1) it holds almost sure convergence

$$\frac{T_n}{n} \to \Delta(\mu) \quad \text{as} \quad n \to \infty.$$

Hence, it follows that $\sup_n T_n = \infty$. Therefore, it follows from the equality

$$P\left(\max_{1\le k<n} T_k \ge c\right) = P\left(\tau_c < n\right)$$

that $P\left(\tau_c < \infty\right) = 1$ for all $c \ge 0$.

Prove statement 2). The quantity τ_c as a function of c doesn't decrease with probability unit. Therefore we have

$$P\left(\tau_\infty = \lim_{c\to\infty} \tau_c \le \infty\right) = 1.$$

On the other hand, for each $m \ge 1$

$$P(\tau_\infty \le m) = \lim_{c\to\infty} P(\tau_c \le m) = \lim_{c\to\infty} P\left(\max_{k\le m} T_k \ge c\right) = 0$$

Hence it follows that $\tau_c \to \infty$ almost sure as $c \to \infty$.

For proving statement 3) it suffices to note

$$\frac{T_{\tau_c - 1}}{\tau_c} < \frac{c}{\tau_c} \le \frac{T_{\tau_c}}{\tau_c} \tag{3}$$

By statement 1) of the proved theorem 1 and Richter's lemma [2] it holds almost sure convergence

$$\frac{T_{\tau_c}}{\tau_c} \to \Delta(\mu) \quad \text{as} \quad c \to \infty.$$

Therefore, statement 3 follows from inequality (3).

Proof of theorem 2. Assume

$$\chi_c = T_{\tau_c} - c = Z_{\tau_c} + \varepsilon_{\tau_c} - c.$$

We have

$$\frac{Z_{\tau_c} - \tau_c\Delta(\mu)}{\Delta'(\mu)\sigma\sqrt{\tau_c}} = \frac{c - \tau_c\Delta(\mu)}{\Delta'(\mu)\sigma\sqrt{\tau_c}} + \frac{\chi_c - \varepsilon_{\tau_c}}{\Delta'(\mu)\sigma\sqrt{\tau_c}} \tag{4}$$

and

$$\frac{Z_n - n\Delta(\mu)}{\Delta'(\mu)\sigma\sqrt{n}} = \frac{X_n - n\mu}{\sigma\sqrt{n}} = X_n^* \,. \tag{5}$$

In the conditions of the proved theorem there is a central limit theorem for Markov chain from the paper [9]. By this theorem, for the considered Markov chain, it holds

$$\lim_{n\to\infty} P\left(\frac{X_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \Phi(x).$$

Hence, by condition (2) and statement 3) of theorem 1, and from Anscombe theorem (lemma1) we have

$$\lim_{n\to\infty} P(X_{\tau_c}^* \le x) = \Phi(x), \tag{6}$$

where

$$X_{\tau_c}^* = \frac{Z_{\tau_c} - \tau_c\Delta(\mu)}{\Delta'(\mu)\sigma\sqrt{\tau_c}}.$$

Now, prove that the second term in equality (4) converges to zero in probability as $c \to \infty$, i.e.

$$\frac{\chi_c - \varepsilon_{\tau_c}}{\sqrt{\tau_c}} \xrightarrow{p} 0 \text{ as } c \to \infty. \tag{7}$$

At first we prove

$$\frac{\varepsilon_n}{\sqrt{n}} \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{8}$$

We have

$$\frac{\varepsilon_n}{\sqrt{n}} = \frac{1}{2}\Delta''(v_n)\left(\frac{X_n}{n} - \mu\right)\left(\frac{X_n - n\mu}{\sqrt{n}}\right).$$

Considering that almost sure

$$\Delta''(v_n) \to \Delta(\mu) \text{ and } \frac{X_n}{n} - \mu \to 0$$

as $n \to \infty$, (8) follows from the last equality.

By condition (2) and lemma 2, the sequence $\frac{\varepsilon_n}{\sqrt{n}}$, $n \geq 1$ is uniformly continuous in probability. Therefore, it follows from (8) by Anscombe theorem that

$$\frac{\varepsilon_{\tau_c}}{\sqrt{\tau_c}} \xrightarrow{p} 0 \text{ as } c \to \infty. \tag{9}$$

Show that

$$\frac{\chi_c}{\sqrt{\tau_c}} \xrightarrow{p} 0 \text{ as } c \to \infty. \tag{10}$$

Really, by definition of τ_c we have

$$\chi_c = Z_{\tau_c} + \varepsilon_{\tau_c} - c \leq Z_{\tau_c} + \varepsilon_{\tau_c} - (Z_{\tau_c - 1} + \varepsilon_{\tau_c - 1}) =$$

$$= \Delta(\mu) - \mu\Delta'(\mu) + \Delta'(\mu)(X_{\tau_c} - X_{\tau_c - 1}) + \varepsilon_{\tau_c} - \varepsilon_{\tau_c - 1}.$$

It is easy to see that by (9) for proving (10), it suffices to show that

$$\frac{X_{\tau_c} - X_{\tau_c - 1}}{\sqrt{\tau_c}} \xrightarrow{p} 0 \text{ as } c \to \infty\ . \tag{11}$$

To this end, we can write

$$\frac{X_n - X_{n-1}}{\sigma\sqrt{n}} = X_n^* - \sqrt{\frac{n-1}{n}}X_{n-1}^* + \frac{\mu}{\sigma\sqrt{n}}.$$

Then, by condition (2) of the proved theorem and lemma 2, the sequence $\frac{X_n - X_{n-1}}{\sqrt{n}}$, $n \geq 1$ is uniformly continuous in probability. Consequently, for each fixed state u of the Markov chain the sequence of normed jumps $\frac{\xi_n(u)}{\sqrt{n}}$, $n \geq 1$ is uniformly continuous in probability.

A family of random variables $\{|\xi_n(u)|\,,\ u \geq U,\ n \geq N\}$ possesses an integrable majorant, since the family of squares of jumps $\left\{\left|\xi_n^2(u)\right|,\ n \geq N,\ u \geq U\right\}$ is uniformly integrable. Therefore

$$\frac{\xi_n(u)}{\sqrt{n}} \xrightarrow{p} 0 \text{ as } n \to \infty$$

uniformly with respect to $u \geq U$.

Then, by the Anscombe mentioned theorem, we have

$$\frac{\xi_{\tau_c}(u)}{\sqrt{\tau_c}} \xrightarrow{p} 0 \quad \text{as} \quad c \to \infty.$$

Hence, (11) follows. Thus, (7) follows from (9) and (11).

Now, from (4), (6) and (7) we find

$$\lim_{c \to \infty} P\left(\frac{c - \tau_c \Delta(\mu)}{\Delta'(\mu)\sigma\sqrt{\tau_c}} \leq x\right) = \Phi(x).$$

By theorem 1 we have

$$\lim_{c \to \infty} P\left(\frac{\tau_c - \dfrac{c}{\Delta(\mu)}}{\Delta'(\mu)\dfrac{\sigma}{\Delta(\mu)}\sqrt{\dfrac{c}{\Delta(\mu)}}} \geq -x\right) = \Phi(x).$$

Hence, the statement of theorem 2 follows.

Reverences

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Received May 22, 2009; Revised September 17, 2009.