Yagub A. SHARIFOV, Shamo I. DJABRAILOV

SINGULAR CONTROLS IN THE SENSE OF PONTRYAGIN'S MAXIMUM PRINCIPLE FOR CONTROLLED SYSTEMS WITH THREE-POINT BOUNDARY CONDITIONS

Abstract

An optimal control problem wherein the system's state is determined from controlled system of ordinary differential equations with three-point boundary conditions is considered in the paper. Admissible controls are chosen from a class of bounded and measurable functions. Validity of Pontryagin's maximum principle is proved for the investigated class of problems. A formula for an increment of a second order functional is calculated. Necessary optimality condition for singular controls in the sense of Pontryagin's maximum principle is obtained in the base of needle-shaped control variation.

1. Introduction. Problem Statement. It is known that the solution of many problems of mechanics and control processes [1] is reduced to two-point, three-point and multi-point boundary value problems. The optimal control problems where a control process is described by two-point boundary conditions are considered in the papers [2-5]. In these papers, first and second order necessary optimality conditions are obtained.

Since in optimal control problems with three-point boundary conditions the solution of the adjoint system undergoes first kind discontinuity at the internal point, direct application of solution methods of two-point boundary value problems to optimal control problems with three-point boundary conditions is impossible. In the paper [6], an optimal control problem wherein system's state is described by means of three-point boundary conditions is investigated and a first order necessary optimality condition is obtained. The goal of the paper is to derive second order necessary optimality condition when a first order necessary optimality condition degenerates.

Let the state of the investigated object be described by a system of differential equations with three-point boundary conditions:

$$\dot{x} = f(x,u,t), \ t \in T = [t_0, t_2] \tag{1}$$

$$Ax(t_0) + Bx(t_1) + Cx(t_2) = D. \tag{2}$$

Here, it is assumed that $x = (x_1, x_2, ..., x_n)$ is an n-dimensional state vector; $f(x,u,t)$ is an n-dimensional given function; $A, B, C \in R^{n\times n}$ and $D \in R^{n\times 1}$ are the known constant matrices; $u = (u_1, u_2, ..., u_r)$ is an r- dimensional control parameter; $t_1 \in (t_0, t_2)$ is a fixed point.

The goal of the optimal control problem is to minimize the functional

$$J(u) = \varphi(x(t_0), x(t_2)) + \int_T F(x,u,t)\,dt \tag{3}$$

on the solution of phase system (2.1) (2.2) in the class of admissible controls

$$U = \{u \in L_\infty^r(T); u(t) \in V, t \in T\}, \tag{4}$$

where the function $\varphi(x,y)$ is continuous and differentiable on $R^n \times R^n$ and the function $F(x,u,t)$ is continuous in totality of its arguments on $R^n \times V \times T$ together with derivatives of the variable x up to second order. Assume that three-point boundary value problem (1), (2) for each admissible control $u(t) \in U,\ t \in T$ has a unique solution $x(t,u)$. Admissible process $\{u(t), x(t,u)\}$ being a solution of problem (1)-(4), i.e. delivering minimal value to functional (3) under restrictions (1), (2), (4) is said to be an optimal process, and $u(t)$ an optimal control.

2. A functional increment formula. Optimal control problem (1)-(4) may be investigated by using different variants of aim functional increment formula on two admissible processes $\{u,x\}$ and $\{\widetilde{u} = u + \Delta u, \widetilde{x} = x + \Delta x\}$.

L. I. Rozonoer's increments method [7] that is already a classic one, allows to get in the considered problem necessary optimality condition of Pontryagin's maximum principle type [8-10]. Notice that while deriving necessary optimality conditions, locality of increment formula is essential, i. e. the remainder terms are estimated by the quantity defining the smallness of the measure of domain of needle-shaped variation of control.

Let $u(t), t \in T$ be a fixed admissible control. Take one more admissible control $\widetilde{u}(t) = u(t) + \Delta u(t),\ t \in T$. Denote by $x(t,u), t \in T$ and $\widetilde{x}(t) = x(t,u) + \Delta x(t) = x(t,\widetilde{u}),\ t \in T$ the solutions of problem (1), (2) corresponding to them. We can determine a boundary value problem in increments for boundary value problem (1), (2):

$$\Delta \dot{x} = \Delta f(x,u,t) \tag{5}$$

$$A\Delta x(t_0) + B\Delta x(t_1) + C\Delta x(t_2) = 0, \tag{6}$$

where

$$\Delta f(x,u,t) = f(\widetilde{x}, \widetilde{u}, t) - f(x,u,t)$$

denotes a complete increment of the function $f(x,u,t)$. For special increments we'll use the denotation

$$\Delta_{\widetilde{u}} f(x,u,t) = f(x,\widetilde{u},t) - f(x,u,t)$$

Compose increment formulae (3) that correspond to admissible controls $\{u, x(t,u)\}$ and $\{\widetilde{u} = u + \Delta u, \widetilde{x} = x(t) + \Delta x(t) = x(t,\widetilde{u})\}$

$$\Delta J(u) = J(\widetilde{u}) - J(u) = \Delta\varphi(x(t_0), x(t_1), x(t_1)) + \int_T \Delta F(x,u,t)\,dt \tag{7}$$

Make some standard operations usually used in derivation of first order necessary optimality conditions.

In formula (7)

- add zero terms

$$\int_T \Big\langle \psi(t), \Delta\dot{x} - \Delta f(x,u,t) \Big\rangle dt \tag{8}$$

and

$$\langle \lambda, A\Delta x(t_0) + B\Delta x(t_1) + C\Delta x(t_2) \rangle, \tag{9}$$

where $\psi(t) \in R^n,\quad t \in T;\ \lambda \in R^n$ are some victor function and constant vector undetermined for the present, a scalar product in $\langle .,. \rangle$ is denoted by R^n;

- introduce Pontryagin's function

$$H(\psi, x, u, t) = \langle \psi(t), f(x, u, t)\rangle - F(x, u, t);$$

- expand the increment $\Delta\varphi(x(t_0), x(t_2))$ by second order Taylor formula

$$\Delta\varphi(x(t_0), x(t_1)) = \left\langle \frac{\partial\varphi}{\partial x(t_0)}, \Delta x(t_0)\right\rangle + \left\langle \frac{\partial\varphi}{\partial x(t_2)}, \Delta x(t_2)\right\rangle +$$
$$+\frac{1}{2}\left\langle \frac{\partial^2\varphi}{\partial x(t_0)^2}\Delta x(t_0) + \frac{\partial^2\varphi}{\partial x(t_2)\partial x(t_0)}\Delta x(t_2), \Delta x(t_0)\right\rangle +$$
$$+\frac{1}{2}\left\langle \frac{\partial^2\varphi}{\partial x(t_2)^2}\Delta x(t_2) + \frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_2)}\Delta x(t_0), \Delta x(t_2)\right\rangle +$$
$$+o_\varphi\left(\|\Delta x(t_0)\|^2, \|\Delta x(t_2)\|^2\right). \tag{10}$$

Considering (8)-(10) in (7), we have:

$$\Delta J(u) = -\int_T \Delta_{\widetilde{u}} H(\psi, x, u, t)\,dt - \int_T \left\langle \Delta_{\widetilde{u}} \frac{\partial H(\psi, x, u, t)}{\partial x}, \Delta x(t)\right\rangle dt -$$
$$-\int_T \left\langle \dot{\psi} + \frac{\partial H(\psi, x, u, t)}{\partial x}, \Delta x(t)\right\rangle dt +$$
$$+\left\langle \left[\frac{\partial\varphi(x(t_0), x(t_1), x(t_2))}{\partial x(t_0)} - \psi(t_0) + A'\lambda\right], \Delta x(t_0)\right\rangle +$$
$$+\left\langle \left[\frac{\partial\varphi(x(t_0), x(t_1), x(t_2))}{\partial x(t_1)} + \psi(t_1 - 0) - \psi(t_1 + 0) + B'\lambda\right], \Delta x(t_1)\right\rangle +$$
$$+\left\langle \left[\frac{\partial\varphi(x(t_0), x(t_1), x(t_2))}{\partial x(t_2)} + \psi(t_1) + C'\lambda\right], \Delta x(t_2)\right\rangle +$$
$$+0_\varphi(\|\Delta x(t_0)\|, \|\Delta x(t_2)\|) - \int_T 0_H(\|\Delta x(t)\|)\,dt. \tag{11}$$

Since in (11) the vector function $\psi(t) \in R^n, t \in T$ and constant vector $\lambda \in R^n$ were arbitrary, now we can determine them as solutions of the following linear differential equation with boundary conditions (stationary state condition of Lagrange function by the state).

$$\psi = -\frac{\partial H(\psi, x, u, t)}{\partial x} \tag{12}$$

$$\psi(t_0) = -\frac{\partial\varphi(x(t_0), x(t_2))}{\partial x(t_0)} + A'\lambda \tag{13}$$

$$\psi(t_1 + 0) - \psi(t_1 - 0) = B'\lambda \tag{14}$$

$$\psi(t_2) = -\frac{\partial\varphi(x(t_0), x(t_2))}{\partial x(t_2)} - C'\lambda. \tag{15}$$

Boundary value problem (12)-(15) is called an adjoint system. For finding the solution of system (12)-(15) it is necessary to find the vector-function $\psi(t) \in R^n, t \in$

T and constant vector $\lambda \in R^n$ such that they satisfy differential equation (12) and boundary conditions (13)-(15). Condition (14) shows that in the general case, the solution of system (12) undergoes first kind discontinuity at the point $t = t_1$.

Assume that the condition

$$\det(A + B + C) \neq 0.$$

is fulfilled. Then assuming

$$\left[\left\|[A+B+C]^{-1}B\right\|(t_1-t_0)+\left\|[A+B+C]^{-1}C+E\right\|(t_2-t_0)\right]K<1,$$

we can get the estimation

$$\max_{[t_0,t_2]}\|\Delta x(t)\| \le \left\{\left[\left\|[A+B+C]^{-1}B\right\|(t_1-t_0)+\right.\right.$$

$$\left.\left.+\left\|[A+B+C]^{-1}C+E\right\|(t_2-t_0)\right]K\right\}\times$$

$$\times\left[\left\|[A+B+C]^{-1}B\right\|+\left\|[A+B+C]^{-1}C+E\right\|\right]\int_{t_0}^{t_2}|\Delta_{\tilde{u}}f(x,u,t)|\,dt. \qquad (16)$$

Formula (16) is said to be an increment formula estimation of the solution of boundary value problem (1), (2) by two different controlling functions.

On the other hand, it follows from equalities (5), (6) that $\Delta x(t)$ is a solution of the following linearized system

$$\Delta\dot{x}(t) = \frac{\partial f(x,u,t)}{\partial x}\Delta x(t) + \Delta_{\tilde{u}}f(x,u,t) + \eta_1(t), \qquad (17)$$

$$A\Delta x(t_0) + B\Delta x(t_1) + C\Delta x(t_2) = 0, \qquad (18)$$

where by definition

$$\eta_1(t) = 0_f(\|\Delta x(t)\|).$$

Let the matrix-function $\Phi(t)$, $t \in T$ be a solution of the following matrix differential equation

$$\dot{\Phi}(t) = \frac{\partial f(x,u,t)}{\partial x}\Phi(t)$$

with the initial condition

$$\Phi(t_0) = E,$$

where E is a unique matrix of dimension $n \times n$.

Then, we can represent any solution of problem (12), (18) in the form:

$$\Delta x(t) = \Phi(t)\Delta x(t_0) + \int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)\Delta_{\tilde{u}}f(x,u,\tau)\,d\tau + \eta_2(t), \qquad (19)$$

where

$$\eta_2(t)=\int_{t_0}^{t}\Phi(t)\,\Phi^{-1}(\tau)\,\eta(\tau)\,d\tau.$$

Require the function (19) satisfy condition (18). Then we get

$$[A+B\Phi(t_1)+C\Phi(t_2)]\,\Delta x(t_0)=-B\Phi(t_1)\int_0^1\Phi^{-1}(\tau)\,\Delta_{\widetilde{u}}f(x,u,\tau)\,d\tau-$$

$$-C\Phi(t_2)\int_{t_0}^{t_2}\Phi^{-1}(\tau)\,\Delta_{\widetilde{u}}f(x,u,\tau)\,d\tau+\eta_3(t)$$

where

$$\eta_3=B\eta_2(t_1)+C\eta(t_2).$$

Hence

$$\Delta x(t_0)=\int_{t_0}^{t_2}S(t)\,\Phi^{-1}(t)\,\Delta_{\widetilde{u}}f(x,u,\tau)\,d\tau+\eta_4 \tag{20}$$

where

$$\eta_4=[A+B\Phi(t_1)+D\Phi(t_2)]^{-1}\,[B\eta_2(t_1)+C\eta_2(t_2)]\,, \tag{21}$$

$$S(t)=S_1\alpha_1(t)+S_2\alpha_2(t)\,,$$

$$S_1=-\,[A+B\Phi(t_1)+C\Phi(t_2)]^{-1}\,[B\Phi(t_1)+D\Phi(t_2)]\,,$$

$$S_2=-\,[A+B\Phi(t_1)+C\Phi(t_2)]^{-1}\,D\Phi(t_2)\,,$$

$\alpha_1(t)$ and $\alpha_2(t)$ are characteristic functions of the sections $[t_0,t_1]$ and $[t_1,t_2]$, respectively. Considering (20) in (19), we have

$$\Delta x(t)=-\Phi(t)\int_{t_0}^{t_2}S(t)\,\Phi^{-1}(t)\,\Delta_{\widetilde{u}}f(x,u,\tau)\,dt+$$

$$+\int_{t_0}^{t}\Phi(t)\,\Phi^{-1}(\tau)\,\Delta_{\widetilde{u}}f(x,u,\tau)\,d\tau+\eta_5(t) \tag{22}$$

where

$$\eta_5(t)=\Phi(t)\,\eta_4+\eta_2(t)\,.$$

Finally, from the latter one we get

$$\Delta x(t_2)=\Phi(t_2)\int_{t_0}^{t_2}[S(t)+E]\,\Phi^{-1}(\tau)\,\Delta_{\widetilde{u}}f(x,u,\tau)\,d\tau+\eta_5(t)\,. \tag{23}$$

Now, we rewrite the terms contained in (11) in the form:

$$\left\langle \Delta x(t_0)'\,\frac{\partial^2\varphi}{\partial x(t_0)^2},\Delta x(t_0)\right\rangle=\int_T\int_T\Big\langle \Delta'_{\widetilde{u}}f(x,u,\tau)\,\Phi^{-1'}(\tau)\,S(t)'\times$$

$$\times\frac{\partial^2\varphi}{\partial x\,(t_0)^2}S\,(s)\,\Phi^{-1}\,(s)\,,\Delta_{\widetilde{u}}f\,(x,y,s)\Big\rangle\,d\tau ds+$$

$$+\left\langle\eta_4'\frac{\partial^2\varphi}{\partial x\,(t_0)^2},\Delta x\,(t_0)\right\rangle+\left\langle\Delta x\,(t_0)'\,\frac{\partial^2\varphi}{\partial x\,(t_0)^2},\eta_4\right\rangle,\tag{24}$$

$$\left\langle\Delta x\,(t_2)'\,\frac{\partial^2\varphi}{\partial x\,(t_0)^2},\Delta x\,(t_2)\right\rangle=\int\limits_T\int\limits_T\Big\langle\Delta_{\widetilde{u}}'f\,(x,u,\tau)\,\Phi^{-1'}\,(\tau)\,[S\,(t)+E]\,\Phi\,(t_2)'\times$$

$$\times\frac{\partial^2\varphi}{\partial x\,(t_2)^2}\Phi\,(t_2)\,[S\,(s)+E]\,\Phi^{-1}\,(s)\,,\Delta_{\widetilde{u}}f\,(x,u,s)\Big\rangle\,d\tau ds+$$

$$+\left\langle\eta_5\,(t_2)'\,\frac{\partial^2\varphi}{\partial x\,(t_2)^2},\Delta x\,(t_2)\right\rangle+\left\langle\Delta x\,(t_2)'\,\frac{\partial^2\varphi}{\partial x\,(t_2)^2},\eta_5\,(t_2)\right\rangle\tag{25}$$

$$\left\langle\Delta x\,(t_0)'\,\frac{\partial^2\varphi}{\partial x\,(t_0)\,\partial x\,(t_2)},\Delta x\,(t_2)\right\rangle=\int\limits_T\int\limits_T\Big\langle\Delta_{\widetilde{u}}'f\,(x,u,\tau)\,\Phi^{-1'}\,(\tau)\,S\,(t)'\times$$

$$\times\frac{\partial^2\varphi}{\partial x\,(t_0)\,\partial x\,(t_2)}\Phi\,(t_2)\,[S\,(s)+E]\,\Phi^{-1}\,(s)\,,\Delta_{\widetilde{u}}f\,(x,u,s)\Big\rangle\,d\tau ds+$$

$$+\left\langle\eta_4{}'\frac{\partial^2\varphi}{\partial x\,(t_0)\,\partial x\,(t_2)},\Delta x\,(t_2)\right\rangle+\left\langle\Delta x\,(t_0)'\,\frac{\partial^2\varphi}{\partial x\,(t_0)\,\partial x\,(t_2)},\eta_5\,(t_2)\right\rangle\tag{26}$$

$$\left\langle\Delta x\,(t_2)'\,\frac{\partial^2\varphi}{\partial x\,(t_2)\,\partial x\,(t_0)},\Delta x\,(t_0)\right\rangle=$$

$$=\int\limits_T\int\limits_T\Big\langle\Delta_{\widetilde{u}}f\,(x,u,\tau)\,\Phi^{-1'}\,(\tau)\,[S\,(t)+E]'\,\Phi\,(t_2)\times$$

$$\times\frac{\partial^2\varphi}{\partial x\,(t_2)\,\partial x\,(t_0)}S\,(t)\,\Phi^{-1}\,(s)\,,\Delta_{\widetilde{u}}f\,(x,u,s)\Big\rangle\,d\tau ds+$$

$$+\left\langle\eta_5\,(t_2)'\,\frac{\partial^2\varphi}{\partial x\,(t_0)\,\partial x\,(t_2)},\Delta x\,(t_0)\right\rangle+\left\langle\Delta x\,(t_2)'\,\frac{\partial^2\varphi}{\partial x\,(t_0)\,\partial x\,(t_2)},\eta_4\right\rangle\tag{27}$$

$$\int\limits_T\left\langle\Delta x\,(t)'\,\frac{\partial^2 H\,(\psi,x,u,\tau)}{\partial x^2},\Delta x\,(t)\right\rangle dt=$$

$$=\int\limits_T\int\limits_T\Big\langle\Delta_{\widetilde{u}}'f\,(x,u,\tau)\Big[\Phi^{-1}\,(\tau)'\int\limits_T\Phi\,(t)'\,\frac{\partial H\,(\psi,x,u,t)}{\partial x}\Phi\,(t)\,d\tau S\,(s)\,\Phi^{-1}\,(s)+$$

$$+\Phi^{-1'}\,(\tau)\int\limits_{\max(\tau,s)}^{t_1}\Phi'\,(t)\,\frac{\partial^2 H\,(\psi,x,u,\tau)}{\partial x^2}\Phi\,(t)\,dt\Phi^{-1}\,(s)-$$

$$-\Phi^{-1'}\,(\tau)\int\limits_{\tau}^{t_1}\Phi'\,(\xi)\,\frac{\partial^2 H\,(\psi,x,u,\xi)}{\partial x^2}\Phi\,(\xi)\,(dt)\,S\,(s)\,\Phi^{-1}\,(s)-$$

$$-\Phi^{-1}(\tau)^{1} S(\tau)\int_{s}^{t_1}\Phi'(\xi)\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi\Phi^{-1}(s)\Bigg]\,d\tau, f(x,u,s)\Bigg\rangle ds. \quad (28)$$

Using (24)-(28) following [10], we introduce the function

$$R(\tau,s)=\Phi^{-1'}(\tau)S(\tau)'\frac{\partial^2\varphi}{\partial x(t_1)^2}S(s)^{-1}A\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)[S(\tau)+E]'\Phi(t_2)'\frac{\partial^2\varphi}{\partial x(t_0)^2}\Phi(t_2)[S(s)+E]\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)S(\tau)'\frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}\Phi(t_2)[S(s)+E]\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)S(\tau)'\frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}\Phi(t_2)[S(s)+E]\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)[S(s)+E]'\Phi(t_2)'\frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}S(s)\Phi^{-1}(s)+$$

$$+\Phi^{-1'}(\tau)\int_{\max(\tau,s)}^{t_1}\Phi'(t)\frac{\partial^2 H(\psi,x,u,\tau)}{\partial x^2}\Phi(t)\,dt\Phi^{-1}(s)-$$

$$-\Phi^{-1'}(\tau)\int_{\tau}^{t_2}\Phi'(\xi)\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi S(s)\Phi^{-1}(s)-$$

$$-\Phi^{-1}(\tau)S(\tau)'\int_{s}^{t}\Phi'(\xi)\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)\,d\xi\Phi^{-1}(s). \quad (29)$$

Using denotation (29) for functional's increment, we get the terminal formula

$$\Delta J(u)=-\int_T\Delta_{\widetilde{u}}H(\psi,x,u,t)\,dt-\int_T\left\langle\Delta_{\widetilde{u}}\frac{\partial H(\psi,x,u,t)\varphi}{\partial x},\Delta x(t)\right\rangle dt$$

$$-\frac{1}{2}\int_T\int_T\left\langle\Delta'_{\widetilde{u}}f(x,u,\tau)R(\tau,s),\Delta_{\widetilde{u}}f(x,u,s)\right\rangle dtds+\eta(u,\Delta u), \quad (30)$$

here $\eta(u,\Delta u)$ is an expression linearly dependent on $\eta_i(t)\, i=1-6$. (Because of its bulky form we don't give it here).

3. Derivation of necessary optimality conditions. Now, pass to direct derivation of Pontryagin's maximum principle.

Consider increment formula of the aim function on needle-shaped variation of adminissible control. We choose the variation parameters in the following way: let $\theta\in(t_0,t_1)\cup(t_1,t_2),\varepsilon\in(0,t_0-\theta)$ and $\nu\in V$

Variation interval $(\theta-\varepsilon,\theta)\in T$ is obvious. We determine needle shaped variation of control as follows $u=u(t)$:

$$\Delta_\varepsilon u(t)=\begin{cases}\nu-u(t), & t\in(\theta-\varepsilon,\theta)\\ \\ 0, & t\in T\backslash(\theta-\varepsilon,\theta)\end{cases} \tag{31}$$

Let $\widetilde{u}(t)=u_\varepsilon(t)=u(t)+\Delta_\varepsilon u(t)$ and $\Delta_\varepsilon x(t)=x(t,u_\varepsilon)-x(t,u)$. Necessary optimality condition, i.e. Pontryagin's maximum principle will follow from the increment formula if we'll show that for $\widetilde{u}(t)=u(t)+\Delta_\varepsilon u(t)$, the increment of the solution $\Delta_\varepsilon x(t)$ is of order ε. This follows from estimation (16). Indeed, if in estimation (16) $u(t)=u_\varepsilon(t)$, then we get

$$|\Delta_\varepsilon x(t)|\le M\varepsilon,\quad t\in T,\quad M=const>0, \tag{32}$$

that means

$$\Delta_\varepsilon x(t)=x(t,u_\varepsilon)-x(t,u)\sim\varepsilon. \tag{33}$$

Now, let $\{u(t),x=x(t,u)\}$ be an optimal process, select $\widetilde{u}(t)=u_\varepsilon(t)$ from (31) and obtain Pontryagin's maximum principle. Thus, we proved the following theorem.

Theorem 1. *Let the admissible process $\{u(t),x(t)=x(t,u)\}$ be optimal in optimal control problem (1)-(4) and $\psi(t)=\psi(t,u)$ be a solution of adjoint system (12)-(15). Then for all $\nu\in v$* and *$\theta\in T$*

$$\Delta_\nu H(\psi(\theta),x(\theta),u(\theta)\theta)\le 0. \tag{34}$$

Inequality (34) is Pontryagin's maximum principle for optimal control problem (1)-(4) and is a first order necessary optimality condition. This condition gives restricted information on controls which are suspicious for optimality. There are cases when condition (34) is fulfilled trivially, i.e. degenerates. In this case, it is desirable to have new necessary optimality conditions allowing to reveal non-optimality of those admissible controls for which Poniryagin's maximum principle degenerates.

Definition. *Admissible control $u(t)$ is called singular in Pontryagin's maximum principle if for all $\nu\in U$, $\theta\in[t_0,t_2]$*

$$\Delta_\nu H(\psi,x,u,t)\le 0. \tag{35}$$

Fulfilment of (35) makes necessary to obtain second order optimality conditions. Allowing for (31)-(33) and (35), from increment formula (30) we get the following equality

$$\Delta J(u)=-\int_T\left\langle\Delta_{\widetilde{u}}\frac{\partial H(\psi,x,u,t)}{\partial x},\Delta x(t)\right\rangle dt$$

$$-\frac{1}{2}\int_T\int_T\left\langle\Delta'_{\widetilde{u}}f(x,u,\tau)R(\tau,s),\Delta_{\widetilde{u}}f(x,u,s)\right\rangle dtds+\eta(u,\Delta u) \tag{36}$$

Theorem 2. *For optimality of the singular control $u(t)$ in problem (1)-(4), the inequality*

$$\Delta'_\nu\frac{\partial H(\psi(\theta),x(\theta),u(\theta),\theta)\varphi}{\partial x}K(\theta,\theta)\Delta_\nu f(x(\theta),u(\theta),\theta)+$$

$$+\Delta_\nu f\left(x\left(\theta\right),u\left(\theta\right),\theta\right)R\left(\theta,\theta\right)\Delta_\nu f\left(x\left(\theta\right),u\left(\theta\right),\theta\right)\leq 0$$

should be fulfilled for all $\nu \in U$

$$K\left(t,\tau\right)=\begin{cases}\Phi\left(t\right)\left[E-S\left(\tau\right)\right]\Phi^{-1}\left(\tau\right), & t_0\leq t\leq s\\ -\Phi\left(t\right)S\left(\tau\right)\Phi^{-1}\left(\tau\right), & s\leq t\leq t_2\end{cases}$$

The proof of the theorem follows from equality (36).

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Yagub A. Sharifov
Institute of Cybernetics of NAS Azeribaijan.
9,F.Agayev str., 370141, Baku, Azerbaijan.
Tel: (99450) 631 55 49.

Shamo I. Djabrailov
Baku State University,
23, Z.I.Khalilov str., AZ 1148, Baku, Azerbaijan.
Tel: (99450) 368 64 09

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