Gulnar M. SALMANOVA

NON-AXISYMMETRIC OSCILLATIONS OF IDEAL LIQUID IN ELASTIC MOMENT SHELL

Abstract

Non-axisymmetric propagation of small amplitude waves in two-phase barotropic bubble liquid enclosed in elastic cylindrical moment shell is considered. Kirchhoff-Love plane sections conjecture is used. For numerical calculation in a long bength approximation, mixture in the form of water involving small addition of air is taken as an example. Influence of volume content of bubbles and number of wave formation on wave characteristics is revealed.

While investigating the problems of dynamics of the system "deformable shell-liquid", it is necessary to consider shell equations with regard to influence of liquid moving in the space on shell dynamics. Specific character of such problems is that two interconnected problems should be solved: to consider liquid flow and shell motion under the action of hydrodynamical forces. Furthermore, on the interface, it is necessary to consider conjugation conditions instead of classic conditions. As a result, the system "shell-liquid" is described by such a complicated system of equations that in general case it is very difficult to solve it. Therefore, in constructing solutions, it is required a definite schematization of phenomenon by introducing a conjecture for liquid and shell.

Non-axisymmetric propagation of small amplitude waves in two-phase barotropic bubble liquid enclosed in elastic thin-shelled cylindrical moment shell whose behavior is described by equations using Kirchhoff-Love plane sections conjecture is considered.

For numerical calculation in a long wave length approximation, mixture in the form of water involving small addition of air is taken as an example.

Influence of volume content of bubbles and number of wave formation on wave characteristics is revealed.

1. Basic relations and problem statement. A system of equations describing propagation of waves in liquid containing deformable shell contains shell and liquid motion equations, boundedness of descred functions and continuity of velocity components on a liquid and shell contact boundary.

Let an annular shell of radius R and thickness $2h$ be given in unperturbed state. In the cylindrical system of coordinates, while considering non-axisymmetric perturbations, we write shell motion equation on which hydrodynamical stress $(0,0,q)$ of liquid acts, in the form [1]:

$$\frac{1}{R^2}\frac{\partial\psi}{\partial\theta}+\frac{1}{R^2}w+\frac{\nu}{R}\frac{\partial u}{\partial x}-\left.\frac{\left(1-\nu^2\right)}{2Eh}q\right|_{r=R}+4h_*^2\left[R^2\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial\theta^2}+\frac{1}{R^2}\frac{\partial^4 w}{\partial\theta^4}\right]-$$

$$-4h_*^2\left[\frac{\partial^3\psi}{\partial x^2\partial\theta}+\frac{1}{R^2}\frac{\partial^3\psi}{\partial\theta^3}\right]+\left(1-\nu^2\right)\frac{\rho_*}{E}\frac{\partial^2 w}{\partial t^2}=0,$$

$$\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{R^2}\frac{\partial w}{\partial\theta}+\frac{1-\nu}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{1+\nu}{2R}\frac{\partial^2 u}{\partial x\partial\theta}-4h_*^2\left[\frac{\partial^3 w}{\partial x^2\partial\theta}+\frac{1}{R^2}\frac{\partial^3 w}{\partial\theta^3}\right]+$$

$$+4h_*^2\left[\frac{1-\nu}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}\right]-\left(1-\nu^2\right)\frac{\rho_*}{E}\frac{\partial^2\psi}{\partial t^2}=0,$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2R^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1+\nu}{2R}\frac{\partial^2\psi}{\partial x\partial\theta}+\frac{\nu}{R}\frac{\partial w}{\partial\theta}-\left(1-\nu^2\right)\frac{\rho_*}{E}\frac{\partial^2 u}{\partial t^2}=0. \tag{1.1}$$

$\{u(x,\theta,t),\psi(x,\theta,t),w(x,\theta,t)\}$ are axial, peripheral and radial components of a displacement vector, respectively, ρ_* is material's density, E is Young's elasticity modulus, ν is Poisson ratio, h is a half of shell's thickness.

Here, by thin-shellness of the shell, the quantity

$$h_*^2=\frac{h^2}{12R^2}<<1.$$

Mixture's flow is accepted as potential. Then, for defining velocity potential $\varphi(x,\theta,r,t)$ we have the equation [2]:

$$\Delta\varphi=\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2}, \tag{1.2}$$

where $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{\partial^2}{\partial r^2}$ is a Laplace operator in a cylindrical system of coordinates. For a known potential φ hydrodynamic stress and a liquid flow velocity vector are determined as

$$q=-\rho_f\frac{\partial\varphi}{\partial t}, \tag{1.3}$$

$$\overrightarrow{v}=\operatorname{grad}\varphi. \tag{1.4}$$

Sound velocity square a^2 and gas liquid medium density ρ_f are written by the following formulae [3]:

$$a^2=\frac{1}{\alpha_2\left(1-\alpha_2\right)}\left(\frac{\rho_1^0}{\rho_1^0-\rho_2^0}\right)^2\frac{p}{\rho_1^0}, \tag{1.5}$$

$$\rho_f=\left(1-\alpha_2\right)\rho_1^0+\alpha_2\rho_2^0. \tag{1.6}$$

In (1.5)-(1.6), α_2 is volume content of bubbles, ρ_1^0,ρ_2^0 are real densities of carrying phase and dispersible, p is static pressure in liquid. The index 0 above denotes the value of the parameter in equilibrium state.

Contact condition.

To complete the problem statement, it is necessary to formulate contact conditions connecting liquid and shell motion. These conditions depend on the adopted conjecture on thickness of the shell wall and liquid's model.

In the considered case, kinematic impermeability condition should be fulfilled on the median surface of the shell. Based on (1.3), the expression for radial component of velocity equals $\partial\varphi/\partial r$. Now, taking into account assumption on neutral buoyancy, the conjugation condition will be of the form [3]:

$$\frac{\partial\varphi}{\partial r}=\frac{\partial w}{\partial t}\quad\text{for}\quad r=R. \tag{1.7}$$

Thus, the considered problem described by equations (1.1),(1.2),(1.3) and (1.7) is completely formulated.

2. Solution of hydrodynamics equation. In the sequel, we'll assume all the desired functions are proportional to time multiplier $\exp(i\omega t)$, where ω is annular frequency, $i=\sqrt{-1}$ is an imaginary unit. Thus, it is required to find the solution of equation (1.2) for the case when

$$\varphi=\varphi_1(x)\,\varphi_2(r)\cos(n\theta)\exp(i\omega t) \tag{2.1}$$

Substituting (2.1) in (1.2), we get:

$$\varphi_1''(x)\,\varphi_2(r)-\frac{n^2}{r^2}\varphi_2(r)\,\varphi_1(x)+\frac{1}{r}\varphi_2'(r)\,\varphi_1(x)+\varphi_2''(r)\,\varphi_1(x)=-\frac{\omega^2}{a^2}\varphi_1(x)\,\varphi_2(r).$$

or

$$\frac{\varphi_2''(r)}{\varphi_2(r)}+\frac{1}{r}\frac{\varphi_2'(r)}{\varphi_2(r)}-\frac{n^2}{r^2}=-\frac{\varphi_1''(x)}{\varphi_1(x)}-\frac{\omega^2}{a^2}=\lambda^2. \tag{2.2}$$

The last equality whose left hand side depends only on r, and the right one only on x is possible only in the case if the common quantity of relations (2.2) will be a constant. Denote this constant by λ^2. Then from equality (2.2) we get two ordinary differential equations

$$\varphi_1''(x)+\left(\lambda^2+\frac{\omega^2}{a^2}\right)\varphi_1(x)=0, \tag{2.3}$$

$$r^2\varphi_2''(r)+r\varphi_2'(r)-n^2\varphi_2(r)-\lambda^2r^2\varphi_2(r)=0. \tag{2.4}$$

Applying the method of separation of variables and considering only the wave travelling in the positive direction of the axis x, finally for the function φ we get:

$$\varphi=AJ_n(i\lambda r)\exp\left\{-i\sqrt{\lambda^2+\frac{\omega^2}{a^2}}x\right\}\exp(i\omega t)\cos(n\theta), \tag{2.5}$$

A is an integration function, n is the number of wave formation.

3. Dispersion equation. Shell motion differential equation (1.1) is satisfied in substituting therein the solutions of the form

$$u=u_0\exp\left\{-i\sqrt{\lambda^2+\frac{\omega^2}{a^2}}x\right\}\exp(i\omega t)\cos(n\theta), \tag{3.1}$$

$$w=w_0\exp\left\{-i\sqrt{\lambda^2+\frac{\omega^2}{a^2}}x\right\}\exp(i\omega t)\cos(n\theta), \tag{3.2}$$

$$\psi = \psi_0 \exp\left\{-i\sqrt{\lambda^2+\frac{\omega^2}{a^2}}x\right\}\exp(i\omega t)\sin(n\theta). \tag{3.3}$$

Here u_0, w_0 and ψ_0 , generally spanking, are complex constants.

Construction of the potential φ will be completed if we use formulae (2.5) and (3.2) in contact condition (1.7). Hence, it directly follows

$$A = \frac{\omega}{\lambda J_n'(i\lambda R)} w_0 \ ,$$

Substituting the obtained expression in formula (2.5), we find:

$$\varphi = \frac{\omega}{\lambda}\frac{J_n(i\lambda r)}{J_n'(i\lambda R)} w_0 \exp\left\{-i\sqrt{\lambda^2+\frac{\omega^2}{a^2}}x\right\}\exp(i\omega t)\cos(n\theta). \tag{3.4}$$

Using formula (1.3), we define the stress q:

$$q = -\frac{i\omega^2\rho_f}{\lambda}\left\{\frac{J_n(i\lambda R)}{J_n'(i\lambda R)}\right\} w_0 \exp\left\{-i\sqrt{\lambda^2+\frac{\omega^2}{a^2}}x\right\}\exp(i\omega t)\cos(n\theta). \tag{3.5}$$

Further substitution of formulae (3.1)-(3.5) in the system of shell motion equations (1.1) gives for the coefficients u_0, w_0 and ψ_0 a system of three linear homogeneous algebraic equations containing ω as a parameter. Introducting the following denotation:

$$M_n = 1 + \frac{1}{2\rho h}\frac{J_n(i\lambda R)}{(i\lambda)\,J_n'(i\lambda R)},$$

$$\mu = \frac{h}{2R},\ \ c_0^2 = \frac{Eh}{2\rho_f R},\ \ \rho = \frac{\rho_*}{\rho_f},$$

we get the equations:

$$\left\{\begin{array}{l}
\left\{\dfrac{n}{R^2} + 4h_*^2 n\left[\left(\lambda^2+\dfrac{\omega^2}{a^2}\right)+\dfrac{n^2}{R^2}\right]\right\}\psi_0 - \dfrac{i\nu}{R}\sqrt{\lambda^2+\dfrac{\omega^2}{a^2}}u_0 + \\
+\left\{\dfrac{1}{R^2} - \left(1-\nu^2\right)\dfrac{\rho\mu}{c_0^2}M_n\omega^2 + 4h_*^2R^2\left[\left(\lambda^2+\dfrac{\omega^2}{a^2}\right)+\dfrac{n^2}{R^2}\right]^2\right\} w_0 = 0, \\
\left\{\dfrac{n}{R^2} + \dfrac{1-\nu}{2}\left(\lambda^2+\dfrac{\omega^2}{a^2}\right) + 4h_*^2\left[\dfrac{1-\nu}{2}\left(\lambda^2+\dfrac{\omega^2}{a^2}\right)+\dfrac{n^2}{R^2}\right] - \right. \\
\left. -\left(1-\nu^2\right)\dfrac{\rho\mu}{c_0^2}\omega^2\right\}\psi_0 - in\dfrac{1+\nu}{2R}\sqrt{\lambda^2+\dfrac{\omega^2}{a^2}}u_0 + \\
+\left\{\dfrac{n}{R^2} + 4h_*^2 n\left[\left(\lambda^2+\dfrac{\omega^2}{a^2}\right)+\dfrac{n^2}{R^2}\right]\right\} w_0 = 0, \\
\dfrac{1+\nu}{2R}in\sqrt{\lambda^2+\dfrac{\omega^2}{a^2}}\psi_0 + \left\{\left(\lambda^2+\dfrac{\omega^2}{a^2}\right) + \dfrac{(1-\nu)\,n^2}{2R^2} - \right. \\
\left. -\left(1-\nu^2\right)\dfrac{\rho\mu}{c_0^2}\omega^2\right\} u_0 + \dfrac{i\nu}{R}\sqrt{\lambda^2+\dfrac{\omega^2}{a^2}}w_0 = 0.
\end{array}\right. \tag{3.6}$$

From the condition on existence of nontrivial solutions for u_0, w_0 and ψ_0 we get that the determinant of system (3.6) equals zero

$$\det \delta_{ij} = 0 \quad \left(i,j=\overline{1,3}\right), \tag{3.7}$$

where the elements δ_{ij} are calculated by the formulae

$$\delta_{11} = \frac{n}{R^2} + 4h_*^2 n\left[\left(\lambda^2 + \frac{\omega^2}{a^2}\right) + \frac{n^2}{R^2}\right], \quad \delta_{12} = -\frac{i\nu}{R}\sqrt{\lambda^2 + \frac{\omega^2}{a^2}},$$

$$\delta_{13} = \frac{1}{R^2} - \left(1-\nu^2\right)\frac{\rho\mu}{c_0^2}M_n\omega^2 + 4h_*^2 R^2\left[\left(\lambda^2 + \frac{\omega^2}{a^2}\right) + \frac{n^2}{R^2}\right]^2;$$

$$\delta_{21} = \frac{n^2}{R^2} + \frac{1-\nu}{2}\left(\lambda^2 + \frac{\omega^2}{a^2}\right) + 4h_*^2\left[\frac{1-\nu}{2}\left(\lambda^2 + \frac{\omega^2}{a^2}\right) + \frac{n^2}{R^2}\right] - \left(1-\nu^2\right)\frac{\rho\mu}{c_0^2}\omega^2,$$

$$\delta_{22} = -in\frac{1+\nu}{2R}\sqrt{\lambda^2 + \frac{\omega^2}{a^2}}, \quad \delta_{23} = \frac{n}{R^2} + 4h_*^2 n\left[\left(\lambda^2 + \frac{\omega^2}{a^2}\right) + \frac{n^2}{R^2}\right];$$

$$\delta_{31} = \frac{1+\nu}{2R}in\sqrt{\lambda^2 + \frac{\omega^2}{a^2}}, \quad \delta_{32} = \left(\lambda^2 + \frac{\omega^2}{a^2}\right) + \frac{(1-\nu)\,n^2}{2R^2} - \left(1-\nu^2\right)\frac{\rho\mu}{c_0^2}\omega^2,$$

$$\delta_{33} = \frac{i\nu}{R}\sqrt{\lambda^2 + \frac{\omega^2}{a^2}}.$$

4. Structure of the mixture. Define concretely liquid and consider mixture consisting of water containing small addition of air bubbles $\alpha_2 = \left\{10^{-2} - 10^{-1}\right\}$ (the case is interesting from practical point of view).

Such a schematization is very important, as water definitely influences on the course of many physico-chemical, biological and technological processes. On the other hand, it always contains ingredients, in particular air bubbles. It is interesting to notice that blod mostly consists of water. Then accept [4].

$$\rho_1^0 = 1\frac{g}{sm^3}, \quad \left(\rho_2^0 = 10^{-3}\frac{g}{sm^3}\right), \; p = 10^6\frac{dn}{sm^2}.$$

5. Numerical experiment and conclusions. We attempt to solve dispersion equation (3.7) following characteristical test data that correspod to the problem. In confirmity to rubber shell we have:

$$\rho_* = 2\ g/sm^3, \quad E = 4\cdot 10^6\frac{dn}{sm^2}, \; R = 2\ sm, \; h = 0,2sm, \; \lambda = 0,01\ sm^{-1}.$$

In dispersion equation (3.7) there is a quantity M_n that contains Bessel functions dependent on the wave number λ. Restricting in considerion of the long wave length process ($\lambda << 1$), the quantities M_n may be represented in a polynomial form. Then for $n = 2;4;6;8;10$ (we are restricted in consideration of these cases), the quantities M_n may be approximately represented in the form

$$\begin{gathered} M_2 = 1 + \frac{R}{4\rho h}, \; M_4 = 1 + \frac{R}{8\rho h}, \; M_6 = 1 + \frac{R}{12\rho h}, \\ M_8 = 1 + \frac{R}{16\rho h}, \; M_{10} = 1 + \frac{R}{20\rho h} \end{gathered} \tag{3.8}$$

($n = 0$ corresponds to the axysymmetric state).

Dispersion equation (3.7) defines three waves. Two roots of this dispersion equation defines longitudinal waves propagating in the shell and liquid. The third root defines the torsional wave in the shell, since ideal liquid doesn't work in shear.

In tables 1,2,3,4, moment ($h_*^2 << 1$) and momentless ($h_*^2 = 0$) cases are compared. Here ω_2 and ω_1 are the values of frequencies of longitudinal waves propagating in the shell and liquid, respectively, ω_3 are the values of torsional wave frequencies.

Table 1.

$n=0$						
α_{20}	ω_1		ω_2		ω_3	
	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$
0	6,2562	6,2562	14,9864	14,9864	8,7706	8,7742
0,02	6,3187	6,3187	15,3225	15,3225	8,8375	8,8412
0,04	6,383	6,383	15,6671	15,6671	8,9031	8,9069
0,06	6,4494	6,4494	16,0199	16,0199	8,9673	8,9712
0,08	6,5179	6,5179	16,3803	16,3803	9,0299	9,0339
0,1	6,5885	6,5885	16,7476	16,7476	9,0909	9,095

Table 2.

$n=2$						
α_{20}	ω_1		ω_2		ω_3	
	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$
0	0,0447	40,5761	1562,7517	1563,8524	877,1185	877,1187
0,02	0,0449	40,7819	1593,8609	1594,9778	886,3431	886,3785
0,04	0,0452	40,9909	1625,8673	1626,9988	895,3911	895,4628
0,06	0,0454	41,203	1658,7367	1659,8799	904,2454	904,3547
0,08	0,0456	41,4184	1692,4207	1693,5715	912,8891	913,0373
0,1	0,0459	41,6372	1726,855	1728,0077	921,3054	921,4938

Table 3.

$n=10$						
α_{20}	ω_1		ω_2		ω_3	
	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$
0	0,00252	1885,5586	7442,0987	7452,6566	4385,2991	4385,2993
0,02	0,00252	1893,5736	7606,0895	7616,6315	4419,7781	4420,4295
0,04	0,00253	1902,5158	7774,2196	7784,4531	4453,5737	4454,9374
0,06	0,00253	1911,3773	7946,2932	7955,6147	4486,6208	4488,813
0,08	0,00254	1920,1502	8122,0457	8129,5309	4518,8528	4522,036
0,1	0,00255	1928,8266	8301,1329	8305,5318	4550,2023	4554,5765

Table 4.

$\alpha_{20}=0.01$						
n	ω_1		ω_2		ω_3	
	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$	$h_*^2=0$	$h_*^2<<1$
2	$4{,}4829\cdot10^{-2}$	40,68	1578,19	1579,3	881,75	881,75
4	$1{,}3633\cdot10^{-2}$	247,33	3053,46	3056,54	1761,83	1761,9
6	$6{,}5466\cdot10^{-3}$	623,58	4539,39	4544,74	2642,03	2642,17
8	$3{,}8393\cdot10^{-3}$	1170,59	6030,19	6038,04	3522,3	3522,52
10	$2{,}5223\cdot10^{-3}$	1889,07	7523,57	7534,13	4402,62	4402,94

Thus, the calculations allow to formulate the following basic conclusions:

-with increase of volume content of bubbles α_2 the frequency ω_2 increases (for $n=2$ approximately 10.4%, for $n=10$, 11%);

-with increase of α_2, the frequency ω_1 increases negligibly;

-with increase of volume content of bubbles, ω_3 changes negligibly.

For example, for $n=2$, ω_3 increases approximately 5%.

-from thin-shellness condition we conclude that momentproperty of the shell practically doesn't influence on ω_2 and ω_3. But as it is seen from the table, the frequency ω_1 changes considerably.

-with increase of wave formation number ω_1, ω_2 and ω_3 increase.

References

[1]. Wallmir A.S. *Shells in liquid and gas flow. Hydro-elasticity problems.* M. 1979, 320 p. (Russian).

[2]. Sedov L.I. *Continuum mechanics.* vol. 1, M. 1976, 535 p. (Russian).

[3]. Gubaidullin A.A., Ivandaev A.I., Nigmatulin R.I., Khabeev N.S. *Waves in bubble liquids.* // Itogi Nauki i techniki. Ser. mekhanika zhidkosti i gaza. M. VINITI, issue 17, 1982, 247 p. (Russian).

[4]. Nigmatulin R.I. *Dynamics of multiphase media.* Part I. M. 1987, 464 p. (Russian).

Gulnar M. Salmanova

Baku State University.

23, Z.I. Khalilov str., AZ1148, Baku, Azerbaijan.

Tel.: (99412) 439 47 20 (off.).

Received September 14, 2009; Revised December 25, 2009.