Bakhtiyar E. RUSTAMOV

LOCAL FAILURE OF BURNING BODY INVOLVING VOLUME FORCES

Abstract

A problem of failure mechanics for a medium weakened by a crack-visible cavity with burning surface is considered. It is assumed that volume forces hold. Quasistatic process of fuel deformation is considered. A model of bridged crack is used. Sufficient condition of stability of "burning-failure" condition is obtained.

Solid fuel finds its wide application in very different fields of engineering. When engines work in solid fuel, sometimes there happens off-desing condition and this leads to explosion. The most extended cause of this phenomenon is that there were inadmissibly great cracklike defects arising as a rule in technological process.

Let a burning body have a crack-visible cavity whose surface burns. We assume that all the reagents of the fuel are uniformly distributed in solid phase and the combustion reaction products are gaseous. We are restricted by the times small in comparison with typical stress relaxation time in fuel and typical fuel heating time because of heat conductivity. By this assumption we can assume that the body is brittle its temperature is constant. We use a bridged crack model that has get experimental affirmation [1-3] for composite materials with a polymer binder when adhesive strength is less than the strength of polymers.

We consider a model of a crack with cohesive forces (bondings) that are continuously distributed in the narrow end area of the crack and have a deformation diagram. Gas flow equatons in the crack-visible cavity are assumed to be local - iso-entropic and irrotational, the gas ideal.

Let the solid fuel occupy the plane Oxy. In the plane Oxy, the crack-visible cavity with end areas is represented by a cut of length 2ℓ along the axis x, the thickness of the cut $h << \ell$. Moreover, the cohesive forces (bondings) will be concentrated in some narrow domain D. The size of this domain is not known beforehand and should be defined from the problem solution. Everywhere in the sequel, we shall take the boundary conditions from the surface of the cavity to the plane $y = 0$ just in the same way as in the theory of thin wing in aerodynamics.

It is assumed that volume forces $F = X + iY$ act on fuel particles. Here X, Y are the given components of the volume force vector.

The boundary conditions for solid fuel are written in the form

$$\begin{aligned} &\sigma_y - i\tau_{xy} = -p(x) \qquad \text{for} \qquad y = 0, \quad \lambda_1 < x < \lambda_2 \\ &\sigma_y - i\tau_{xy} = q_y(x) - iq_{xy}(x) \quad \text{for} \quad y = 0, \quad a \le x \le \lambda_1, \text{ and } \quad \lambda_2 \le x \le b. \end{aligned} \tag{1}$$

Here $d_1 = \lambda_1 - a,\ d_2 = b - \lambda_2$ are the lengths of end areas to be determined; σ_x,σ_y, τ_{xy} are the stress tensor components in cartesian system of coordinates; $p(x)$ is gas pressure in cavity; $q_y(x)$ and $q_{xy}(x)$ are normal and tangential forces in bondings, respectively. The values of stresses q_y and q_{xy} are not known beforehand and have to be determined. We accept the gas flow equation in the crack-visible cavity for a plane stationary case in the form [4].

In the presence of volume forces, we represent the solution in the form of the sum

$$\sigma_x = \sigma_x^0 + \sigma_x^1,\quad \sigma_y = \sigma_y^0 + \sigma_y^1,\ \tau_{xy} = \tau_{xy}^0 + \tau_{xy}^1, \tag{2}$$

where σ_x^0,σ_y^0, τ_{xy}^0 is any special solution of equations of plane theory of equations of elasticity involving volume forces; σ_x^1,σ_y^1, τ_{xy}^1 is a general solution of equations of plane theory of elasticity in the absence of volume forces. Using Kolosov-Muskheleshvili complex potential [5] for stresses, we have

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\left[\Phi(z) - \frac{1}{2(1+k_0)}\frac{\partial Q}{\partial z}\right],$$

$$\sigma_y - \sigma_x + 2i\ \tau_{xy} = 2\left[z\Phi'(z) + \Psi(z) + \frac{1}{2(1+k_0)}\frac{\partial}{\partial z}\left(k_0\overline{Q} - \overline{F}_1\right)\right], \tag{3}$$

that contain two analytic functions $\Phi(z)$ and $\Psi(z)$ of a complex variable $z = x + iy$ and the two functions $Q(z,\overline{z})$ and $F_1(z,\overline{z})$ representing any special solutions of equations

$$\frac{\partial^2 Q}{\partial z \partial \overline{z}} = F(z,\overline{z}),\quad \frac{\partial^2 F_1}{\partial z^2} = \overline{F(z,\overline{z})}. \tag{4}$$

Here $k_0 = 3 - 4\nu$; ν is Poisson's ratio of fuel's material.

The basic relations of the considered problem should be complemented by a relation connecting the opening of lips of end areas of the cavity and forces in bondings. We can represent this relation in the form

$$\left(v^+(x,0) - v^-(x,0)\right) - i\left(u^+(x,0) - u^-(x,0)\right) = C(x,\sigma)\left[q_y(x) - iq_{xy}(x)\right]. \tag{5}$$

Here, the functions $C(x,\sigma)$ may be considered as effective compliance of bondings depending on tension; $(v^+ - v^-)$ and $(u^+ - u^-)$ are normal and tangential components of opening of end area lips; $\sigma = \sqrt{q_y^2 + q_{xy}^2}$ is a of force vector modulus in bondings.

For determining the complex potentials $\Phi(z)$ and $\Omega(z)$ we have the linear conjugation problem

$$\begin{aligned} &[\Phi(x) + \Omega(x)]^+ + [\Phi(x) + \Omega(x)]^- = 2f(x) \\ &[\Phi(x) - \Omega(x)]^+ - [\Phi(x) - \Omega(x)]^- = 0, \end{aligned} \tag{6}$$

where $\Omega(z)=\overline{\Phi}(z)+z\overline{\Phi}'(z)+\overline{\Psi}(z)$; $a\le x\le b$ are the affices of points of lips of crack-visible cavity with end areas;

$$f(x)=\begin{cases} -p(x)+f_0(x) & \text{on the cavity lips} \\ f_0(x)+q_y-iq_{xy} & \text{on the end area lips}\end{cases}$$

$$f_0(x)=\frac{1}{1+k_0}\operatorname{Re}\frac{\partial Q}{\partial z}-\frac{1}{2(1+k_0)}\left(k_0\frac{\partial\overline{Q}}{\partial z}-\frac{\partial\overline{F}_1}{\partial z}\right)\ \text{for } y=0. \tag{7}$$

The general solution of boundary value problem (6) in the class of everywhere bounded functions will be of the form [5]

$$\Phi(z)=\Omega(z)=\frac{\sqrt{(z-a)(z-b)}}{2\pi i}\int_a^b\frac{f(t)\,dt}{\sqrt{(t-a)(t-b)(t-z)}}. \tag{8}$$

The solvability condition of the boundary value problem will be written in the form

$$\int_a^b\frac{f(t)\,dt}{\sqrt{(t-a)(t-b)}}=0,\quad \int_a^b\frac{tf(t)\,dt}{\sqrt{(t-a)(t-b)}}=0. \tag{9}$$

These relations help to determine the unknown sizes of end areas of the crack-visible cavity. The obtained relations $(8),(9)$ contain the unknown stresses $q_y(x)$ and $q_{xy}(x)$ in bondings between the lips in the end areas of crack-visible cavity. To find them we use additional condition (5). By means of the obtained solution (8), we find the derivative of the opening between the lips in end areas of crack-visible cavity

$$\frac{\partial}{\partial x}\left[\left(v^+(x,0)-v^-(x,0)\right)-i\left(u^+(x,0)-u^-(x,0)\right)\right]=$$

$$=-\frac{1+k_0}{2\pi\mu}\sqrt{(b-x)(x-a)}\int_a^b\frac{f(t)\,dt}{\sqrt{(b-t)(t-a)(t-x)}}, \tag{10}$$

where μ is fuel's material shear modulus.

To find forces q_y-iq_{xy} in the bondings of end areas of the cavity we get the complex nonlinear integrodifferential equation

$$-\frac{1+k_0}{2\pi\mu}\sqrt{(b-x)(x-a)}\int_a^b\frac{f(t)\,dt}{\sqrt{(b-t)(t-a)(t-x)}}=$$

$$=\frac{\partial}{\partial x}\left[C(x,\sigma)(q_y-iq_{xy})\right]. \tag{11}$$

After some transformations we get the system of nonlinear integro-differential equations with respect to the unknown functions q_y and q_{xy};

$$-\frac{1}{\pi}\sqrt{(b-x)(x-a)}\int_a^b\frac{f_y(t)\,dt}{\sqrt{(b-t)(t-a)(t-x)}}=\frac{2\mu}{1+k_0}\frac{d}{dx}\left[C(x,\sigma)\,q_y(x)\right], \tag{12}$$

$$-\frac{1}{\pi}\sqrt{(b-x)(x-a)}\int\limits_a^b\frac{f_{xy}(t)\,dt}{\sqrt{(b-t)(t-a)(t-x)}}=\frac{2\mu}{1+k_0}\frac{d}{dx}\left[C(x,\sigma)\,q_{xy}(x)\right], \quad (13)$$

where $f_y(t)=\operatorname{Re}f(t)$; $f_{xy}(t)=\operatorname{Im}f(x)$.

Each of equations $(12),(13)$ is a nonlinear integro-differential equation with Cauchy type kernel and may be solved numerically. To determine stress distribution and sizes of end areas it is necessary to give the law of change of volume forces. Expand the functions $X(x,y)$ and $Y(x,y)$ in Taylor's series at the origin of coordinates, and in this expansion, for simplicity we are restricted by two terms of the expansion. By means of integration of equations (4), we find

$$Q(z,\overline{z})=\int^{z}dz\int^{\overline{z}}F(z,\overline{z})\,d\overline{z},\ \ F_1(z,\overline{z})=\int^{z}dz\int^{\overline{z}}\overline{F(z,\overline{z})}dz. \quad (14)$$

By the found functions $Q(z,\overline{z})$ and $F_1(z,\overline{z})$ according to (7) we find the function $f_0(x)$.

Pass to algebraization of integral equations (12) and (13) with additional conditions (9). At first, in integral equations (12) and (13) and additional conditions (9) all the integration intervals are reduced to one interval $[-1,1]$. For that we use change of variables

$$t=\frac{a+b}{2}+\frac{b-a}{2}\tau,\ \ x=\frac{a+b}{2}+\frac{b-a}{2}\eta.$$

The left hand side of integrodifferential equation (12) for such change of variables takes the form

$$-\frac{1}{\pi}\sqrt{1-\eta^2}\left[\int\limits_{-1}^{1}\frac{q_y(\tau)\,d\tau}{\sqrt{1-\tau^2}(\tau-\eta)}+\int\limits_{-1}^{1}\frac{f_y^*(\tau)\,d\tau}{\sqrt{1-\tau^2}(\tau-\eta)}\right].$$

Correspondingly, for the left hand side of equation (13) we get

$$-\frac{1}{\pi}\sqrt{1-\eta^2}\left[\int\limits_{-1}^{1}\frac{q_{xy}(\tau)\,d\tau}{\sqrt{1-\tau^2}(\tau-\eta)}+\int\limits_{-1}^{1}\frac{f_{xy}^*(\tau)\,d\tau}{\sqrt{1-\tau^2}(\tau-\eta)}\right].$$

Replace the derivative contained in the right hand side of (12) for arbitrary internal node by finite difference approximation

$$\frac{d}{dx}\left[C(x,\sigma)\,q_y(x)\right]=\frac{1}{2\Delta x}\left[C(x_{i+1},\sigma_{i+1})\,q_y(x_{i+1})-C(x_{i-1},\sigma_{i-1})\,q_y(x_{i-1})\right],$$

where $\Delta x=\dfrac{2\ell}{M}$; $2\ell=b-a$.

We treat right hand side of equation (13) in the same way. Moreover, take into account boundary conditions for $\eta=\mp$ $q_y(-\ell)=q_y(\ell)=0$; $q_{xy}(-\ell)=q_{xy}(\ell)=0$

(this corresponds to the conditions $\upsilon^{+}(-\ell,0)–\upsilon^{-}(-\ell,0)=0;\upsilon^{+}(\ell,0)–\upsilon^{-}(\ell,0)=0;$ $u^{+}(-\ell,0)-u^{-}(-\ell,0)=0;\ u^{+}(\ell,0)-u^{-}(\ell,0)=0$).

Using the quadrature formula

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{g(\tau)\,d\tau}{\sqrt{1-\tau^2}\,(\tau-\eta)}=\frac{1}{M\sin\theta}\sum_{k=1}^{M}g_k\sum_{m=0}^{M-1}\cos m\theta_k\sin m\theta$$

$$\tau=\cos\theta;\eta_m=\cos\theta_m;\ \theta_m=\frac{2m-1}{2M}\pi\quad(m=1,2,....,M),$$

all the integrals in (12) and (13) are replaced by finite sums, and the derivatives in the hand side of equations (12) and (13) are replaced by finite-difference approximations. The above reduced formulae enable to replace each singular integro-differential equation by a system of algebraic equations with respect to approximate values of the desired function at the end points. As a results we get

$$-\frac{2}{M}\sum_{\nu=1}^{M}q_{y,\nu}\sum_{k=0}^{M-1}\cos k\theta_k\sin k\theta_m+\sum_{\nu=1}^{M}f^{*}_{y,\nu}\sum_{k=0}^{M-1}\cos k\theta_k\sin k\theta_m=$$

$$=\left[C\left(x_{m+1},\sigma\left(x_{m+1}\right)\right)q_{y,m+1}-C\left(x_{m-1},\sigma\left(x_{m-1}\right)\right)q_{y,m-1}\left(x_{i-1}\right)\right]\frac{(1+k_0)\,M}{8\mu\ell}\quad(15)$$

$$-\frac{2}{M}\sum_{\nu=1}^{M}q_{xy,\nu}\sum_{k=0}^{M-1}\cos k\theta_k\sin k\theta_m+\sum_{\nu=1}^{M}f^{*}_{xy,\nu}\sum_{k=0}^{M-1}\cos k\theta_k\sin k\theta_m=\frac{(1+k_0)\,M}{8\mu\ell}\times$$

$$\times\left[C\left(x_{m+1},\sigma\left(x_{m+1}\right)\right)q_{xy,m+1}-C\left(x_{m-1},\sigma\left(x_{m-1}\right)\right)q_{xy,m-1}\left(x_{i-1}\right)\right],\quad(16)$$

$$(m=1,2,...,M).$$

If we take into account in equality

$$2\sum_{k=0}^{M-1}\cos k\theta_\nu\sin k\theta_m=ctg\frac{\theta_m\mp\theta_\nu}{2},$$

the systems will take the forms

$$\sum_{\nu=1}^{M}A_{m\nu}\left(q_{y,\nu}+f^{*}_{y,\nu}\right)=\frac{(1+k_0)\,M}{8\mu\ell}\times$$

$$\times\left[C\left(x_{m+1},\sigma\right)q_{y,m+1}-C\left(x_{m-1},\sigma\right)q_{y,m-1}\right],\ (m=1,2,...,M)\quad(17)$$

$$\sum_{\nu=1}^{M}A_{m\nu}\left(q_{xy,\nu}+f^{*}_{xy,\nu}\right)=$$

$$=\frac{(1+k_0)\,M}{8\mu\ell}\left[C\left(x_{m+1},\sigma\right)q_{xy,m+1}-C\left(x_{m-1},\sigma\right)q_{xy,m-1}\right],\ (m=1,2,...,M)\quad(18)$$

respectively, where $q_{y,\nu}=q_y(\tau_\nu)$; $q_{xy,\nu}=q_{xy}(\tau_\nu)$; $f^{*}_{y,\nu}=f^{*}_{y}(\tau_\nu)$; $f^{*}_{xy,\nu}=f^{*}_{xy}(\tau_\nu)$;

$$x_{m+1}=\ell\eta_{m+1};\ A_{m\nu}=-\frac{1}{M}ctg\frac{\theta_m\mp\theta_\nu}{2}.$$

The upper sign is taken when the number $|m-\nu|$ is odd, the lower one when it is even.

Now, pass to algebraization of solvability conditions of boundary value problem (9). Separating in them real and imaginary parts, using change of variables and Gauss's quadrature formula, we get the solvability conditions in the form

$$\sum_{\nu=1}^{M} f_y\left(\tau_\nu\right)=0, \quad \sum_{\nu=1}^{M} \tau_\nu f_y\left(\tau_\nu\right)=0, \tag{19}$$

$$\sum_{\nu=1}^{M} f_{xy}\left(\tau_\nu\right)=0, \quad \sum_{\nu=1}^{M} \tau_\nu f_{xy}\left(\tau_\nu\right)=0. \tag{20}$$

As a result of algebraization, instead of each integro-differential equation with additional conditions we get $M+2$ algebraic equations for determining stresses at the nodal points and the sizes of end areas of the crack-visible cavity.

Together with gas dynamics equations the obtained system of equations allows to determine gas pressure intensity in the cavity, stress state of burning fuel involving volume forces under the given characteristics of bondings. The following method is more convenient for numerical solution of these equations: the function $p(x)$ is found in the form of a polynomial with unknown coefficients, the equation for $p(x)$ is satisfied approximately in the sence of the greatest proximity to zero of mean square error. Moreover, the unknown coefficients are determined from the minimum conditions of the obtained function.

Even under linear-elastic bondings, the solving system of equations become nonlinear because of unknown quantities λ_1 and λ_2. The successive approximations method was used for its solution. In each approximation, the system of equations was solved by the Gauss method with choice of the principal element for different values of M. It should be noted that beginning with $M=20$ all the desired quantities do not change essentially. In the case of nonlinear law of deformation of bondings for determining forces at the end areas of the crack-visible cavity the iteration algorithm is also used similar to the method of elastic solutions [6].

The calculations show that for linear law of deformation of bondings, the forces in bondings have always maximal value at the edge of the end area. The similar situation is observed also for the quantities of the crack-visible cavity opening. Opening of the crack-visible cavity at the edge of the end area has maximum for linear laws of deformations and opening of the crack-visible cavity increases due to increase of relative compliance of bondings.

To determine the limit state under which the crack-visible cavity growth happens, we used the critical condition

$$\left|\left(v^{+}-v^{-}\right)-i\left(u^{+}-u^{-}\right)\right|=\delta_c. \tag{21}$$

where δ_c is characteristics of fuel's material resistance to crack development.

It is assumed that disconnection of bondings at the edge of the end area for $x = \lambda$ happens by fulfilling condition (21).

On the basis of the obtained solution and taking into account additional relation (5), the limit condition is written as:

for the left end of the crack-visible cavity

$$C(\lambda_1, \sigma(\lambda_1))\, \sigma(\lambda_1) = \delta_c; \tag{22}$$

for the right end of the crack-visible cavity

$$C(\lambda_2, \sigma(\lambda_2))\, \sigma(\lambda_2) = \delta_c. \tag{23}$$

Under the given characteristics of bondings, the combined solution of the obtained system of equations and condition (21) allows to determine critical intensity of gas pressure in cavity, forces in bondings, the sizes of end areas, parameters of volume force for the limit equilibrium state.

Using the solution of the considered plane stationary problem, we can write the local condition of stability of mode in the form

$$C(x_0, \sigma(x_0))\, \sigma(x_0) < \delta_c, \tag{24}$$

where $x_0 = \lambda_1$ for the left end area, $x_0 = \lambda_2$ for the right end area.

Local condition of mode's stability allows to predict ultimate admissible size of the crack-visible cavity (technological defect) for each specific form of fuel by means of numerical calculation.

References.

[1]. Budianscky B., Evans A.G., Hutchinson J.W. *Fiber - matrix de bonding effects on cracking in aligned fiver ceramic composites* // Int. J.Solid structures, 1995, vol. 32, pp. 315-328.

[2]. Ji H., Gennes P.G. *Adhesion via connector molecules*: The Many-stitch Problem // Macromolecules. 1993, vol. 26, pp. 520-525.

[3]. Goldstein R.V., Bakirov V.F., Perelmuter M.N. *Modeling of the adhesion strength and fracture kinetics of the microelectronic package polymer-polymer joints* // In: Proc. Inst. Phys. Technol. Russian Ac. of Sci. vol. 13. *Modeling and Simulation of Submicron Technology and Devices*, 1997, pp. 115-125.

[4]. Cherepanov G.P. *Mechanics of birttle fracture.* New - York: Mc. Graw-Hill. 1979.

[5]. Muskheleshvili N.I. *Some basic problems of the mathematical theory of elasticity.* Amsterdam: Klumer. 1977.

[6]. Il'yushin A.A. *Plasticity.* Moscow and Leningrad. Gostechizd. 1948 (Russian).

Bakhtiyar E. Rustamov.
Azerbaijan State Academy of Physical Culture and Sport.
Tel.:(99 412) 439 47 20 (off.).

Received September 08, 2009; Revised December 04, 2009.