Aydin I.SHAHBAZOV, Dashgin A. SEYIDOV

CLOSED RANGE AND COMPACT WEIGHTED COMPOSITION OPERATORS ON UNIFORM ALGEBRAS

Abstract

Let X be a compact metric space and let $C(X)$ denote the space of all continuous complex-valued functions on X equipped with the supremum norm. In this paper we investigate compactness and closedness of range of weighted composition operators on a uniformly closed subspaces of $C(X)$ and give as some applications compactness and closedness of range criterion on uniform algebras.

1. Introduction

Let X be a compact metric space and let $C(X)$ denote the space of all continuous complex-valued functions defined on X equipped with the supremum norm. Let $A = A(X)$ be a uniformly closed subspace of $C(X)$. We will consider the operators $T : A \to C(X)$ of the form $T : f \longmapsto u \cdot f \circ \varphi$ (the symbol " $\circ$ " denote the composition of functions), where $u \in C(X)$is a fixed function and $\varphi : X \to X$ is a selfmapping of X which is continuous on the support of function u, i.e., on the open set $S(u) = \{x \in X : u(x) \neq 0\}$ (in particular, we can choose the function u and the selfmapping φ such that the operator T may be acting in $A(X)$, i.e., $T : A(X) \to A(X)$). The operators of these forms are called the weighted composition operators induced by the function u (the weighted function) and by selfmapping φ. Since the endomorphisms of any semisimple commutative Banach algebras (also, any bounded linear operator on a Banach space) can be represented as operators of these forms, so the weighted composition operators are very interesting to study. Composition operators (i.e., the operators of the forms as operator T with the weighted function $u \equiv 1$) and weighted composition operators on the concrete uniform algebras are being investigated from different points of view (such as compactness, nuclearity, spectrum, closedness of range, etc.) by many authors. The aim of this paper is to clarify the compactness and closedness of range conditions of such operators. Kamowitz [1] in particular, gave the compactness criterion for the weighted composition operators acting in the disc-algebra $A(D)$(the uniform algebra of functions analytic on the open unit disc $D = \{z \in C : |z| < 1\}$ of complex plane C and continuous on its closure $\overline{D}$), when $u, \varphi \in A(D)$ and $\|\varphi\| \leq 1$: the operator T of the form $T : f \longmapsto u \cdot f \circ \varphi$ in $A(D)$ is compact if, and only if, φ is constant, or $|\varphi(z)| < 1$ whenever $u(z) \neq 0$. In [2] (see also [3]) was given sufficiently simple necessity condition of compactness of the operator T in general case (i.e., when $A(X)$ has no any additional structures, such as algebraic, analytic, etc.), which in concrete situations for the algebras with good structure turn into compactness criterion. Here we give analogous general compactness criterion as above mentioned necessity condition in the case, when the set of all peak points with respect to $A(X)$ is dense in the set of all peak sets with respect to $A(X)$ (see definition 2.3) and its applications for concrete uniform algebras (in particular, including multidimensional analogues of disc-algebra $A(D)$ and also Banach- $A(D)$ module of functions), which reduces to easily verifiable constructive compactness criterion (note that for an arbitrary compact set the converse of above mentioned necessity condition is not true; see the proof of necessity of Theorem 2.6 in the next section). Moreover, we give the necessary and sufficiently condition for those operators to have closed range.

2. Compactness criterion on uniform algebras

First of all in this section we investigate compactness of operator $T : A(X) \to C(X)$ in general case, when $A(X)$ has no any special structure (such as algebraic, analytic, etc.), further we consider the case when $A(X)$ has an algebraic structure. Except for easy degenerate cases, we will consider the nontrivial weighted composition operators, i.e., we assume that $\varphi \neq const$ and u is no identically zero.

We begin this section with a very simple note. If X is a compact metric space (say with a metric d), then it is clear that the equicontinuity of the family F of the functions $f \in C(X)$ (or the family of mappings to any metric space) is equivalent to such property: if $x_n \to x_0\,(n \to \infty)$, then $f(x_n) \to f(x_0)$ uniformly with respect to F. Indeed, let this property be fulfilled, but we have no equicontinuity. Then for some $\delta > 0$ there exist sequences $\{x'_n\}, \{x''_n\}$ and $\{f_n\} \subset F$ such that $d(x'_n, x''_n) \to 0\,(n \to \infty)$, but $|f_n(x'_n) - f_n(x''_n)| \geq \delta > 0$. Replasing the sequences by the subsequences, if necessary, we can assume that the both subsequences $\{x'_n\}$ and $\{x''_n\}$ are converging to $x_0 \in X$. Then from the above mentioned property follows that for sufficiently big numbers n and for all $f \in F$ take place $|f(x_0) - f(x'_n)| < \frac{1}{2}\delta$, $|f(x_0) - f(x''_n)| < \frac{1}{2}\delta$. But, for the sequences of functions $f = f_n$ it is contradiction. Consequently, from this property follows the equicontinuity of the family F. Converse is obvious.

Lemma 2.1. *If $\varphi : X \to Y$ is a continuous mapping from the compact metric space X onto the compact metric space Y, then the equicontinuity of the family $F \subset C(X)$ is equivalent to equicontinuity of the family $F \circ \varphi = \{f \circ \varphi : f \in F\} \subset C(X)$.*

Proof. Let $F \subset C(X)$ be an equicontinuous family and $x_n \to x_0\,(n \to \infty)$. Since φ is a continuous mapping, then we have that $\varphi(x_n) \to \varphi(x_0)\,(n \to \infty)$. So, $f(\varphi(x_n)) \to f(\varphi(x_0))$ $(n \to \infty)$ uniformly with respect to , i.e., $g(x_n) \to g(x_0)$ $(n \to \infty)$ uniformly on the family $F \circ \varphi$. Consequently, by the above noted property the family $F \circ \varphi$ is equicontinuous. Conversely, let $F \circ \varphi$ be an equicontinuous family, but the family F is not equicontinuous. Then there exist a sequence $y_n \to y_0 \in Y\,(n \to \infty)$ and the functions $f_n \in F$ such that $|f_n(y_n) - f_n(y_0)| \geq \delta > 0$. Put $y_n = \varphi(x_n)$. If it is necessary passing to subsequences we may assume that $x_n \to x_0\,(n \to \infty)$ for some point $x_0 \in X$. Since the family $F \circ \varphi$ is a equicontinuous, so for sufficiently big numbers n we have $|(f \circ \varphi)(x_n) - (f \circ \varphi)(x_0)| < \delta$ for all $f \in F$. But in this case for the functions $f = f_n$ we have contradiction. The lemma is proved.

Lemma2.2. *The operator T is compact if and only if, for each $\varepsilon > 0$ the restriction of the family $U = \{f \in A(X) : \|f\| \leq 1\}$ to compact set $Y_\varepsilon = \{\varphi(x) \in X : |u(x)| \geq \varepsilon\}$ is equicontinuous.*

Proof. Necessity. We assume that T is a compact operator. Let $\varepsilon > 0$. Put $X_\varepsilon = \{x \in X : |u(x)| \geq \varepsilon\}$. Let $S : C(X) \to C(X_\varepsilon)$ be operator, such that for every $f \in C(X)$ according the restriction of the function $u^{-1}f$ on the compact X_ε. Since $|u(x)| \geq \varepsilon$ on the X_ε, so the operator S is well defined and is continuous; therefore an operator $ST : A(X) \to C(X_\varepsilon)$ is compact operator (because T is a compact operator). In particular, we have that the set $ST(U)$ is a relatively compact in $C(X_\varepsilon)$. Consequently, the family $\left\{ (f\circ\varphi)|_{X_\varepsilon} : f\in A(X), \|f\|\le 1\right\}$ (where the symbol $\cdot|_{X_\varepsilon}$ denote a restriction of functions on the compact X_ε) is equicontinuous. Then from Lemma2.1 (using for the compact sets X_ε and Y_ε) we have that a family $\{ f|\, Y_\varepsilon : f\in A(X), \|f\|\le 1\}$ is equicontinuous. Necessity is proved.

Sufficiency. Let $\varepsilon > 0$. Then the compact X may be covered by the open sets $E_1 = \{x\in X : |u(x)| > \varepsilon\}$ and $E_2 = \{x\in X : |u(x)| < 2\varepsilon\}$. Let $\delta > 0$ be a Lebesque number, which according to this cover. This is meaning that every ball with radius less than δ wholly is contained either in E_1, or in E_2 (it is well-known that every cover of the metric compact has a Lebesque number). Since the restriction of the unit ball of $A(X)$ on the compact $\{\varphi(x) : x\in X, |u(x)|\ge\varepsilon\}$ is equicontinuous family, so from Lemma 2.1 we have that the family

$\{f\circ\varphi : f\in A(X), \|f\|\le 1\}$also is equicontinuous on the set E_1. Let δ_1 be a positive number such that, if $x_1,\, x_2\in E_1$ and $d(x_1,x_2) < \delta_1$, then

$$|(f\circ\varphi)(x_1) - (f\circ\varphi)(x_2)| < \varepsilon$$

for all $f\in A(X)$, $\|f\|\le 1$. Without loss of generality we can assume that $|u(x)|\le 1$ everywhere on X and $|u(x_1) - u(x_2)| < \varepsilon$, if $x_1, x_2\in X$, $d(x_1,x_2) < \delta_1$. Therefore for the points $x_1, x_2\in E_2$ with $d(x_1,x_2) < \delta_1$ we have that

$$|u(x_1)\, f(\varphi(x_1)) - u(x_2)\, f(\varphi(x_2))| < 2\varepsilon.$$

Further, we may assume that $\delta_1 < \delta$. Then if, $d(x_1,x_2) < \delta_1$, but $\{x_1,x_2\}$ is not contained in E_1, so we have that $\{x_1,x_2\}\in E_2$; consequently we have that,$|u(x_1)\, f(\varphi(x_1)) - u(x_2)\, f(\varphi(x_2))| < 4\varepsilon$, because $|(f\circ\varphi)(x)|\le 1$ and $|u(x)| < 2\varepsilon$ everywhere on the set E. The lemma is proved.

Definition 2.3. *A closed subset $E\subset X$ is called a peak set with respect to $A(X)$, if there exists a sequence $\{f_n\}\subset A(X)$, such that $\|f_n\| = f_n(x) = 1$ for all n and all $x\in E$, moreover, outside any neighbourhood of the set E the sequence $\{f_n\}$ tends to 0 uniformly. A peak set consisting of only one point is called a peak point.*

Definition 2.4. *Two points x_1, x_2 of a topological space X are called compactly connected, if there exists a connected compact set $E\subset X$ such that it contains both of these points. It can be easily seen that the compactly connected relation is a equivalence relation. Equivalence classes of this relation is called compactly connected componenets of X.*

Definition 2.5. *Let $A(X)$ be a uniformly closed subspace of $C(X)$ (in particular , a uniform algebra) . A mapping $\varphi : X\to X$ is called a compositor on $A(X)$ if $f\circ\varphi\in A(X)$ whenever $f\in A(X)$. A function $u\in C(X)$ is called a multiplicator with respect to $A(X)$ if $u\cdot f\in A(X)$ for all functions $f\in A(X)$. We denote the set of all compositors on $A(X)$ by $C_{A(X)}$ and the set of all multiplicators with respect to $A(X)$ by $M_{A(X)}$.*

We denote the set of all peak sets with respect to $A(X)$ by $S(A(X))$ and the set of all peak points with respect to $A(X)$ by $S_0(A(X))$. We will assume that the set $S_0(A(X))$ is dense in $S(A(X))$ and the number of the compactly connected components of $\varphi(Y)\subset S(A(X))$ is finitely many, also the restriction of the unit ball of $A(X)$ on the compact subsets of the form $\varphi(Y)$ (where Y is compact subset of $S(u)$) of the set $X \backslash S(A(X))$ is an equicontinuous family. Then under this conditions for weighted composition operators induced by the mapping $\varphi \in C_{A(X)}$ and by the function $u \in M_{A(X)}$ from the Lemma 2.2 directly we obtain following theorem:

Theorem 2.6. *If $u \in M_{A(X)}$ and $\varphi \in C_{A(X)}$, then the weighted composition operator $T : A(X) \to A(X)$, $f \longmapsto u \cdot f \circ \varphi$ is compact if and only if, for every compactly connected component Y of $S(u)$ and for any peak set E with respect to $A(X)$, we have either $\varphi(Y) \cap E = \varnothing$, or $\varphi(Y) \subset E$.*

Proof. Necessity. It is directly corollary of the Lemma 2[2] (see also, Theorem 1.5 [3]). (note that for an arbitrary compact set the converse of this statement is not true; indeed if is compact with only one limit point, then this statement holds for any weighted composition operator, because there is no connected subset in X other than one-point sets).

Sufficiency. Since for every connected compact Y of $S(u)$ such that $\varphi(Y) \subset X \backslash S(A(X))$ we have that the restriction of unit ball of $A(X)$ on $\varphi(Y)$ is equicontinuous and from the condition $\varphi(Y) \cap S(A(X)) \neq \varnothing$ follows that $\varphi(Y) \subset S(A(X))$, so sufficiently to check equicontinuity of the family $\{f \circ \varphi : f \in A(X), \|f\| \leq 1\}$ only on such compact sets $Y \subset S(u)$ which $\varphi(Y) \subset S(A(X))$. But in this case for every compactly connected component Y_i $(i = 1, ..., n)$ of $\varphi(Y)$ is singleton (indeed, in otherwise case, since the compact Y_i is connected and the set of peak points is dense in $S(A(X))$, so there exists peak point which its intersection with Y_i is not empty; this is contradiction to condition of the theorem). Consequently, for any compact $Y \in S(u)$ we have that $\varphi(Y)$ is consist of finitely number of points, so from the Lemma 2.2 we have that the theorem is correctly. The theorem is proved.

Remark 2.7. Note that when X is a locally connected compact set (we recall that the compact set X is locally connected if any point of X has a fundamental system of connected compact neighbourhoods) and $A(X)$ is a uniformly closed subspace of $C(X)$, then the number of the compactly connected components of $\varphi(Y) \subset S(A(X))$ is finitely many (indeed, for every $\varepsilon > 0$ from the locally connected assumption we have that there exist a finitely many connected compact sets $Y_1, ..., Y_n$, such that $Y_i \in S(u)$ and $\bigcup_{i=1}^{n} Y_i \supset \{x \in S(u) : |u(x)| \geq \varepsilon\}$. Consequently, for this uniformly closed subspaces $A(X)$ of $C(X)$ such that the set $S_0(A(X))$ is dense in $S(A(X))$ and the restriction of the unit ball of $A(X)$ on the compact subsets of the form $\varphi(Y) \subset X \backslash S(A(X))$ (where Y is compact subset of $S(u)$) is equicontinuity family, so the Theorem 2.6 is true without the assumption the number of the compactly connected components of $\varphi(Y) \subset S(A(X))$ is finitely many.

Let $A(X)$ be a uniform algebra. We note that in this case we may defined the notations of the peak set and the peak point in the following way: a peak set of the algebra $A(X)$ is a closed subset E of X for which there exists a function f in the algebra with $\|f\| = f(x) = 1$ for $x \in E$ and $|f(x)| < 1$ for $x \in X \backslash E$; a singleton peak set is a peak point. It is a well-known theorem of Bishop that for a uniform algebra $A(X)$ on a compact metrizable space X every peak set of $A(X)$ contains a peak point of $A(X)$ and so the set of peak points of $A(X)$ is a boundary (a subset E of X is called a boundary with respect to $A(X)$ if every function $f \in A(X)$ attains its maximum modulus on E; evidently every boundary must contain $S_0(A(X))$) and therefore is dense in the Shilov boundary of $A(X\ )$(the Shilov boundary of $A(X)$ is smallest closed boundary with respect to $A(X)$ and it is denoted by $Sh(A(X))$), i.e., $\overline{S}_0(A(X)) = Sh(A(X))$. It is well known that the Shilov boundary for function algebras (in particular, for uniform algebras) exists and it is in fact the intersection of all closed boundaries. So, for every unifom algebra $A(X)$ with the property such that the restriction of the unit ball of $A(X)$ on the compact subsets of the set $X\backslash Sh(A(X))$ of the form $\varphi(Y)$ (where Y is compact subset of $S(u)$) is equicontinuous family and number of compactly connected components of $\varphi(Y) \subset S(A(X))$ is finitely many, we have the Theorem 2.6 is correct. Therefore for this uniform algebras we have following easily verifiable compactness criterion:

Theorem 2.8. *If $u \in M_{A(X)}$ and $\varphi \in C_{A(X)}$, then the weighted composition operator $T : A(X) \to A(X)$, $f \longmapsto u \cdot f \circ \varphi$ is compact if and only if, for every compactly connected component Y of $S(u)$ we have either $\varphi(Y)$ is singleton, or $\varphi(Y) \subset X\backslash S(A(X))$.*

In particular, for the uniform algebras $A(X)$ (for example the algebra $C(X)$, the disc-algebra $A(\partial D)$ viewed as the algebra of continuous functions on the boundary ∂D of the unit disc D of complex plane and so) which $S_0(A(X)) = X = Sh(A(X))$, since $X\backslash Sh(A(X))$ is empty set, so the compactness of the nontrivial weighted composition operator $T : A(X) \to A(X)$, $f \longmapsto u \cdot f \circ \varphi$ induced by the mapping $\varphi \in C_{A(X)}$ and by the function $u \in M_{A(X)}$ under the conditions of Theorem 2.8 is equivalent to the property such as, for every compact subset $Y \subset S(u)$, the set $\varphi(Y)$ is finite. For example from this we obtain as corollary the following compactness criterion for the weighted composition operators induced by the continuous mapping $\varphi : X \to X$ and by the continuous function $u \in C(X)$ on the algebra $C(X)$ (see also [2], [3]):

Corollary 2.9. *The weighted composition operator T of the form $f \longmapsto u \cdot f \circ \varphi$ is compact on $C(X)$ if, and only if, for each compact subset Y of $S(u)$ the set $\varphi(Y)$ is finite. In particular, when $S(u) = X$ and X is connected compact set, then T is compact on $C(X)$ if, and only if, φ is constant.*

When the compact X is connected set, unlike to $C(X)$ in analytical situations a nontrivial weighted endomorphism may be a compact operator, as in the case disc-algebra. Let the uniform algebra $A(X)$ has an analytical structure, $S_0(A(X)) = Sh(A(X)) = \partial X$ and $\varphi(O) \subset O$, for every connected open subset O of G, where $G = X\backslash S_0(A(X))$ is a nonempty open set (where ∂X is a topologically boundary of compact set X; a typical example is given by the disc-algebra $A(D)$, where $X = \overline{D}$, $S_0(A(X)) = \partial\overline{D} = \overline{D}\backslash D$, $G = D$). So, from the compactness principle (Montel's theorem; see, for example [4], p.222) and from the Theorem 2.8 we immediately obtain the following theorem, which is the generalization of Kamowitz compactness criterion for the nontrivial weighted composition operators in the disc-algebra:

Theorem 2.10. *If $u \in M_{A(X)}$ and $\varphi \in C_{A(X)}$, then the nontrivial weighted composition operator $T : A(X) \to A(X)$, $f \mapsto u \cdot f \circ \varphi$ is compact if and only if, for every compactly connected component X we have either φ is constant mapping, or $\varphi(x) \in G$ whenever $x \in S(u)$.*

Corollary 2.11. *Under the conditions of the Theorem 2.9 when X is connected compact set then the nontrivial weighted composition operator $T : A(X) \to A(X)$, $f \mapsto u \cdot f \circ \varphi$, is compact if and only if $\varphi(x) \in G$, whenever $x \in S(u)$.*

We may replace the disc in Kamowitz theorem by more general domains. In multidimensional case different domains give rise to additional properties. The case of ball is easier than the case of poly-disc (because the Shilov's boundary of poly-disc is contained in the topological boundary as a proper subset). Let $A(B^n)$ be the algebra of analytic functions in the interior of the unit ball $B^n = \left\{ z = (z_1, ..., z_n) \in C^n : \sum_{k-1}^{n} |z_k|^2 < 1 \right\}$ of n-dimensional complex plane C^n and continuous on its closure. Since every point of the topological boundary of ball is a peak point, so Corollary 2.10 enables us to generalize Kamowitz Theorem (see also [2]):

Corollary 2.12. *The operator of weighted composition induced by* u, φ *on* $A(B^n)$ *(*u *and* φ *are analytic) is compact if, and only if, either* $\varphi = const$, *or* $\|\varphi(z)\| < 1$*(Eucledian norm) for all* $z \in S(u)$

Remark 2.13. Note that Corollary 2.10 enables us to characterize compactness of weighted composition operators on a wide class of functions- $A(X)$- Banach module spaces of functions, when the set of peak points is invariant. For example (in the simple case), let $D \subset C^n$ $(n \geq 1)$ be a connected bounded domain and $A(D)$ is the uniform algebra of functions analytic on D and continuous on its closure $\overline{D}$. We consider a normed space $A \subset Hol(D)$, such that A contains $A(D)$ and is $A(D)$-Banach module, where $Hol(D)$ is the space of holomorphic functions on the domain D with closed-open topology (we assume that also the inclusion $A \subset Hol(D)$ is continuous). It is clear that $S_0(A) = S_0(A(D))$. Since $Hol(D)$ is a Montel space (any bounded subset is relatively compact), then from above mentioned result it is clear that, the weighted composition operators on the space A induced by any function $u \in A$ and by any holomorphic mapping $\varphi : \overline{D} \to D$ is compact.

3. Closedness of range of weighted composition operators on uniformly closed subspaces of with analytical structure

In this section we will investigated the nontrivial weighted composition operators on the uniformly closed subspaces $A(X)$ of $C(X)$ which have analytical structure. For simplicity we will assume that the compact X is connected, the subspace $A(X)$ has analytical structure and $S(A(X)) = S_0(A(X)) = \partial X$ is boundary with respect to $A(X)$. We also assume that $G = X \backslash \partial X$ is nonempty set and it is invariant with respect to the mapping $\varphi : X \to X$.

Theorem 3.1. I*f* $u \in M_{A(X)}$ *and* $\varphi \in C_{A(X)}$, *then the nontrivial weighted composition operator* $T : A(X) \to A(X)$, $f \longmapsto u \cdot f \circ \varphi$ *has closed range if and only if there exists a compact subset* K *of* $\partial X \cap S(u)$ *such that* $\varphi(K) \supset \partial X$.

Proof. Since from the uniqueness theorem for analytic functions we have that the kernel of nontrivial weighted composition operators on uniformly closed subspaces which have an analytical structure are trivial, then we have that the operator has closed range if and only if, there exists a positive constant δ such that $\delta \|f\| \leq \|Tf\|$ for all $f \in A(X)$ (see [5], TheoremIV.5.9, also see [6], p. 487]). Using this inequality, we will prove the theorem.

Sufficiency. Suppose that there exists a compact subset $K \subset \partial X \cap S(u)$ such that $\varphi(K) \supset \partial X$. So, it is clear that there exists a positive constant δ such that $|u(x)| \geq \delta$ for all $x \in K$. Then for every $f \in A(X)$, we have:

$$\|f\| = \sup\{|f(x)| : x \in \partial X\} \leq \sup\{|f(x)| : x \in \varphi(K)\} =$$

$$= \sup\{|f(\varphi(x))| : x \in K\} = \sup\left\{\frac{1}{|u(x)|}|Tf(x)| : x \in K\right\} \leq$$

$$\leq \frac{1}{\delta}\sup\{|Tf(x)| : x \in K\} \leq \frac{1}{\delta}\sup\{|Tf(x)| : x \in X\} = \frac{1}{\delta}\|Tf\|.$$

So, for this constant $\delta > 0$ we have $\delta\|f\| \leq \|Tf\|$ for all $f \in A(X)$, i.e., the operator T has closed range.

Necessity. Suppose that the operator T has closed range. Then there exists a positive constant δ such that $\delta\|f\| \leq \|Tf\|$ for all $f \in A(X)$. Put $K_1 = \{x \in \partial X : |u(x)| \geq \delta_1\} \subset \partial X \cap S(u)$, where δ_1 is any positive constant such that $\delta_1 < \delta$. We will show that the set $\varphi(K_1)$ contains the set of all peak points of $A(X)$. For this end, assume that $\varphi(K_1)$ does not contains ∂X. Then there exists a peak point a of $A(X)$, such that $a \notin \varphi(K_1)$. So, there exists a sequence of functions $\{f_n\} \subset A(X)$ such that $\|f_n\| = f_n(a) = 1$ for all n and outside any neighbourhood of the point a the sequence $\{f_n\}$ tends to 0 uniformly. Hence, for sufficiently large n we obtain that $|f_n(x)| \leq \frac{\delta_1}{\|u\|}$ for all $x \in \varphi(K_1)$ (because the point a has a neighbourhood such that the intersection with compact $\varphi(K_1)$ is empty). For this function f_n we have

$$|(Tf_n)(x)| = |u(x)| \cdot |f_n(\varphi(x))| \leq \begin{cases} \|u\|\frac{\delta_1}{\|u\|} = \delta_1, \text{ if } x \in K_1, \\ \\ \delta_1\|f_n\| = \delta_1, \text{ if } x \in \partial X \backslash K_1. \end{cases}$$

So,

$$\|Tf_n\| = \{|(Tf_n)(x)| : x \in \partial X\} \leq \delta_1 < \delta = \delta\|f_n\|.$$

But this contradicts to the condition such that for all $f \in A(X)$ holds $\delta\|f\| \leq \|Tf\|$, consequently we have that $\partial X \subset \varphi(K_1)$. The theorem is proved.

From this theorem we have easily verifiable necessary and sufficient conditions for the weighted composition operators on the classical uniform algebras which have analytically structure and the set of peak points coincides with the topologically boundary of the compact.

For example, in the case ball-algebra $A(B^n)$ $(n \geq 1)$ we have following criterion of closedness of range of nontrivial weighted composition operators.

Corollary 3.2. *The operator of weighted composition induced by u, φ on $A(B^n)$ (u and φ are analytic) has closed range if and only if there exists a positive constant δ such that $\varphi(\{z \in \partial B^n : |u(z)| \geq \delta\})$contains the boundary ∂B^n of the unit ball B^n.*

Corollary 3.3. *The composition operator $f \longmapsto f \circ \varphi$ on the ball-algebra $A(B^n)$ induced by a nonconstant analytic mapping $\varphi : B^n \to B^n$ $(n \geq 1)$ has closed range if and only if, the set $\varphi(\partial B^n)$ contains the boundary ∂B^n of the unit ball B^n.*

Proof. Indeed, if u is the constant function 1, then the set$\{z \in \partial B^n : |u(z)| \geq \delta\}$ is the whole ∂B^n of for every constant $\delta \leq 1$ and is empty set otherwise. Hence Corollary 3.3 follows from Corollary 3.2.

Remark 3.4. Note that, under the conditions of Corollary 3.3 it is not necessary that an analytic mapping φ is surjective when the operator composition has closed range. For example, in simple case $(n = 1)$ we may define a conformal mapping $\varphi : D \to R_\alpha$ which maps the unit disc D to its proper subset $R_\alpha = \{(r, \theta) : \alpha < r < 1, 0 < \theta < 2\pi\}$, where α is any positive number, less than 1. Then

φ has a continuous extension to the closed unit disc $\overline{D}$ and $\varphi\left(\overline{D}\right)=\{(r,\theta):\alpha\leq r\leq 1, 0\leq\theta\leq 2\pi\}=\overline{R}_{\alpha}$, which contains the unit circle. Consequently, by Corollary 3.3 the composition operator induced by the mapping φ has a closed range, but φ is clearly not surjective.

Corollary 3.5. *The nontrivial multiplication operator* $f\longmapsto u\cdot f$ *on* $A(B^n)$ $(n\geq 1)$ *induced by the function* $u\in A(B^n)$ *has closed range if and only if* $u(z)\neq 0$, *for all* $z\in\partial B^n$.

Proof. Since u is continuous and nonzero on the compact set ∂B^n only in the case when there exists a positive number δ such that $|u(z)|\geq\delta$ for all $z\in\partial B^n$, that is $\partial B^n\subset\{z\in\partial B^n:|u(z)|\geq\delta\}$.

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Aydin I. Shahbazov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
E-mail: aydinshahbazov@yahoo.com.

Dashgin A. Seyidov
Nakhchivan State University.
University campus, AZ 7000, Nachchivan, Azerbaijan.
Tel.: (99412) 439 60 69 (off.).

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