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ON BEHAVIOR OF SOLUTIONS OF HIGNER ORDER DEGENERATE ELLIPTIC EQUATIONS

Abstract

Behavior of solutions of higher order elliptic equations is studied.

The goal of the paper is to study behavior of solutions of the Direchlet problem for degenerate divergent quasi-linear elliptic equations in the vicinity of the boundary.

For linear elliptic and parabolic equations the questions on behavior of solutions near the boundary were studied in the papers of O.A. Oleinik and his followers [1] − [2] . For quasilinear equations, similar results were obtained in the papers of A.F. Tedeev, A.E. Shishkov [3], T.S. Gadjiev [4]. Behavior of solutions in the vicinity of a boundary point for elliptic equations of second order was studied in the papers of V.G. Mazya and G.M. Verzhbinskii as well, where a wide reference was cited. S. Bonafade [5] and others studied quality properties of solutions for degenerate equations.

We obtained some estimations that are analogies of Saint-Venant's principle known in theory of elasticity. By means of these estimations we obtained estimations on behavior of solutions and their derivatives in bounded domains up to boundary.

In the bounded domain $\Omega \subset R^n$ $n \geq 2$, consider a generalized solution from the Sobolev space $\overset{\circ}{W}{}^{m}_{p,\omega}(\Omega)$ of the Dirichlet problem for the equation

$$\sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha A_\alpha (x, u, \nabla u, ..., \nabla^m u) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha F_\alpha (x) , \qquad (1)$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n, m \geq 1.$

Assume that the coefficients $A_\alpha (x, \xi)$ are measurable with respect to $x \in \overline{\Omega}$, continuous with respect to $\xi \in R^M$ (M is the number of different multi-indices of length no more than m) and satisfy the conditions

$$\sum_{|\alpha|=m} A_\alpha (x, \xi) \xi^m_\alpha \geq \omega (x) |\xi_m|^p - c_1 \omega (x) \sum_{i=1}^{m-1} |\xi_i|^p - f_1 (x)$$

$$|A_\alpha (x, \xi)| \leq c_2 \omega (x) \sum_{i=0}^{m} \left|\xi^i\right|^{p-1} - f_2 (x) , \qquad (2)$$

where $\xi = \left(\xi^0, ..., \xi^m\right)$, $\xi^i = \left(\xi^i_\alpha\right)$, $|\alpha| = i,\ c_1, c_2 > 0,\ p > 1,$

$$f_1 (x) \in L_{\frac{p}{p-1},loc} (\Omega) ,\ f_2 (x) \in L_{1,loc} (\Omega)\, F_\alpha \in L_{\frac{p}{p-1},loc} (\Omega) .$$

Assume that $\omega (x) , x \in \Omega$ is a measurable non negative function satisfying the conditions: $\omega \in L_{1,loc} (\Omega)$, and for any $\rho > 0$ and some

$$\sigma > 1 \int_{\Omega_\rho} \omega^{-1/(\sigma-1)} dx < \infty, ess \sup_{x\in\Omega_\rho} \omega (x) \leq c_3 \rho^{n(\sigma-1)} \left(\int_{\Omega_\rho} \omega^{-\frac{1}{(\sigma-1)}} dx \right)^{1-\sigma} . \qquad (3)$$

Here $\Omega_\rho = \Omega \cap B_\rho$, $B_\rho = \{x : |x| < \rho\}$, c_i are positive constants dependent only on the problem's data. In particular, it follows from condition (3) that $\omega \in A_\sigma$ (*see* [6]), i.e.

$$\int_{\Omega_\rho} \omega dx \left[\int_{\Omega_\rho} \omega^{-\frac{1}{(\sigma-1)}} dx \right]^{\sigma-1} \leq C_4 \rho^{n\sigma} \tag{4}$$

(3) also yields the estimation

$$ess \sup_{x\in\Omega_\rho} \omega(x) \leq c_5 \rho^{-n} \left(\int_{\Omega_\rho} \omega dx \right). \tag{5}$$

Furthermore, assume that

$$\frac{\omega(\Omega_s)}{\omega(\Omega_h)} \leq c_6 \left(\frac{s}{h}\right)^{n\mu}, \tag{6}$$

$\mu < 1 + p/n$, for any $s \geq h > 0$, where $\omega(\Omega_s) = \int\limits_{\Omega_s} \omega(x)\, dx$.

Let $0 \in \partial\Omega$, $S_\rho = \Omega \cap \partial\Omega_\rho$. $K(\rho_1, \rho_2) \equiv \Omega_{\rho 2} \backslash \Omega_{\rho 1}$.

Our main goal is to obtain estimations of behavior of the integral of energy $I_\rho = \int\limits_{\Omega_\rho} \varpi(x) |\nabla^m u|^p dx$, dependent on geometry Ω in the vicinity of the point 0 for small ρ. We'll describe geometry $\partial\Omega$ with weight nonlinear basic frequency $\lambda_p^p(r)$ of section S_r

$$\lambda_p^p(r) = \inf \left(\int_{S_r} \omega(x) |\nabla_S v|^p ds \right) \left(\int_{S_r} \omega(x) |v|^p ds \right)^{-1}, \tag{7}$$

where the lower bound is taken by all continuously differentiable functions in the vicinity of S_r that vanish on $\partial\Omega$; $\nabla_S v(x)$ is a projection of the vector $\nabla v(x)$ on a tangential plane to S_r at the point x. For $p = 2$ and $\omega(x) = 1$ the number $\lambda_2^2(r)$ the first eigen value of the Beltrame-Laplace operator on S_r for $p \neq 2$ $\lambda_p^p(r)$ was studied in various papers. Some examples, calculations or its lower estimations for some concrete sets are for example [7].

Here $W_{p,\omega}^m(\Omega)$ is a closure of the functions from $C^m\left(\overline{\Omega}\right)$ with respect to the norm

$$\|u\|_{W_{p,\omega}^m(\Omega)} = \left(\int_\Omega \omega(x) \sum_{|\alpha|\leq m} |D^\alpha u|^p dx \right)^{\frac{1}{p}}$$

$\overset{\circ}{W}{}_{p,\omega}^m$ is a closure of the functions from $C_0^\infty(\Omega)$ in the norm $W_{p,\omega}^m(\Omega)$.

The function $u(x) \in \overset{\circ}{W}{}_{p,\omega}^m(\Omega)$ is said to be a generalized solution of the Dirichlet problem for equation (1) if the integral identity

$$\int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x, u, ..., D^m u) D^\alpha \eta dx = \int_\Omega \sum_{|\alpha|\leq m} F_\alpha(x) D^\alpha \eta dx \tag{8}$$

is fulfilled for the arbitrary function $\eta(x) \in C_0^\infty(\Omega)$.

Lemma 1. *Let* $u(x) \in \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$ *and* $f(K(x))$ *be a measurable non-negative function locally bounded in* Ω. *Then the inequality*

$$\int\limits_{K(\rho_1,\rho_2)} \left|\nabla^j u\right|^p \lambda_p^{(m-j)p}(K(x))\, f(K(x))\,\omega(x)\,dx \leq$$

$$\leq \frac{h_2}{h_1} \int\limits_{K(\rho_1,\rho_2)} |\nabla^m u|\, f(K(x))\,\varpi(x)\,dx,$$

is valid, where $j \leq m$.

Lemma 2. *Assume that the continuous non-decreasing on* $(0, r_0)$ *function* $J(r)$ *satisfies the inequality*

$$J(r(1-\varphi_1(r))) < \lambda J(r) + h(r),\ \forall r \in (0, r_0),\ 0 < \lambda < 1 \tag{9}$$

and the estimation

$$h\left(re^{-\frac{\overline{\varphi}_0(r)}{1-c_0-\delta}}\right) < c_7 \exp\left(-\delta \ln \lambda^{-1} \int\limits_r^{r_0} \frac{d\tau}{\tau\overline{\varphi}_0(\tau)}\right) h(r_0),\ 0 < \delta < 1 - c_0 \tag{10}$$

is valid for $h(r)$.

Then the estimation

$$J\left(re^{-\frac{\overline{\varphi}_0(r)}{1-c_0-\delta}}\right) < c_8(c_7, \nu) \times$$

$$\times \exp\left(-\delta \ln(\lambda+\nu)^{-1} \int\limits_r^{r_0} \frac{d\tau}{\tau\overline{\varphi}_0(\tau)}\right)(J(r_0) + h(r_0)),\ \nu > 0 \tag{11}$$

is valid for $J(r)$.

Lemma 3. *Assume that the non-negative on* $(0, r_0)$ *function* $J(r)$ *satisfies the inequality*

$$J(\varepsilon(r)r) < (1-\varphi(r))\, J(r) + h(r)\,\varphi(r) \tag{12}$$

and the estimation

$$h(r) < c_9 \exp\left(-\frac{1}{\ln \varepsilon^{-1}} \int\limits_r^{r_0} \frac{\overline{\varphi}(\tau)}{\tau} d\tau\right) h(r_0) \tag{13}$$

is valid for $h(r)$.

Then the estimation

$$J(r) < c_{10}(c_9, \nu) \exp\left(-\frac{1-\nu}{\ln \varepsilon^{-1}} \int\limits_r^{r_0} \frac{\overline{\varphi}(\tau)}{\tau} d\tau\right)(J(r_0) + h(r_0)),\ \forall \nu > 0 \tag{14}$$

is valid for $J(r)$.

Further, we'll divide the considered domains into two classes. The first class is "narrow domains", i.e. such domains whose complement in the vicinity of the point 0 is sufficiently massive, for example, it contains some cone with a vertex at this point. In the terms of frequency of the set, this class of domains satisfy the condition

$$r\lambda_p(r) > d_1 > 0,\ \forall r \in (0, r_0),\quad r_0 > 0. \tag{A}$$

The second class contains "wide domains", i.e. the domains that for example have "inside spinode" at the point 0. In the terms of frequency of the set, this class by domains satisfies the condition

$$r\lambda_p(r) < d_2 < \infty,\ \forall r \in (0, r_0). \tag{B}$$

Define the function $\psi(r)$ on $(0, r_0)$ by the inequality

$$\inf_{r\psi(r)<|x|<r} \lambda_p(|x|)(r - r\psi(r))\varpi(x) \geq \mu > 0, \tag{15}$$

where μ is such that $0 < 1 - c_0 < \psi(r) < 1$. For monotonically decreasing functions $\lambda_p(r)$ (that we often meet in applications), inequality (15) takes the following form

$$r\lambda_p(r)(1 - r\psi(r))\omega(x) \geq \mu,\ \text{for } \varphi(r) \equiv 1 - \psi(r) \geq \mu\omega^{-1}(x)(r\lambda_p(r))^{-1}. \tag{16}$$

Make the following denotation

$$J(r) = \int_{\Omega_r} \varpi(x)|D^m u|^p\, dx,$$

$$G(r) = \int_{\Omega_r} \left(\sum_{|\alpha|\leq m} \omega(x)(|F_\alpha| + |f_2|)^{\frac{p}{p-1}} \lambda_p^{-\frac{m-|\alpha|}{p-1}p}(|x|) + |f_1| \right) dx.$$

Theorem 1. *Let $u(x) \in \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$ be a generalized solution of the Dirichlet problem for equation* (1). *Assume that the coefficients of the equation satisfy condition* (2)*, the domain Ω the condition* (A)*, the weight $\omega(x)$ the conditions* (3), (6)*. Let $\overline{\psi}$ be an arbitrary continuous non-increasing on $(0, r_0)$ function satisfying the inequality $0 < 1 - c_0 < \overline{\psi}(r) \leq \psi(r) < 1$, where $\psi(r)$ is determined from inequality* (15)*, and assume that the condition*

$$G\left(r\exp\left(-\frac{1-\overline{\psi}(r)}{1-c_0-\theta}\right)\right) < c_{11}\exp\left(-\theta\ln\beta^{-1}\int_r^{r_0}\frac{d\tau}{\tau\left(1-\overline{\psi}(\tau)\right)}\right)G(r_0), \tag{17}$$

is fulfilled for $c_{11} > 0,\ \theta < 1 - c_0,\ \beta = const < 1$. Then the estimation

$$J\left(r\exp\left(-\frac{1-\overline{\psi}(r)}{1-c_0-\theta}\right)\right) <$$

$$< c_{12}(c_{11},\nu)\exp\left(-\theta\ln(\beta+\nu)^{-1}\int_r^{r_0}\frac{d\tau}{\tau\left(1-\overline{\psi}(\tau)\right)}\right)(J(r_0) + G(r_0)) \tag{18}$$

is valid for $J(r)$ *at* $\forall \nu > 0$.

Proof of Theorem 1. In the course of the proof we'll use Hopf functions type special shear functions. Namely, let $\xi(r)$ be m times continuously differentiable function, $0 < \xi(r) < 1$, $0 < r < 1$, $\xi(r) = 1$ for $r \leq 0$, $\xi(r) = 0$ for $r \geq 1$ $a_j = \max \left| \xi^{(j)}(r) \right|$. Denote $\xi_{\psi(r)}(r) = \xi\left(\dfrac{r-\psi(r)}{1-\psi(r)}\right)$, where $0 < \psi(r) < 1$ is an arbitrary measurable function on $(0, r_0)$. The following estimations are true for this shear function

$$\left| D_x^j \xi_{\psi(r)} \left(\frac{g(x)}{r} \right) \right| \leq \frac{a_j}{[r(1-\psi(r))]^j}, r\psi(r) < g(x) < r, \ j > 1$$

$$D_x^j \xi_{\psi(r)} \left(\frac{g(x)}{r} \right) = 0 \text{ for } g(x) < r\psi(r), g(x) > r, \ j > 1. \tag{19}$$

Into integral identity (8) we put $\eta(x) = u(x)\, \xi_{\psi(r)} \left(\dfrac{g(x)}{r} \right)$.

Continuing the theorem's proof, we have

$$\sum_{|\alpha|=m} \int_{\Omega_r} A_\alpha(x, u, ..., D^m u)\, D^\alpha u \cdot \xi dx = \left(\sum_{\substack{|\alpha|=m \\ |\beta|<m}} + \sum_{\substack{|\alpha|<m \\ |\beta|\leq\alpha}} \right) \times$$

$$\times \int_{\Omega_r} C_{\alpha\beta}(m,n)\, A_\alpha(x, ..., D^m u) \cdot D^\alpha u \cdot D^{\alpha-\beta} \xi dx +$$

$$+ \int_{\Omega_r} \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq\alpha}} C_\beta(m,n)\, F_\alpha(x)\, D^\beta u D^{\alpha-\beta} \xi dx, \tag{20}$$

where the vector $\alpha - \beta = (\alpha_i - \beta_i)$, $i = 1, ..., n$. Using condition (2), we estimate the integrals in (20) and get

$$\int_{\Omega_{r\psi(r)}} \omega(x) \left| D^m u \right|^p dx <$$

$$< \int_{\Omega_r} [c_2 \varpi(x) \sum_{|\alpha|<m} \left| D^\alpha u \right|^p + c_3 k_1 \omega(x) \left(\sum_{|\alpha|\leq m} \left| D^\alpha u \right| \right)^{p-1} \left(\sum_{|\alpha|<m} \left| D^\alpha u \right| \right) +$$

$$+ \sum_{|\alpha|<m} \left| f_2(x) \right| \left| D^\alpha u \right| + \sum_{|\alpha|\leq m} \left| F_\alpha(x) \right| \left| D^\alpha u \right|]\, \xi dx +$$

$$+ \int_{\Omega_r \cap \Omega_{r\psi(r)}} [c_3 k_2 \omega(x) \left(\sum_{|\alpha|\leq m} \left| D^\alpha u \right| \right)^p . \tag{21}$$

$$\sum_{|\alpha|\leq m} \sum_{|\beta|\leq|\alpha|} \left| D^{\alpha-\beta} u \right| \left| D^\beta \xi \right| + \sum_{|\alpha|\leq m} \sum_{|\beta|\leq|\alpha|} \left| f_2(x) \right| \left| D^{\alpha-\beta} u \right| \left| D^\beta \xi \right| + \sum_{|\alpha|\leq m}$$

$$\sum_{|\beta|\leq\alpha}|F_\alpha(x)|\cdot\left|D^{\alpha-\beta}u\right|\left|D^\beta\xi\right|dx+\int_{\Omega_r}|f_1(x)|\,\xi dx.$$

Applying the Holder, Young inequality with ε and Lemma 1, estimate the right hand side by (21) and get

$$J(r\psi(r))<J(r)\left[c_2k_2\sup_{0<\tau<r_0}\left(\sum_{|\alpha|=m}\lambda_p^{-\alpha p}(\tau)\right)+\right.$$

$$\left.+c_3k_1k_3\sup_{0<\tau<r_0}\left(\lambda_p^{-1}(\tau)\right)\cdot\sup_{0<\tau<r_0}\left(\sum_{|\alpha|\leq m}\lambda_p^{-\alpha p}(\tau)\right)+\varepsilon\right]+$$

$$+\varepsilon^{1-p}k_4\int_{\Omega_r}\left[\sum_{|\alpha|<m}|f_2(x)|^{\frac{p}{p-1}}\lambda_p^{-\frac{m-|\alpha|}{p-1}p}(\rho)+\sum_{|\alpha|\leq m}|F_\alpha|^{\frac{p}{p-1}}\lambda_p^{-\frac{m-|\alpha|}{p-1}p}(\rho)\right]dx+$$

$$+c_3k_2k_5\sup_{0<\tau<r_0}\left(\sum_{|\alpha|\leq m}\lambda_p^{-pi}(\tau)\right)^{\frac{p}{p-1}}\left(\int_{\Omega_r\setminus\Omega_{r\psi(r)}}|D^mu|^p\,dx\right)^{\frac{p}{p-1}}\times$$

$$\times\left(\int_{\Omega_r\setminus\Omega_{r\psi(r)}}|D^mu|^p\sum_{|\alpha|\leq m}\sum_{|\beta|\leq|\alpha|}\left(\frac{\omega(x)\,D^\beta\xi}{\lambda^{m-|\alpha|+|\beta|}(\rho)}\right)^p dx\right)^{\frac{1}{p}}+$$

$$+k_6\left[\int_{\Omega_r\setminus\Omega_{r\psi(r)}}\left(\sum_{|\alpha|\leq m}(|f_2(x)|+|F_\alpha|)^{\frac{p}{p-1}}\cdot\lambda_p^{\frac{m-|\alpha|}{p-1}p}(\rho)\,dx\right)\right]^{\frac{p}{p-1}}\tag{22}$$

$$\left(\int_{\Omega_r\setminus\Omega_{r\psi(r)}}|D^mu|^p\sum_{|\alpha|\leq m}\sum_{|\beta|\leq|\alpha|}\left(\frac{\omega(x)\,D^\beta\xi}{\lambda^{m-|\alpha|+|\beta|}(\rho)}\right)^p dx\right)^{\frac{1}{p}}+\int_{\Omega_r}|f_1(x)|\,dx$$

here k_i are constants dependent only on p,m and n.

Since the domain Ω satisfies condition (A), then for arbitrary small δ there exists r_0 such that for $r\in(0,r_0)$, $\lambda_p(r)>\delta^{-1}$. Then allowing for (15) and (19), from (22) we get

$$J(r\psi(r))<\left(2c_2k_2\delta^p+c_3k_1k_3\delta+\varepsilon\right)J(r)+\left(\varepsilon^{1-p}k_4+k_6k_7^p\right)$$

$$\int_{\Omega_r}\left(\sum_{|\alpha|\leq m}(|f_2(x)|+|F_\alpha|)^{\frac{p}{p-1}}\lambda_p^{-\frac{m-|\alpha|}{p-1}p}(\rho)+|f_1(x)|\right)dx+\tag{23}$$

$$+\left(2c_3k_2k_5k_7+k_6\right)\left(J(r)-J(r\psi(r))\right),$$

where $k_7 = 2\left(\sum\limits_{i=0}^{m}\left(\frac{a_i}{\mu^i}\right)^p\right)^{\frac{1}{p}}$. We choose δ and ε so small that

$$2c_2k_2\delta^p + c_3k_1k_3\delta + \varepsilon < 2^{-1}.$$

Then from (23) we have

$$J\left(r\psi\left(r\right)\right) < \beta J\left(r\right) + k_8 G\left(r\right), \tag{24}$$

where $\beta = \dfrac{2^{-1} + 2c_3k_2k_5k_7 + k_6}{1 + 2c_3k_2k_5k_7 + k_6} < 1.$

Validity of the theorem follows from estimation (21) and Lemma 2.

The theorem is proved.

Assume $\psi\left(r\right) \equiv d$, $0 < d < 1$. Denote $\overline{\lambda}_p\left(r\right) = \inf\limits_{dr<\tau<r} \lambda_p\left(\tau\right)$. Usually, in applications $\lambda_p\left(r\right)$ is a non-increasing function, therefore $\widetilde{\lambda}_p\left(r\right) = \lambda_p\left(r\right).$

Theorem 2. *Let $u\left(x\right) \in \overset{\circ}{W}{}^{m}_{p,\omega}\left(\Omega\right)$ be a generalized solution of the Dirichlet problem for equation (1). Assume that the coefficients of the equation satisfy the conditions (2), the domain Ω the condition (B), the weight $\omega\left(x\right)$ the conditions (3), (6). Let $\overline{\varphi}\left(r\right)$ be an arbitrary non-decreasing function on $\left(0, r_0\right)$ satisfying the inequality $\widetilde{\varphi}\left(r\right) < \varphi\left(r\right) \equiv r\widetilde{\lambda}_p\left(r\right)$ and assume that the condition*

$$G\left(r\right) < c_{13}\exp\left(-\frac{\left(1-d\right)^m}{2c_{12}\ln d^{-1}}\int\limits_r^{r_0}\frac{\overline{\varphi}^m\left(\tau\right)d\tau}{\tau}\right)G\left(r_0\right) \tag{25}$$

is fulfilled.

Then for $J\left(r\right)$ it is valid the following estimation

$$J\left(r\right) < c_{14}\exp\left(-\frac{\left(1-\nu\right)\left(1-d\right)^m}{2c_{12}\ln d^{-1}}\int\limits_r^{r_0}\frac{\overline{\varphi}^m\left(\tau\right)d\tau}{\tau}\right)\left(J\left(r_0\right) + G\left(r_0\right)\right) \tag{26}$$

at $\forall\nu > 0$.

Proof. For proof we return back to inequality (22). As we took $\psi\left(r\right) \equiv d$, $0 < d < 1$,then $\xi = \xi_d\left(\dfrac{g\left(x\right)}{r}\right)$ respectively in (22). Using the fact that by the conditions as $\tau \to 0$, $\lambda_p\left(\tau\right) \to \infty$ and the Young inequality for any $\delta > 0$ on the interval $\left(0, r_0\left(\delta\right)\right)$, we get

$$\begin{gathered} J\left(dr\right) < \left(2c_2k_2\delta^p + c_3k_1k_3\delta + \varepsilon\right)J\left(r\right) + \varepsilon^{1-p}k_4G\left(r\right) + \left(r\widetilde{\lambda}_p\left(r\right)\right)^{-m}\times \\ \times\left(1-d\right)^{-m}\left(2ac_3k_8 + ak_9\right)\left(J\left(r\right) - J\left(dr\right)\right) + \\ + \left(r\widetilde{\lambda}_p\left(r\right)\right)^{-m}\left(1-d\right)^{-m}ak_7k_{10}\left(G\left(r\right) - G\left(dr\right)\right), \end{gathered} \tag{27}$$

where $a = \max a_i$, $i = \overline{1,m}$. Assuming δ and ε sufficiently small (27), we get

$$J\left(dr\right) < 2^{-1}J\left(r\right) + k_{10}G\left(r\right) + k_{11}\left(r\widetilde{\lambda}_p\left(r\right)\right)^{-m}\times$$

$$\times (1-d)^{-m} (J(r) - J(dr)) + k_{12} \left(r\widetilde{\lambda}_p (r) \right)^{-m} (1-d)^{-m} (G(r) - G(dr)). \quad (28)$$

Hence, after simple calculations we get

$$J(dr) + lG(dr) < (1 - g(r)) (J(r) + lG(r)) + KG(r), \quad (29)$$

where

$$l(r) \equiv \frac{k_{12}}{k_{11} + \varphi^m (r) (1-d)^m}, \quad g(r) = \frac{\varphi^m (r) (1-d)^m}{2k_{11} + \varphi^m (r) (1-d)^m},$$

$$K(r) = \varphi^m (r) (1-d)^m \left(\frac{k_{10}}{k_{11} + \varphi^m (r) (1-d)^m} + \frac{k_{12}}{2 (k_{11} + \varphi^m (r) (1-d)^m)^2} \right),$$

$$\varphi(r) = r\widetilde{\lambda}_p (r) < c_1 < \infty.$$

The estimations $0 < l(r) < k_{12} k_{11}^{-1}, K(r) < c(k_{10}, k_{11}, k_{12}) g(r)$, $2^{-1} (k_{11} + c_1)^{-1} \varphi^m (r) (1-d)^m < g(r) < 2^{-1}$ are true. On the basis of these estimations, from inequality (29) and Lemma 3 we get estimation (26).

The theorem is proved.

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