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MEAN OSCILLATION, Φ-OSCILLATION AND HARMONIC OSCILLATION

Abstract

In the paper, the notion of Φ-oscillation is introduced and its relations with mean and harmonic oscillations are studied. Bilateral estimations connecting the indicated quantities are obtained.

1. Some estimations in the terms of mean oscillation

Let R^n denote an $n-$dimensional Euclidean space of the points $x=(x_1,x_2,...,x_n)$, where $x_1,x_2,...,x_n\in R$; $B(a,r):=\{x\in R^n\colon |x-a|\le r\}$ be a closed ball in R^n of radius $r>0$ centered at the point $a\in R^n$; N be a set of all natural numbers, P_k be a totality of all polynomials in R^n of at most k degree. By $L^p_{loc}(R^n)$ $(1\le p<\infty)$ we denote a class of all locally summable functions of p degree, and by $L^p_{loc}(R^n)$ a class of all locally bounded functions determined in R^n.

Let $x=(x_1,x_2,...,x_n)\in R^n$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, $x^\nu=x_1^{\nu_1}\cdot x_2^{\nu_2}...x_n^{\nu_n}$, $|\nu|=\nu_1+\nu_2+...+\nu_n$, $\nu_i(i=1,2,...,n)$ be entire non-negative numbers. Apply the orthogonalization process with respect to the scalar product

$$(f,g)\colon=|B(0,1)|^{-1}\cdot\int\limits_{B(0,1)}f(t)g(t)dt$$

to the system of power functions $\{x^\nu\}$, $|\nu|\le k$ located in partially lexicographic order (see [4]), where $|B(a,r)|$ denotes the volume of the ball $B(a,r)$, $k\in$ $\in N\cup\{0\}$. We denote the result of the orthogonalization process by $\{\varphi_\nu\}$, $|\nu|\le$ $\le k$. The system $\{\varphi_\nu\}$, $|\nu|\le k$ is orthogonal and normalized.

For the function $f\in L^1_{loc}(R^n)$ we put [2], [3]

$$P_{k,B(a,r)}f(x)\colon=\sum_{|\nu|\le k}\left(\frac{1}{|B(a,r)|}\int\limits_{B(a,r)}f(t)\varphi_\nu\left(\frac{t-a}{r}\right)dt\right)\varphi_\nu\left(\frac{x-a}{r}\right).$$

It is seen from definition that $P_{k,B(a,r)}f$ is a polynomial of at most k degree.

Let $f\in L^p_{loc}(R^n)$ $(1\le p\le\infty)$, $k\in N$. Introduce the following denotation

$$\mu^k_f(x;r)_P\colon=\inf_{\pi\in P_{k-1}}\|f-\pi\|_{L^p(B(x,r))},\quad r>0,\quad x\in R^n,$$

$$O_k\left(f,B(x,r)\right)_p\colon=\left\|f-P_{k-1,B(x,r)}f\right\|_{L^P(B(x,r))},\quad r>0,\quad x\in R^n.$$

It is known that [5]

$$\exists c>0\quad\forall x\in R^n\quad\forall r>0\colon\quad \mu^k_f(x;r)_p\le O_k(f,B(x,r))_p\le c\cdot\mu^k_f(x;r)_p.\qquad(1.1)$$

This relation may be written in the following form as well:[1]

$$\mu^k_f(x;r)_p\approx O_k(f,B(x,r))_p\qquad(r>0,\quad x\in R^n).$$

[1]If the functions f and g are determined on the set $X\subset R^m$, the conditions $f(x)=O(g(x))$, $(x\in X)$ and $g(x)=O(f(x))$ $(x\in X)$ are fulfilled, this is written as: $f(x)\approx g(x)$ $(x\in X)$.

For the function $f \in L^p_{loc}(R^n) \quad (1 \le p \le \infty)$ we denote

$$\Omega_k(f, B(x,r))_p := \left(\frac{1}{|B(a,r)|} \int\limits_{B(a,r)} \left| f(t) - P_{k-1,B(a,r)} f(t) \right|^P dt \right)^{1/p}, \quad 1 \le p < \infty,$$

$$\Omega_k(f, B(a,r))_\infty := ess \sup \left\{ \left| f(t) - P_{k-1,B(a,r)} f(t) \right| : t \in B(a,r) \right\}.$$

It is easy to see that

$$O_k(f, B(x,r))_p = |B(x,r)|^{1/p} \cdot \Omega_k(f, B(x,r))_p, \quad (x \in R^n, \quad r > 0). \qquad (1.2)$$

By the proposal 1.2 from [7], the function $\mu^k_f(x;r)_p$ monotonically increases with respect to the argument r. Taking this fact into account, from relation (1.1) we get:

$$\mu^k_f(x;\delta)_p \approx \sup \left\{ O_k(f, B(x,r))_p : \ r \le \delta \right\} \quad (x \in R^n, \delta > 0). \qquad (1.3)$$

Applying the Holder inequality, we can prove the following statement:

Lemma 1.1. *Let $f \in L^q_{loc}(R^n) \quad (1 \le p < q \le \infty)$. Then the inequality*

$$O_k(f, B(x,r))_p \le |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \cdot O_k(f, B(x,r))_q, \ (x \in R^n; r > 0). \qquad (1.4)$$

is true.

In the sequel, we'll use the following denotation

$$m^k_f(x;\delta)_p := sup \left\{ \Omega_k(f, B(x,r))_p : \ 0 < r \le \delta \right\} \quad (x \in R^n, \delta > 0),$$

$$M^k_f(\delta)_p := \sup \left\{ m^k_f(x;\delta)_p : \ x \in R^n \right\} \quad (\delta > 0), \quad 1 \le p \le \infty, \quad k \in N.$$

We can write inequality (1.4) in the form:

$$|B(x,r)|^{-\frac{1}{p}} \cdot O_k(f, B(x,r))_p \le |B(x,r)|^{-\frac{1}{q}} \cdot O_k(f, B(x,r))_q \quad (x \in R^n, r > 0).$$

Hence, by relation (1.2) we get that if $f \in L^q_{loc}(R^n) \quad (1 \le p \le q \le \infty), \ x \in$ $\in R^n$ and $r > 0$, the inequality

$$\Omega_k(f, B(x,r))_p \le \Omega_k(f, B(x,r))_q.$$

is true.

Passing to supremum, hence we get

$$m^k_f(x;\delta)_p \le m^k_f(x;\delta)_q \quad (x \in R^n, \delta > 0). \qquad (1.5)$$

Theorem 1.1. *Let $f \in L^1_{loc}(R^n), \ \alpha > 0, \ \ k \in N, \ \ k < \alpha + 1, \ \ x_0 \in R^n$. Then the inequality*

$$\int\limits_{R^n} \frac{\left| f(x) - P_{k-1,B(x_0,r)} f(x) \right|}{r^{n+\alpha} + |x - x_0|^{n+\alpha}} dx \le c \cdot \int\limits_r^\infty \frac{\mu^k_f(x_0;t)_1}{t^{n+\alpha+1}} dt, \qquad (1.6)$$

is true for any $r > 0$, where $c > 0$ is independent of f, x_0 and r.

Proof. Applying elementary trans formations, we get

$$A{:}=\int\limits_{R^n}\frac{\left|f(x)-P_{k-1,B(x_0,r)}f(x)\right|}{r^{n+\alpha}+|x-x_0|^{n+\alpha}}dx=r^{-\alpha}\int\limits_{R^n}\frac{\left|f(x)-P_{k-1,B(x_0,r)}f(x)\right|}{1+\left(\frac{|x-x_0|}{r}\right)^{n+\alpha}}\cdot\frac{dx}{r^n}.$$

Having made change of variables $x-x_0=rt$, we get

$$A=r^{-\alpha}\int\limits_{R^n}\frac{\left|f(x_0+rt)-P_{k-1,B(x_0,r)}f(x_0+rt)\right|}{1+|t|^{n+\alpha}}dt=r^{-\alpha}\int\limits_{R^n}\frac{|g(t)|}{1+|t|^{n+\alpha}}dt, \tag{1.7}$$

where $g(t)=g_r(t):=f(x_0+rt)-P_{k-1,B(x_0,r)}f(x_0+rt)$ is denoted.

Applying theorem1 from [6] to the last integral of equality (1.7), we have

$$A\le c\cdot r^{-\alpha}\left(\int\limits_{B(0,1)}|g(t)|\,dt+\int\limits_1^\infty\frac{\mu_g^k(0;t)_1}{t^{n+\alpha+1}}dt\right), \tag{1.8}$$

where the constant $c>0$ is independent of g.

Further, by means of change of variables by the formula $x_0+rt=y$ we get

$$\begin{aligned}\int\limits_{B(0,1)}|g(t)|\,dt&=\int\limits_{B(0,1)}\left|f(x_0+rt)-P_{k-1,B(x_0,r)}f(x_0+rt)\right|dt=\\&=r^{-n}\int\limits_{B(x_0,r)}\left|f(y)-P_{k-1,B(x_0,r)}f(y)\right|dy=\\&=c_0\cdot\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\left|f(y)-P_{k-1,B(x_0,r)}f(y)\right|dy=\\&=c_0\cdot\Omega_k(f,B(x_0,r))_1=c_0\,|B(x_0,r)|^{-1}\cdot O_k(f,B(x_0,r))_1,\end{aligned} \tag{1.9}$$

where $c_0=|B(0,1)|$ is the volume of a unit ball in R^n. In the last transition we used equality (1.2).

Before we estimate the quantity $\mu_g^k(0,t)_1$, we establish some auxiliary relations. It is easy to see that if $y_0\in R^n$ is an arbitrary point, $B(a,r)$ is an arbitrary ball, $f_{y_0}(t){:}=f(y_0+t)$, then

$$P_{k-1,B(a.r)}(f_{y_0})(x)=P_{k-1,B(a+y_0,r)}f(x+y_0). \tag{1.10}$$

Further, by means of equality (1.10) we get (for $1\le p<\infty$)

$$\begin{aligned}O_k(f_{y_0},B(a,r))_p&=\left(\int\limits_{B(a,r)}\left|f_{y_0}(x)-P_{k-1,B(a,r)}f_{y_0}(x)\right|^p dx\right)^{\frac1p}=\\&=\left(\int\limits_{B(a,r)}\left|f(y_0+x)-P_{k-1,B(a+y_0,r)}f(y_0+x)\right|^p dx\right)^{\frac1p}=\\&=\left(\int\limits_{B(a+y_0,r)}\left|f(t)-P_{k-1,B(a+y_0,r)}f(t)\right|^p dt\right)^{\frac1p}=O_k(f,B(a+y_0,r))_p.\end{aligned} \tag{1.11}$$

Arguments for the case $p=\infty$ are similar.

In particular, it follows from equality (1.11) that the relation

$$\mu^k_{f_{y_0}}(a,r)_p \approx \mu^k_f(a+y_0;r)_p \quad (r>0).$$

is true.

Let $(\tau_\delta f)(x)=f(\delta x),\ \ x\in R^n,\ \ \delta>0$. Then we have

$$P_{k-1,B(0,r)}(\tau_\delta f)(x)=\sum_{|\nu|\le k-1}\left(\frac{1}{|B(0,r)|}\int\limits_{B(0,\delta r)} f(t)\varphi_\nu\left(\frac{t}{\delta r}\right)\frac{dt}{\delta^n}\right)\varphi_\nu\left(\frac{\delta x}{\delta r}\right)=$$

$$=\sum_{|\nu|\le k-1}\left(\frac{1}{|B(0,\delta r)|}\int\limits_{B(0,\delta r)} f(t)\varphi_\nu\left(\frac{t}{\delta r}\right)dt\right)\varphi_\nu\left(\frac{\delta x}{\delta r}\right)=P_{k-1,B(0.\delta r)}f(\delta x).$$

Hence, applying the rule of change of variables, we get

$$O_k(\tau_\delta f,B(0,r))_p=\left(\int\limits_{B(0,r)}\left|f(\delta x)-P_{k-1,B(0,r)}(\tau_\delta f)(x)\right|^p dx\right)^{\frac{1}{p}}=$$

$$=\left(\int\limits_{B(0,r)}\left|f(\delta x)-P_{k-1,B(0,\delta r)}f(\delta x)\right|^p dx\right)^{\frac{1}{p}}= \qquad (1.12)$$

$$=\frac{1}{\delta^{n/p}}\left(\int\limits_{B(0,\delta r)}\left|f(t)-P_{k-1,B(0,\delta r)}f(t)\right|^p dt\right)^{\frac{1}{p}}=\frac{1}{\delta^{n/p}}O_k(f,B(0;\delta r))_p,$$

with appropriate modification in the case $p=\infty$. In particular, it follows from relation (1.12) that

$$\mu^k_{\tau_\delta f}(0,r)_p \approx \frac{1}{\delta^{n/p}}\mu^k_f(0;\delta r)_p \quad (r>0,\ \ \delta>0).$$

Let $h(t){:=}\,f(t)-P_{k-1,B(x_0.r)}f(t)$. Then $g(t)=h(x_0+rt)=h_{x_0}(rt)=(\tau_r(h_{x_0}))(t)$.

Therefore, considering equalities (1.11) and (1.12), we get

$$P_{k-1,B(0,t)}g(y)=P_{k-1,B(0,t)}\left(\tau_r(h_{x_0})\right)(y)=P_{k-1,B(0,rt)}(h_{x_0})(ry)=$$

$$=P_{k-1,B(x_0,rt)}h(x_0+ry)=\left(P_{k-1,B(x_0,rt)}h\right)(x_0+ry)=$$

$$=\left(P_{k-1,B(x_0,rt)}f-P_{k-1,B(x_0,r)}f\right)(x_0+ry)=$$

$$=P_{k-1,B(x_0,rt)}f(x_0+ry)-P_{k-1,B(x_0,r)}f(x_0+ry).$$

Hence it follows that

$$\left|g(y)-P_{k-1,B(0,t)}g(y)\right|=\left|f(x_0+ry)-P_{k-1,B(x_0,rt)}f(x_0+ry)\right|.$$

Therefore, for $1 \le p < \infty$

$$O_k(g, B(0,t))_p = \left(\int\limits_{B(0,t)} \left| f(x_0 + ry) - P_{k-1,B(x_0,rt)} f(x_0 + ry) \right|^p dy \right)^{\frac{1}{p}}.$$

After substitution of $x_0 + ry = u$, hence we get

$$O_k(g, B(0,t))_p = \left(\int\limits_{B(0,rt)} \left| f(u) - P_{k-1,B(x_0,rt)} f(u) \right|^p \frac{du}{r^n} \right)^{\frac{1}{p}} =$$

$$= \frac{1}{r^{n/p}} O_k(f, B(x_0, rt))_p.$$

The arguments for the case $p = \infty$ are similar.

Hence it follows that

$$\begin{array}{c} \mu_g^k(0,t)_p = \mu_{g_r}^k(0,t)_p \approx \frac{1}{r^{n/p}} O_k(f, B(x_0, rt))_p \approx \\ \approx \frac{1}{r^{n/p}} \cdot \mu_f^k(x_0; rt)_p, \quad 1 \le p \le \infty. \end{array} \tag{1.13}$$

Considering relations (1.9) and (1.13), from inequality (1.8) we get

$$A \le c \cdot r^{-\alpha} \left(c_0 \cdot |B(x_0, r|^{-1} O_k(f, B(x_0, r))_1 + c_1 \int\limits_1^\infty \frac{\mu_f^k(x_0; rt)_1}{t^{n+\alpha+1}} \frac{dt}{r^n} \right) \le$$

$$\le c_2 r^{-\alpha-n} \left(\mu_f^k(x_0, r)_1 + r^{n+\alpha} \int\limits_r^\infty \frac{\mu_f^k(x_0; x)_1}{x^{n+\alpha+1}} dx \right).$$

Thus,

$$A \le c_2 \left(r^{-\alpha-n} \mu_f^k(x_0, r)_1 + \int\limits_r^\infty \frac{\mu_f^k(x_0; x)_1}{x^{n+\alpha+1}} dx \right), \tag{1.14}$$

where c_2 is a positive constant independent of f, x_0 and r. It is known that the function $\mu_f^k(x_0, x)_1$ monotonically increases with respect to the argument x. Therefore, we have

$$\int\limits_r^\infty \frac{\mu_f^k(x_0; x)_1}{x^{n+\alpha+1}} dx \ge \mu_f^k(x_0; r)_1 \int\limits_r^\infty x^{-n-\alpha-1} dx = \frac{1}{n+\alpha} \frac{1}{r^{n+\alpha}} \mu_f^k(x_0, r)_1.$$

Taking this into account, from inequality (1.14) we get

$$A \le c_3 \int\limits_r^\infty \frac{\mu_f^k(x_0; x)_1}{x^{n+\alpha+1}} dx, \quad r \in (0, +\infty),$$

where c_3 is some positive constant independent of f, x_0 and r.

This is inequality (1.6). The theorem is proved.

Corollary 1.1. *Let* $f \in L^p_{loc}(R^n)$ $(1 \le p \le \infty)$, $\alpha > 0$, $k \in N$, $k < < \alpha + 1$, $x_0 \in R^n$. *Then the inequality*

$$r^{-n} \int_{R^n} \frac{1}{1 + \left(\frac{|x - x_0|}{r}\right)^{n+\alpha}} \left| f(x) - P_{k-1,B(x_0,r)} f(x) \right| dx \le \le c r^{\alpha} \int_r^{\infty} \frac{m_f^k(x_0;t)_p}{t^{\alpha+1}} dt, \tag{1.15}$$

holds for any $r > 0$, *where* $c > 0$ *is independent of* f, x_0 *and* r.

2. Φ-oscillation and mean oscillation

Let $\Phi(x)$ $(x \in R^n)$ be a function summable in R^n, such that $\Phi(x) \ge 0$ $(x \in \mathbb{R}^n)$ and

$$\int_{R^n} \Phi(x) dx = 1.$$

Introduce the following denotation.

$$\Phi_r(x) := r^{-n} \Phi\left(\frac{x}{r}\right) \qquad (r > 0, \quad x \in R^n);$$

$$\Omega_{k,\Phi}(f, B(x;r)) := \int_{R^n} \Phi_r(x - t) \left| f(t) - P_{k-1,B(x,r)} f(t) \right| dt,$$

where $f \in L^1_{loc}(R^n)$, $k \in N$.

$\Omega_{k,\Phi}(f, B(x;r))$ is said to be Φ-oscillation of k-th order of the function f in the ball $B(x,r)$.

Furthermore, let

$$h_f^{k,\Phi}(x;\delta) := sup \left\{ \Omega_{k,\Phi}(f, B(x;r)) : \quad 0 < r \le \delta \right\}, \quad \delta > 0, \quad x \in R^n,$$

$$H_f^{k,\Phi}(\delta) := \sup \left\{ h_f^{k,\Phi}(x;\delta) : \quad x \in R^n \right\}, \qquad \delta > 0.$$

It is obvious that the functions $h_f^{k,\Phi}(x;\delta)$ and $H_f^{k,\Phi}(\delta)$ monotonically increase with respect to the argument $\delta \in (0; +\infty)$

Let

$$\Phi(x) \equiv \Phi^{(\alpha)}(x) : \ = c(n;\alpha) \frac{1}{1 + |x|^{n+\alpha}}, \quad \alpha > 0,$$

where $c(n, \alpha)$ is a constant such that

$$\int_{R^n} \Phi^{(\alpha)}(x) dx = 1.$$

Introduce the following denotation

$$\Omega_{k,\alpha}(f, B(x;r)) := \Omega_{k,\Phi^{(\alpha)}}(f, B(x;r)),$$

$$h_f^{k,\alpha}(x;\delta) := h_f^{k,\Phi^{(\alpha)}}(x;\delta), \quad H_f^{k,\alpha}(\delta) : H_f^{k,\Phi^{(\alpha)}}(\delta).$$

If $\Phi(x) \equiv \dfrac{1}{|B(0,1)|} \cdot X_{B(0,1)}(x)$, where X_E is a characteristic function of the set $E \subset R^n$, then

$$\Phi_r(x-t) = r^{-n}\Phi\left(\frac{x-t}{r}\right) = \frac{1}{r^n\,|B(0,1)|}X_{B(0,1)}\left(\frac{x-t}{r}\right) =$$

$$= \frac{1}{|B(x,r)|}X_{B(x,r)}(t) = \begin{cases} \frac{1}{|B(x,r)|} & \text{for } \ t \in B(x,r) \\ 0 & \text{for } \ t \notin B(x,r). \end{cases}$$

Therefore, for this function $\Phi(x)$ we have

$$\Omega_{k,\Phi}(f, B(x;r)) = \int\limits_{R^n} \Phi_r(x-t)\left|f(t) - P_{k-1,B(x,r)}f(t)\right| dt =$$

$$= \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} \left|f(t) - P_{k-1,B(x,r)}f(t)\right| dt = \Omega_k(f, B(x,r))_1,$$

where $\Omega_k(f, B(x;r))_1$ is a k-th order mean oscillation of the function f in the ball $B(x,r)$ in the metric of the space L^1.

It is easy to see that $\Phi^{(1)}(x) \approx P(x), \quad x \in R^n$, where $P(x) := c_n \cdot \dfrac{1}{\left(1+|x|^2\right)^{\frac{n+1}{2}}}$ is a Poisson kernel for the case R^n; here $c_n = \Gamma\left(\dfrac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}$. Therefore the relation

$$\Omega_{k,P}(f, B(x;r)) \approx \Omega_{k,\Phi^{(1)}}(f, B(x;r)) \qquad (r > 0; \quad x \in R^n),$$

where the constant in " $\approx$ " relation is independent of $f \in L^1_{loc}(R^n)$, is fulfilled.

After the introduced denotation, we can write inequality (1.15) in the form:

$$\Omega_{k,\alpha}(f, B(x_0,r)) \le c \cdot r^\alpha \int\limits_r^\infty \frac{m_f^k(x_0;t)_p}{t^{\alpha+1}}dt, \quad r > 0. \tag{2.1}$$

We can show that if $\varphi(t)$ monotonically increases on the interval $(0,+\infty)$, accepts only non-negative values and $\alpha > 0$, the function

$$F(r) := r^\alpha \int\limits_r^\infty \frac{\varphi(t)}{t^{\alpha+1}}dt, \quad r > 0,$$

also monotonically increases on the interval $(0,+\infty)$.

Considering monotone increase of the function $F(r)$, from inequality (2.1) we get

$$h_f^{k,\alpha}(x_0;\delta) \le c \cdot \delta^\alpha \int\limits_\delta^\infty \frac{m_f^k(x_0;t)_p}{t^{\alpha+1}}dt, \quad \delta > 0.$$

Thus, we proved

Proposal 2.1. Let $f \in L^p_{loc}(R^n)$ $(1 \le p \le \infty)$, $\alpha > 0$, $k \in N$, $k < \alpha + 1$, $x_0 \in R^n$. Then, the inequality

$$h_f^{k,\alpha}(x_0;\delta) \le c \cdot \delta^\alpha \int_\delta^\infty \frac{m_f^k(x_0;t)_p}{t^{\alpha+1}} dt, \quad \delta > 0, \tag{2.2}$$

is true, where the positive constant c is independent of f, x_0 and δ.

Hence, passing to supremum with respect to $x_0 \in R^n$, we get

Proposal 2.2. Let $f \in L^p_{loc}(R^n)$ $(1 \le p \le \infty)$, $\alpha > 0$, $k \in N$, $k < \alpha + 1$. Then the inequality

$$H_f^{k,\alpha}(\delta) \le c \cdot \delta^\alpha \int_\delta^\infty \frac{M_f^k(t)_p}{t^{\alpha+1}} dt, \quad \delta > 0, \tag{2.3}$$

where $c > 0$ is independent of f, x_0 and δ, is true.

Remark. Further, we'll mainly consider inequalities (2.2) and (2.3) for $p = 1$, therefore the case $1 < p \le \infty$ easily follows from the case $p = 1$ by the inequalities $m_f^k(x;t)_1 \le m_f^k(x,t)_p$ and $M_f^k(t)_1 \le M_f^k(t)_p$, $t \in (0,+\infty)$ (see (1.5)). Furthermore, instead of $m_f^k(x;r)_1$ and $M_f^k(r)_1$ we'll oftenly write $m_f^k(x;r)$ and $M_f^k(r)$, respectively.

It is also true the following

Proposal 2.3. Let $f \in L^1_{loc}(R^n)$, $\alpha > 0$, $k \in N$. Then the inequalities

$$m_f^k(x;\delta) \le c \cdot h_f^{k,\alpha}(x,\delta) \quad (x \in R^n,\ \delta > 0), \tag{2.4}$$

$$M_f^k(\delta) \le c \cdot H_f^{k;\alpha}(\delta) \quad (\delta > 0), \tag{2.5}$$

where $c > 0$ is independent of f, x and δ, are valid.

3. Harmonic oscillation and its relation with mean oscillation

Let $P(x)$ be a Poisson kernel for R^n, $P_r(x) := r^{-n} P\left(\frac{x}{r}\right)$ $(r > 0)$ and let $f \in L^1_{loc}(R^n)$, $P_r f(x) := (P_r * f)(x) = \int_{R^n} f(t) P_r(x-t) dt$. The quantity

$$\int_{R^n} |f(t) - P_r f(x)|\, P_r(x-t) dt$$

is said to be harmonic oscillation of the function f (see [1]). We also introduce the following denotation:

$$h_f(x;\delta) := \sup_{0<r\le\delta} \int_{R^n} |f(t) - P_r f(x)|\, P_r(x-t) dt \quad (x \in R^n,\ \delta > 0),$$

$$H_f(\delta) := \sup\{h_f(x;\delta) : x \in R^n\}, \quad \delta > 0.$$

Lemma 3.1. *Let $f \in L^1_{loc}(R^n)$. Then the relation*

$$\int_{R^n} |f(t) - P_r f(x)|\, P_r(x-t)dt \approx \int_{R^n} \left|f(t) - f_{B(x,r)}\right| P_r(x-t)dt, \quad (x \in R^n, \;\; r > 0),$$

is true, where $f_B := \frac{1}{|B|}\int_B f(t)dt$ *and the constants in the relation " $\approx$ " depend only on dimension n.*

Notice that for $k = 1$, the polynomial $P_{k-1,B(x,r)}f(t)$ coincides with $f_{B(x,r)}$. Therefore lemma, 3.1 shows that

$$\int_{R^n} |f(t) - P_r f(x)|\, P_r(x-t)dt \approx \Omega_{1,p}(f, B(x,r)) \approx$$

$$\approx \Omega_{1,1}(f, B(x,r)), \quad (x \in R^n, \; r > 0).$$

Hence, it follows that

$$h_f(x;\delta) \approx h_f^{1,1}(x;\delta) \quad (x \in R^n, \; \delta > 0),$$

$$H_f(\delta) \approx H_f^{1,1}(\delta) \qquad (\delta > 0). \tag{3.1}$$

It should be noted that a variant of the characteristics $H_f(\delta)$ for periodic functions is met in [1].

Applying the Proposals 2.1, 2.2, 2.3 and considering relation (3.1), we get the following statement.

Theorem 3.1. *Let $f \in L^1_{loc}(R^n)$. Then the following inequalities*

$$m_f^1(x;\delta) \le c_1 \cdot h_f(x;\delta) \le c_2 \cdot \delta \int_\delta^\infty \frac{m_f^1(x;t)_p}{t^2} dt, \quad (x \in R^n, \; \delta > 0);$$

$$M_f^1(\delta) \le c_1 \cdot H_f(\delta) \le c_2 \cdot \delta \int_\delta^\infty \frac{M_f^1(t)}{t^2} dt, \quad (\delta > 0),$$

are true, where $c_1 > 0$ and $c_2 > 0$ are independent of f, x and δ.

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