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ON THE FIRST PASSAGE TIME OF ONE-SIDED NONLINEAR BOUNDARY BY THE TRAJECTORY OF MARKOV CHAIN

Abstract

In paper, a theorem on Kolmogorov's strong law for the first passage time of Markov chain for the nonlinear boundary dependent on growing parameter is proved.

1. Introduction. Let $X = \{X_n, n \geq 0\}$ be a realvalued inhomogeneous in time Markov chain with transition probability

$$P\left(X_{n+1} \in B / X_n = x\right) = P_n\left(x, B\right), \tag{1}$$

where $x \in R = (-\infty, \infty)$ and $B \in \beta(R)$ is σ algebra of Borel subsets R.

We consider a family of the first passage moments

$$\tau_a = \inf\left\{n : X_n > f_a(n)\right\} \tag{2}$$

of the Markov chain X for the nonlinear boundary $f_a(t)$, $t \geq 0$ dependent on some growing parameter $a > 0$, and $f_a(1) \uparrow \infty$ as $a \to \infty$. When the Markov chain $X_n = \sum_{k=1}^{n} \xi_k$ is generated by the sums of independent identical random variables, there are many results in references on asymptotic behavior of distribution of the first passage moment τ_a. Statement of the results in this direction are in the papers [1], [7], [10], [15] and in monographs [2], [3], [19]. These works are on the base of the theory of boundary problems for random walks.

The boundary problems for the Markov chain were studied in the recent papers [4], [5] in which the linear boundary value problems (i.e. when $f_a(t) \equiv a$) related with the first passage time τ_a are studied.

At present, there are many papers [6], [8], [9], [11], [12], [14], [26], [17] devoted to the boundary value problems of the Markov chain, i.e. to the problems related with achievement of the boundary by trajectory the Markov chain. These papers give ground to speak on the existence theory of boundary problems for the Markov chain. In the paper, Kolmogorov's strong law is proved for the first passage moment of τ_a of the form (2) of the Markov chain. In linear statement $f_a(t) \equiv a$ this problem was studied in [16]. Similar problems for the ordinary process of summation of independent identical random variables were studied in the papers [1], [3], [15] and [19].

2. Conditions and formulation of basic results. We'll assume that the function $f_a(t)$ is of the form $f_a(t) = ag(t)$, where the positive function $g(t)$ is continuous, concave and monotonically increases for $t \geq T$, ($T > 0$ is a large number) Furthermore, the function $g(t)$ regularly varies at the infinity with the index $0 \leq \beta < 1$, i. e. it has the form $g(t) = t^{\beta} L(t)$, where $L(t)$ is a slowly changing function at the infinity [18] $L(t) = const$, $L(t) = \ln t$, $L(t) = \ln\ln t$, $L(t) = arctgt$ are typical examples for the function $L(t)$.

For the Markov chain we'll assume that it is a chain with a drift asympyotically homogeneous in time and in space, i.e. the jump $\xi_n(x)$ of the chain from the state x

at moment n satisfies the condition: $E\xi_n(x)$ converges as $n, x \to \infty$ to some number $\mu \in R$, and at this time the existence of mathematical expectation $E\xi_n(x)$ is not supposed for all the values of n and x. In this case, they say that the Markov chain X has a mean drift μ asymptotically homogeneous in time and in space (see [13]). We'll suppose $\mu > 0$.

Notice that the distribution of the jump $\xi_n(x)$ of the chain X may be given by the equality (1), i.e.

$$P(x + \xi_n(x) \in B) = P_n(x, B).$$

By $N_a = N_a(\mu)$ we denote the solution of the equation $f_a(n) = n\mu$ that exists and unique for sufficiently large a by the made assumptions for the function $g(t)$ and $N_a \to \infty$ as $a \to \infty$. In particular, for the boundary $f_a(t) = at^\beta$ we get

$$N_a = N_a(\mu) = \left(\frac{a}{\mu}\right)^{\frac{1}{1-\beta}}, \quad 0 \le \beta < 1.$$

It holds

Theorem. *Let the above listed conditions for the boundary $f_a(t)$ be satisfied and the Markov chain X have a mean positive drift $m > 0$ asymptotically homogeneous in time and in space, moreover $X_n \overset{a.s.}{\to} \infty$ as $n \to \infty$. Assume that for some spatial level U and time M the family of random variables $\{|\xi_n(u)|, u \ge U, n \ge M\}$ possesses an integrable majorant, i.e. there exists a non-negative random variable ξ with $E\xi < \infty$ such that*

$$\sup_{\substack{u \ge U \\ n \ge M}} p(|\xi_n(u)| \ge x) \le P(\xi \ge x) \qquad (3)$$

for any $x \in R$. Then

$$\frac{\tau_a}{N_a} \overset{a.s.}{\to} 1 \text{ as } a \to \infty.$$

The following results follow from this theorem.

Corollary 1. *Let the conditions of the theorem be fulfilled. Then,*

$$\frac{X_{\tau_a}}{f_a(N_a)} \overset{a.s.}{\to} 1 \text{ or } \frac{X_{\tau_a}}{N_a} \overset{a.s.}{\to} \mu \quad \text{as} \quad a \to \infty.$$

Corollary 2. *Let the conditions of the theorem be fulfilled and $E\xi^r < \infty$ for some $r \ge 1$. Then,*

$$\frac{\xi_{\tau_a}^{(u)}}{(N_a)^{1/r}} \overset{a.s.}{\to} 0 \quad \text{as} \quad a \to \infty.$$

uniformly with respect to $u \ge U$.

Remark 1. The ordinary summation process $X_n = \xi_1 + ... + \xi_n$ of independent identical random variables ξ_k, $k \ge 1$ with a positive mean value $\mu = E\xi_1 > 0$ and the random walk $X_{n+1} = \max\left(0, X_n + \xi_{n+1}\right)$ with delay in zero [13] give the simplest examples for the Markov chains satisfying the conditions of theorem 1.

Remark 2. The theorem and its corollaries for an ordinary random walk are proved in the papers [1], [3].

Remark 3. The condition $X_n \overset{a.s.}{\to} \infty$ as $n \to \infty$ in the theorem is equivalent to irrevocability of the chain with denumerable set of states ([13], [20]). This condition shows that the theorem and its corollaries are very substantial for the irrevocable

Markov chains, though it is not obviously supposed in the theorem if the Markov chain X is positive revocable or zero revocable or irrevocable [20] (see also [13]).

3. Theorem's proof. For the proof we'll need the following additional statements.

Lemma 1. *When fulfilling the conditions of the theorem for the Markov chain X_n it holds*

$$\frac{X_n}{n} \stackrel{a.s.}{\to} \mu \quad as \quad n \to \infty.$$

The affirmation of this lemma is in theorem 2 from the paper [13] on strong law of large numbers for the Markov chain.

Lemma 2. *Let the function $f_a(t)$ be continuous and increase monotonically, moreover $f_a(1) \uparrow \infty$ as $a \to \infty$ and let this conditions of the theorem for the Markov chain X be fulfilled.*

Then

1) τ_a is a stop moment with respect to the flow of σ- algebra $F_n = \sigma(X_k, k \le n)$, $n \ge 1$;

2) $\tau_a \stackrel{a.s.}{\to} \infty$ as $n \to \infty$;

3) $\frac{g(\tau_a)}{\tau_a} \stackrel{a.s.}{\to} 0$ as $n \to \infty$.

Proof. Statement 1) follows from the following equality

$$\{\tau_a = n\} = \{X_0 \le f_a(0), ..., X_{n-1} \le f_a(n-1), X_n > f_a(n)\} \in F_n$$

for each $n \ge 0$.

Prove statement 2) Consider the first passage time

$$t_a = \inf\{n : X_n > f_a(1)\}.$$

It is clear that by the monotonicity of the boundary $f_a(t)$ it holds

$$\tau_a \ge t_a. \tag{4}$$

Show that $t_a \stackrel{a.s.}{\to} \infty$ as $n \to \infty$. Let $Z_n = \max\{X_0, X_1, ..., X_n\}$. We have

$$\{t_a > n\} = \{Z_n \le f_a(1)\}. \tag{5}$$

Taking into account $Z_n \stackrel{a.s.}{\to} \infty, n \to \infty$, from (5) we get $P\{t_a < \infty\} = 1$ for all $a > 0$. The variable t_a increases as the function a Therefore, $P\left(\lim_{a\to\infty} t_a = t_\infty \le \infty\right) = 1$, since $f_a(1) \uparrow \infty$ as $a \to \infty$. It follows from (5) that

$$P(t_\infty \le n) = \lim_{a\to\infty} P(Z_n > f_a(1)) = 0$$

for all $n \ge 0$ and consequently, $P(t_\infty = \infty) = 1$. Hence by (4) we get statement 2) of lemma 2.

For proving statement 3) it suffices to notice that $t^{-\varepsilon}L(t) \to 0$ as $t \to \infty$ for any $\varepsilon > 0$ [18].

Lemma 3. *Let $\xi_n, n \ge 1$ be a sequence of random variables and $\theta_a, a > 0$ be a familly if positive integer random variables such that $\theta_a \stackrel{a.s.}{\to} \infty$ as $n \to \infty$. If $\xi_n \stackrel{a.s.}{\to} \xi$ as $n \to \infty$, then $\xi_{\theta_a} \stackrel{a.s.}{\to} \xi$ as $n \to \infty$.*

Proof. Denote

$$A=\left\{\omega:\xi_n \underset{n\to\infty}{\to} \xi\right\},\quad B=\left\{\omega:\xi_{\theta_a} \underset{n\to\infty}{\to} \xi\right\} \text{ and } D=\left\{\omega:\theta_a \underset{n\to\infty}{\to} \infty\right\}.$$

It is easy to understand that $A\cap D\subset B$. By the condition $P(A)=P(D)=1$. Then it follows from the equality

$$P(A\cup D)=P(A)+P(D)-P(A\cap D)$$

that $P(A\cap D)=1$, since $P(A\cup D)=1$. Therefore $P(B)=1$.

Lemma 4. *Let the positive number functions $f(t)$ and $g(t)$be given such that $f(t)\to\infty$, $g(t)\to\infty$ as $t\to\infty$ and for some slowly changing function $L(x), x>0$ the relation for $t\neq 0$*

$$\left(\frac{f(t)}{g(t)}\right)^r \frac{L(f(t))}{L(g(t))}\to 1 \text{ as } t\to\infty$$

be fulfilled

Then,

$$\frac{f(t)}{g(t)}\to 1 \text{ as } t\to\infty.$$

The statement of this lemma follows from lemma 2.2 of the paper ([3], see. Supplemet B).

Lemma 5. *Let the Markov chain X satisfy the conditions of the theorem and θ_a, $a>0$ be a family of positive integer random variables such that $\theta_a \overset{a.s.}{\to} \infty$ as $t\to\infty$. Then,*

1) $\dfrac{X_{\theta_a}}{\theta_a} \overset{a.s.}{\to} \infty$ as $t\to\infty$

2) If $E\xi^r<\infty$ for some $r>0$, then $\dfrac{\xi_{\theta_a}(u)}{(\theta_a)^{1/r}} \overset{a.s.}{\to} 0$ as $t\to\infty$ uniformly with respect to $u\geq U$.

Proof. Statement 1) follows from lemmas 1 and 3. Prove statement 2). If follows from the condition $E\xi^r<\infty$ that

$$\sum_{n=1}^{\infty} P(\xi^r>n)<\infty \tag{6}$$

(5) yields

$$\sum_{n=1}^{\infty} P\left(\xi>\varepsilon n^{1/r}\right)<\infty \tag{7}$$

for all $\varepsilon>0$.

By (3), from (6) we get $\sum\limits_{n=1}^{\infty} P\left(|\xi_n(u)|>\varepsilon n^{1/r}\right)<\infty$ for all $u\geq U$.

Then by the theorem on almost sure convergence ([3]) we have

$$\frac{\xi_n(u)}{n^{1/r}} \overset{a.s.}{\to} \infty \text{ as } n\to\infty \tag{8}$$

uniformly with respect to $u\geq U$.

Statement 2) of the proved lemma we get from (7) and lemma 3.

Proof of the theorem. By the definition of the first passage moment τ_a we have

$$f_a(\tau_a) < X_{\tau_a} \quad \text{and} \quad X_{\tau_a - 1} \le f_a(\tau_a - 1).$$

By the monotonicity of $f_a(t)$, for these inequlities we get

$$f_a(\tau_a) < X_{\tau_a} \le f_a(\tau_a - 1) + X_{\tau_a} - X_{\tau_a - 1} \le f(\tau_a) + X_{\tau_a} - X_{\tau_a - 1}. \tag{9}$$

It follows from lemma 1 that

$$\frac{X_n - X_{n-1}}{\tau_a} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

By lemmas 1,2 and 3 we have

$$\frac{X_{\tau_a} - X_{\tau_a - 1}}{\tau_a} \overset{a.s.}{\to} 0 \quad \text{and} \quad \frac{X_{\tau_a}}{\tau_a} \overset{a.s.}{\to} \mu \quad \text{as} \quad n \to \infty.$$

Therefore, from (7) and statemenr 1 of lemma 5 we find

$$\frac{f_a(\tau_a)}{\tau_a} \overset{a.s.}{\to} \mu \quad \text{as} \quad n \to \infty.$$

In view of $f_a(N_a) = \mu N_a$, from (8) we get

$$\frac{N_a f_a(\tau_a)}{\tau_a f(N_a)} \overset{a.s.}{\to} 1 \quad \text{as} \quad n \to \infty \quad \text{or} \quad \left(\frac{\tau_a}{N_a}\right)^{\beta - 1} \frac{L(\tau_a)}{L(N_a)} \overset{a.s.}{\to} 1 \quad \text{as} \quad n \to \infty.$$

By lemma 4, from the last relation we complete the proof of theorem 1.

Proof of Corollary 1. We have

$$\frac{X_{\tau_a}}{f_a(N_a)} = \frac{X_{\tau_a}}{\mu N_a} = \frac{\tau_a}{N_a} \frac{X_{\tau_a}}{\mu \tau_a}.$$

Hence, applying theorem 1 and statement 2) of lemma 2 and statement 1) of lemma 5 we get the statement of Corollary 1.

To prove Corollary 2 it suffices to note the following equality

$$\frac{\xi_{\tau_a}^{(u)}}{(N_a)^{1/r}} = \frac{\xi_{\tau_a}^{(u)}}{(\tau_a)^{1/r}} \left(\frac{\tau_a}{N_a}\right)^{1/r}$$

and apply theorem 1) and statement 2) of lemma 5.

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