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# SOME PROPERTIES OF THE SOLUTIONS TO THE HYPERBOLIC EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY

**Abstract**

*In this article we established the behaviour of the global (in time) solution to the mixed problems for nonlinear hyperbolic equations of the second order if $t \to +\infty$. Also some nonexistence conditions of the global solution to the examined problems are received.*

Semilinear hyperbolic equation of the form

$$u_{tt} - \Delta u + \alpha u_t + \beta |u_t|^{p_1-2} u_t + \gamma |u|^{p_2-2} u = f(x,t), \tag{1}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$, and $p_1, p_2 \in (1, +\infty)$, was considered by many authors (see for instance [1-3]). Over the last years problems for nonlinear partial differential equations have being actively studied in some special classes of functions namely in generalized Lebesgue and Sobolev spaces [4], [5]. In paper [6], the mixed problem for a hyperbolic equation of the third order with the power nonlinearities in the main part is examined. Conditions are obtained providing the existence and uniqueness of the problems' solution in the generalized Sobolev spaces.

This article deals with properties of mixed problem's solutions for certain generalization of the equation (**??**) in which nonlinearity exponents depend on spacial variables. The behaviour of the global solution to such problems is researched if $t \to +\infty$. Besides under some conditions the nonexistence of the global solution is shown.

Let $n \in \mathbb{N}$, $T > 0$ be some fixed numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain of $C^{0,1}$ class with a boundary $\partial\Omega$, $Q_\tau = \Omega \times (0, \tau)$, $\Omega_\tau = Q_T \cap \{t = \tau\}$, $\tau \in [0, T]$, $Q_\infty = \Omega \times (0, +\infty)$, $p_1, p_2 \in L^\infty(\Omega)$.

Consider the following problem

$$u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} + c_1(x) u_t + c_2(x) u + g_1(x) |u_t|^{p_1(x)-2} u_t +$$

$$+ g_2(x) |u|^{p_2(x)-2} u = f(x,t), \quad (x,t) \in Q_T, \tag{2}$$

$$u|_{\partial\Omega \times (0,T)} = 0, \tag{3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{4}$$

Let $p \in L^\infty(\Omega)$, $p_0 = \operatorname{ess\,inf}_{x \in \Omega} p(x)$, $p^0 = \operatorname{ess\,sup}_{x \in \Omega} p(x)$, $1 < p_0 \leq p^0 < +\infty$, and $1/p(x) + 1/p'(x) = 1$, $x \in \Omega$. By definition, put $\rho_p(v, \Omega) = \int_\Omega |v(x)|^{p(x)}\, dx$. A $L^{p(x)}(\Omega)$ is called a generalized Lebesgue space if for all $v \in L^{p(x)}(\Omega)$ the following conditions hold: (*i*) $v : \Omega \to \mathbb{R}^1$ is a measurable function; (*ii*) $\rho_p(v, \Omega) < +\infty$. From our conditions $1 < p_0 \leq p^0 < +\infty$ it follows that $L^{p(x)}(\Omega)$ is a reflexive space (see [4], p. 600). $L^{p(x)}(\Omega)$ is a Banach space with respect to the norm

$$||v; L^{p(x)}(\Omega)|| = \inf \left\{ \lambda > 0 : \ \rho_p(v/\lambda, \Omega) \leq 1 \right\},$$

$[L^{p(x)}(\Omega)]^* = L^{p'(x)}(\Omega)$, and $L^{r(x)}(\Omega) \subset L^{p(x)}(\Omega)$ if $r(x) \geq p(x)$ (see [4], p. 599-600).

We assume that the following conditions hold:

**(A)**: $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$, $i,j = \overline{1,n}$,

$a_0|\xi|^2 \leq \sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq a^0|\xi|^2$ for every $\xi \in \mathbb{R}^n$ and for a.e. $x \in \Omega$, where $a_0, a^0 > 0$;

**(C)**: $c_1, c_2 \in L^\infty(\Omega)$, $c_{i,0} \leq c_i(x) \leq c_i^0$ for a.e. $x \in \Omega$, where $c_{i,0}, c_i^0 \in \mathbb{R}^1$, $i = 1,2$;

**(G)**: $g_1, g_2 \in L^\infty(\Omega)$, $g_{i,0} \leq g_i(x) \leq g_i^0$ for a.e. $x \in \Omega$, where $g_{i,0}, g_i^0 \in \mathbb{R}^1$, $i = 1,2$;

**(F)**: $f : Q_\infty \to \mathbb{R}^1$ and for every $T > 0$ we have $f, f_t \in L^2(Q_T)$;

**(P)**: $p_1, p_2 \in L^\infty(\Omega)$, $1 < p_{i,0} \leq p_i^0 < +\infty$, where $p_{i,0} = \operatorname{ess\,inf}\limits_{x\in\Omega} p_i(x)$, $p_i^0 = \operatorname{ess\,sup}\limits_{x\in\Omega} p_i(x)$, $i = 1,2$;

**(U)**: $u_0 \in H_0^1(\Omega) \cap L^{p_2(x)}(\Omega)$, $u_1 \in H_0^1(\Omega) \cap L^{p_1(x)}(\Omega)$.

Using Galerkin's method we may prove next theorem.

**Theorem 1.** *Suppose that conditions* ***(A),(C),(G),(F),(P),(U),*** $g_{1,0} > 0$, $g_0^2 > 0$, *and* $2 < p_{2,0} \leq p_2^0 \leq \dfrac{2n-2}{n-2}$, *if* $n \geq 3$, *are satisfied. If, additionally,* $a_{ij,x_k} \in L^\infty(\Omega)$ $(i,j,k = \overline{1,n})$, $u_0 \in H^2(\Omega) \cap L^{2(p_2(x)-1)}(\Omega)$, $u_1 \in L^{2(p_1(x)-1)}(\Omega)$, *then problem* (**??**)-(**??**) *has a global solution* $u$ *such that for every* $T > 0$ *we have* $u \in L^\infty(0,T; H_0^1(\Omega) \cap L^{p_2(x)}(\Omega))$, $u_t \in L^\infty(0,T; H_0^1(\Omega) \cap L^{p_1(x)}(\Omega))$, $u_{tt} \in L^2(Q_T)$.

Now we consider the behaviour of the solution to problem (**??**)-(**??**). Let us introduce a function

$$E(t) = \frac{1}{2}\int\limits_\Omega \left[|u_t(x,t)|^2 + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}(x,t)u_{x_j}(x,t) + c_2(x)|u(x,t)|^2\right] dx +$$

$$+\int\limits_\Omega \frac{g_2(x)}{p(x)}\,|u(x,t)|^{p_2(x)}\,dx, \quad t \geq 0,$$

where $u$ is a global solution to problem (**??**)-(**??**).

First let us study a behaviour of the global solution $u$ to problem (**??**)-(**??**) if $t \to +\infty$. Let $p_1 = p_2 = p$. By definition, put

$$W = \frac{p^0 g_{1,0}}{g_1^0(p^0-1)}, \quad H = \frac{c_1^0+2}{2c_{1,0}}, \quad L = \min\left\{1,\ W,\ 1/H,\ a_0/\gamma\right\}, \tag{5}$$

where $\gamma = \gamma(\Omega) > 0$ is a constant from the Friedrichs inequality, namely,

$$\int\limits_\Omega |v|^2\,dx \leq \gamma \int\limits_\Omega \sum_{i=1}^{n} |v_{x_i}|^2\,dx \quad \forall\, v \in H_0^1(\Omega). \tag{6}$$

We will say that the condition **(V)** holds, if

**(V)**: $K_1 \equiv 1 - \dfrac{c_1^0\gamma}{a_0} > 0$, $K_2 \equiv p_0 - \dfrac{g_1^0}{g_{2,0}} > 0$.

The following theorem is established.

**Theorem 2.** *Suppose that conditions* ***(A), (C), (G), (U), (V)*** *are satisfied,* $c_2 \equiv 0$, $c_{1,0}, c_1^0 > 0$, *and* $g_{i,0}, g_i^0 > 0$, *where* $i = 1,2$. *If* $p_1 = p_2 = p$, *and* $u$ *is the* *global solution to problem (**??**)-(**??**), then there exist constants* $C > 0$, $\omega > 0$ *such that*

$$E(t) \le CE(0)e^{-\omega t}, \quad t > 0. \tag{7}$$

**Proof.** Let $u$ be the global solution to problem (**??**)-(**??**). Then for all $v \in H_0^1(\Omega) \cap L^{p(x)}(\Omega)$ we get

$$\int\limits_{\Omega_t} \left[ u_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + c_1(x) u_t v + \right.$$

$$\left. + g_1(x)|u_t|^{p(x)-2} u_t v + g_2(x)|u|^{p(x)-2} uv \right] dx = 0. \tag{8}$$

Using equality (**??**) with $v = u_t$, we obtain

$$\frac{dE}{dt} = \int\limits_{\Omega_t} \left[ u_{tt}u_t + \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j t} + g_2|u|^{p(x)-2} uu_t \right] dx = -\int\limits_{\Omega_t} \left[ c_1|u_t|^2 + g_1|u_t|^{p(x)} \right] dx. \tag{9}$$

Let us consider a function

$$J(t) = E(t) + \varepsilon \int\limits_{\Omega_t} uu_t \, dx, \quad \varepsilon > 0, \quad t \ge 0. \tag{10}$$

From (**??**), (**??**) and (**??**) with $v = u$, we have

$$\frac{dJ}{dt} = \frac{dE}{dt} + \varepsilon \int\limits_{\Omega_t} |u_t|^2 \, dx + \varepsilon \int\limits_{\Omega_t} u_{tt} u \, dx - \int\limits_{\Omega_t} \left[ c_1|u_t|^2 + g_1|u_t|^{p(x)} \right] dx +$$

$$+ \varepsilon \int\limits_{\Omega_t} |u_t|^2 \, dx - \varepsilon \int\limits_{\Omega_t} \left[ \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} + c_1 u_t u + g_1 |u_t|^{p(x)-2} u_t u + g_2 |u|^{p(x)} \right] dx. \tag{11}$$

Taking into account conditions **(A)**, **(C)**, **(G)** and using some transformations, we obtain

$$\frac{dJ}{dt} \le -\frac{K_1 \varepsilon}{2} \int\limits_{\Omega_t} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \, dx - c_{1,0} \left(1 - H\varepsilon\right) \int\limits_{\Omega_t} |u_t|^2 \, dx -$$

$$- \left(1 - \frac{\varepsilon}{W}\right) \int\limits_{\Omega_t} g_1 |u_t|^{p(x)} \, dx - K_2 \varepsilon \int\limits_{\Omega_t} \frac{g_2}{p} |u|^{p(x)} \, dx \tag{12}$$

(see (**??**) and **(V)**). In view of **(V)**, and choosing $\varepsilon \in \big(0, \min\{W, 1/H\}\big)$, from (**??**) we get the inequality

$$\frac{dJ}{dt} \le -K_3(\varepsilon) E(t), \tag{13}$$

where $K_3(\varepsilon) > 0$.

Now we will obtain the additional estimates. From (**??**) and (**??**) we find

$$J(t) \le E(t) + \frac{\varepsilon}{2} \int\limits_{\Omega_t} \left[ |u|^2 + |u_t|^2 \right] dx \le$$

$$\leq E(t) + \frac{\varepsilon}{2} \int\limits_{\Omega_t} |u_t|^2 \, dx + \frac{\varepsilon\gamma}{2a_0} \int\limits_{\Omega_t} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \, dx \leq$$

$$\leq \int\limits_{\Omega_t} \left[ \frac{1}{2} \left( 1 + \frac{\varepsilon\gamma}{a_0} \right) \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} + \frac{1}{2}(1+\varepsilon)|u_t|^2 + \frac{1}{p} \, g_2 |u|^{p(x)} \right] dx \leq A(\varepsilon) E(t), \quad (14)$$

where $A(\varepsilon) = \max\{1 + \varepsilon\gamma/a_0, 1 + \varepsilon, 1\}$.

In a similar way for $0 < \varepsilon < \min\{1, a_0/\gamma\}$ we obtain the estimate

$$J(t) \geq \int\limits_{\Omega_t} \left[ \frac{1}{2} \left( 1 - \frac{\varepsilon\gamma}{a_0} \right) \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} + \frac{1}{2}(1 - \varepsilon)|u_t|^2 + \frac{1}{p} \, g_2 |u|^{p(x)} \right] dx \geq N(\varepsilon) E(t),$$

where $N(\varepsilon) = \min\{1 - \varepsilon\gamma/a_0, 1 - \varepsilon, 1\}$.

Therefore if $\varepsilon \in (0, L)$ (see (**??**)), then taking into account (**??**), (**??**), we get

$$\frac{dJ}{dt} \leq -\frac{K_3(\varepsilon)}{A(\varepsilon)} J(t).$$

Hence

$$J(t) \leq J(0) \, e^{-\omega t}, \quad \text{where} \quad \omega = \frac{K_3(\varepsilon)}{A(\varepsilon)}.$$

Since $J(0) \leq A(\varepsilon)E(0)$ and $J(t) \geq N(\varepsilon)E(t)$, we see that

$$E(t) \leq \frac{J(t)}{N(\varepsilon)} \leq \frac{J(0)}{N(\varepsilon)} \, e^{-\omega t} \leq \frac{A(\varepsilon)}{N(\varepsilon)} E(0) \, e^{-\omega t}.$$

So this implies that (**??**) holds for $C = \dfrac{A(\varepsilon)}{N(\varepsilon)}$. This completes the proof of Theorem 2.

Now let us consider the case $g_2 < 0$. Assume that $p_1 = \text{const}$, $p_2 = \text{const}$, and use such denotation as

$$||v||_r = ||v||_{L^r(\Omega)} = \left( \int\limits_{\Omega} |v(x)|^r \, dx \right)^{1/r}, \quad r > 1.$$

To prove the following result we need the auxiliary lemma.

**Lemma ([2]).** *Let $c_{2,0}, c_2^0 \geq 0$, $q \geq 2$ and, additionally, $q \leq \dfrac{2n}{n-2}$ if $n \geq 3$. Then there exists constant $M = M(\Omega) > 0$ such that for all $v \in H_0^1(\Omega)$ and for all $s \in [2, q]$ the next inequality holds:*

$$||v||_q^s \leq M \int\limits_{\Omega_t} \Big[ \sum_{i,j=1}^{n} a_{ij} v_{x_i} v_{x_j} + c_2 |v|^2 + |v|^q \Big] \, dx.$$

**Theorem 3.** *Suppose that the conditions **(A)**, **(C)**, **(G)** hold, and $2 < p_1 < p_2$ if $n = 1$ or $n = 2$, $2 < p_1 < p_2 \leq \dfrac{2n}{n-2}$ if $n \geq 3$, $c_{1,0} \geq 0$, $c_{2,0} \geq 0$, $g_{1,0} > 0$,*

$g_{2,0} < 0$, $u_1 \in H_0^1(\Omega)$, $u_0 \in L^2(\Omega)$, $f \equiv 0$, $E(0) < 0$; *then problem (??)-(??) can not have a global solution, namely, there exist the points* $T_0 > 0$ *and* $\varepsilon > 0$ *such that*

$$\lim_{t\to T_0-0}\left[||u(t);L^{p_2}(\Omega)||^{1+\frac{p_2}{2}}+\varepsilon\int_\Omega u_t(x,t)u(x,t)\,dx\right]=+\infty. \tag{15}$$

**Proof.** Assume that theorem's conditions hold. By definition, put $\lambda = -E(0)$, $H(t) = -E(t)$, where $t \geq 0$. Then $\lambda > 0$, $E'(t) \leq 0$ (see (**??**)),

$$H'(t)=\int_{\Omega_t}\left[c_1|u_t|^2+g_1|u_t|^{p_1}\right]dx\geq 0,\quad t\geq 0. \tag{16}$$

Let

$$L(t)=H^{1-\alpha}(t)+\varepsilon\int_{\Omega_t}u_tu\,dx,\quad \alpha\in(0,1),\quad \varepsilon>0,\quad t\geq 0.$$

Hence

$$L'(t)=(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\int_{\Omega_t}\left[|u_t|^2+u_{tt}u\right]dx.$$

From (**??**) it follows that

$$L'(t)=(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\int_{\Omega_t}|u_t|^2dx-$$

$$-\varepsilon\int_{\Omega_t}\left[\sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}+c_1u_tu+c_2|u|^2+g_1|u_t|^{p_1-2}u_tu+g_2|u|^{p_2}\right]dx.$$

Taking into account (**??**), lemma's statement, and theorem's conditions, we will obtain the following estimate

$$L'(t)\geq\Big(1-\alpha-\frac{\varepsilon}{2\delta_1}\Big)H^{-\alpha}(t)\int_{\Omega_t}c_1|u_t|^2\,dx+\Big(1-\alpha-\frac{\varepsilon}{\delta_2}\Big)H^{-\alpha}(t)\int_{\Omega_t}g_1|u_t|^{p_1}\,dx+$$

$$+\varepsilon\int_{\Omega_t}|u_t|^2\,dx-\varepsilon\Big(1+\delta_1K_4+\delta_2^{p_1-1}K_5\Big)\int_{\Omega_t}\Big[\sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}+c_2|u|^2\Big]dx+$$

$$+\,\varepsilon\Big(\frac{\delta_1K_4+\delta_2^{p_1-1}K_5}{g_{2,0}}-1\Big)\int_{\Omega_t}g_2|u|^{p_2}dx, \tag{17}$$

where $\alpha\in\left(0,\dfrac{p_2-p_1}{p_2(p_1-1)}\right)$, $\delta_1,\delta_2>0$, $\varepsilon,K_4,K_5>0$. If $\delta_1,\delta_2>0$ are sufficiently small, then using the equality

$$\int_{\Omega_t}\Big[\sum_{i,j=1}^n a_{ij}u_{x_i}u_{x_j}+c_2|u|^2\Big]dx=-2H(t)-\int_{\Omega_t}|u_t|^2\,dx-\frac{2}{p_2}\int_{\Omega_t}g_2|u|^{p_2}\,dx,$$

from (**??**) we get

$$L'(t) \geq C_1(\varepsilon)\Big[H(t) + ||u||_{p_2}^{p_2} + ||u_t||_2^2\Big], \quad t \geq 0, \tag{18}$$

where $\varepsilon = \varepsilon(\delta) > 0$ is sufficiently small, $C_1(\varepsilon) > 0$ is some constant. It is easy to see that

$$\Big[L(t)\Big]^{\frac{1}{1-\alpha}} \leq C_2(\varepsilon)\Big[H(t) + ||u_t||_2^2 + ||u||_{p_2}^{\frac{2}{1-2\alpha}}\Big], \quad t \geq 0, \tag{19}$$

where $C_2(\varepsilon) > 0$.

Setting $\alpha = \dfrac{p_2-2}{2p_2}$ $\Big($notice that $\dfrac{p_2-2}{2p_2} < \dfrac{p_2-p_1}{p_2(p_1-1)}$, $\dfrac{1}{1-\alpha} = 2 - \dfrac{4}{p_2+2}$, and $\dfrac{2}{1-2\alpha} = p_2\Big)$, by (**??**), (**??**) we obtain the inequality

$$L'(t) \geq C_3(\varepsilon)\Big[L(t)\Big]^{2-\frac{4}{p_2+2}}, \quad t \geq 0, \tag{20}$$

where $C_3(\varepsilon) > 0$.

Note that if $\varepsilon > 0$ is sufficiently small, then $L(0) = \lambda^{1-\alpha} + \varepsilon \int\limits_\Omega u_0 u_1\, dx \geq \dfrac{\lambda^{1-\alpha}}{2}$. Therefore (**??**) implies that for all $\tau \in (0, T_0)$ the following estimate holds

$$L(\tau) \geq \Big[\frac{p_2+2}{C_3(p_2-2)(T_0-\tau)}\Big]^{\frac{p_2+2}{p_2-2}},$$

where $T_0 = \dfrac{2^{\frac{p_2-2}{p_2+2}}(p_2+2)}{\lambda^{\frac{p_2-2}{2p_2}}C_3(p_2-2)}$.

It is obvious that $L(\tau) \to +\infty$ if $\tau \to T_0 - 0$ and (**??**) holds which concludes the proof of Theorem 3.

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