Yagub Y. MAMMADOV

ON THE BOUNDEDNESS OF THE MAXIMAL OPERATOR IN MORREY SPACES ASSOCIATED WITH THE DUNKL OPERATOR ON THE REAL LINE

Abstract

On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. We consider the generalized shift operator, associated with the Dunkl operator. We study some embeddings into the Morrey space (Dunkl-type Morrey spaces) associated with the Dunkl operator on $\mathbb{R}$. We obtain the boundedness of the Dunkl-type maximal operator in the Dunkl-type Morrey spaces. As applications we get boundedness of the Dunkl-type maximal operator in the Dunkl-type Besov-Morrey spaces.

1. Introduction

On the real line, the Dunkl operators Λ_α are differential-difference operators introduced in 1989 by Dunkl [8]. For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right)$$

Note that $\Lambda_{-1/2} = d/dx$.

In this paper we consider the generalized shift operator, generated by the Dunkl operator Λ_α in terms of which the maximal operator (Dunkl-type maximal operator) in the Morrey space (Dunkl-type Morrey space) associated with the Dunkl operator on $\mathbb{R}$ is investigated. We obtain the boundedness of the Dunkl-type maximal operator in the Dunkl-type Morrey spaces.

The paper organized as follows. In Section 2, we present some definitions and auxiliary results. In section 3, we give some embeddings into the Dunkl-type Morrey spaces. In Section 4, we give the our main result on the boundedness of the Dunkl-type maximal operator in the Dunkl-type Morrey spaces. As applications of this result, we prove the boundedness of the Dunkl-type maximal operator in the Besov-Morrey spaces (Dunkl-type Besov-Morrey spaces) associated with the Dunkl operator on $\mathbb{R}$.

2. Preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(d\mu_\alpha)(\mathbb{R})$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_{\infty,\alpha}} = ess \sup_{x\in\mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r \, \left(\mu_\alpha \{x \in \mathbb{R} \ : \ |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha} \text{ for all } f \in L_{p,\alpha}(\mathbb{R}).$$

For all x, y, $z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, \, y \in \mathbb{R}\setminus 0, \\ 0 & \text{otherwice} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } |z| \in A_{x,y}, \\ 0 & \text{otherwice,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

Proposition 1 (see Rösler [17]). *The signed kernel W_α is even and satisfies the following properties*

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)| \, d\mu_\alpha(z) \le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z) & \text{if } x, \, y \in \mathbb{R}\setminus 0, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases}$$

Definition 1. *For x, $y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see ref. [17, 18])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_1(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta$$

$$+c_\alpha \int_0^\pi f_o\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_2(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with $c_\alpha = \Gamma(\alpha+1)/(\sqrt{\pi}\,\Gamma(\alpha+1/2))$,

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta \;\text{ and }\; h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{\sqrt{x^2+y^2-2|xy|\cos\theta}} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

Proposition 2 (see Soltani [15]).

(i) If f is an even positive continuous function, then $\tau_x f$ is positive.

(ii) For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}.$$

3. Dunkl-type Morrey spaces

Let $B(x,r) = \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-r\}, |x|+r[\}$ and $r > 0$. Then $B(0,r) =]-r,r[$ and

$$\mu_\alpha(]-r,r[) = b_\alpha\, r^{2\alpha+2},$$

where $b_\alpha = \left[2^{\alpha+1}(\alpha+1)\,\Gamma(\alpha+1)\right]^{-1}$.

Definition 2. *Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha+2$. We denote by $L_{p,\lambda,\alpha}(\mathbb{R})$ Morrey space ($\equiv$ Dunkl-type Morrey space), associated with the Dunkl operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm*

$$\|f\|_{p,\lambda,\alpha} = \sup_{x\in\mathbb{R},\, r>0} \left(r^{-\lambda} \int_{B(0,r)} \tau_x |f(y)|^p \, d\mu_\alpha(y) \right)^{1/p}.$$

Note that

$$L_{p,\alpha}(\mathbb{R}) \subset\!\!\succ L_{p,0,\alpha}(\mathbb{R}),$$

$$\|f\|_{L_{p,0,\alpha}} \leq 4\,\|f\|_{L_{p,\alpha}}$$

and if $\lambda < 0$ or $\lambda > 2\alpha+2$, then $L_{p,\lambda,\alpha}(\mathbb{R}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Definition 3. *Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha+2$. We denote by $WL_{p,\lambda,\alpha}(\mathbb{R})$ weak Dunkl-type Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with finite norm*

$$\|f\|_{WL_{p,\lambda,\alpha}} = \sup_{t>0} t \sup_{x\in\mathbb{R},\, r>0} \left(r^{-\lambda} \int_{\{y\in B(0,r):\, \tau_x|f(y)|>t\}} d\mu_\alpha(y) \right)^{1/p}.$$

We note that

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset WL_{p,\lambda,\alpha}(\mathbb{R}) \text{ and } \|f\|_{WL_{p,\lambda,\alpha}} \le \|f\|_{p,\lambda,\alpha}$$

Lemma 1 [11]. *Let* $1 \le p < \infty$. *Then*

$$L_{p,2\alpha+2,\alpha}(\mathbb{R}) = L_{\infty}(\mathbb{R})$$

and

$$\|f\|_{p,2\alpha+2,\alpha} = b_\alpha^{1/p} \|f\|_\infty .$$

On the Dunkl-type Morrey spaces the following embedding is valid.

Lemma 2 [11]. *Let* $0 \le \lambda < 2\alpha + 2$ *and* $0 < \beta \le 2\alpha + 2 - \lambda$. *Then for* $p = \dfrac{2\alpha + 2 - \lambda}{\beta}$

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset L_{1,2\alpha+2-\beta,\alpha}(\mathbb{R}) \quad \textit{and} \quad \|f\|_{1,2\alpha+2-\beta,\alpha} \le b_\alpha^{1/p'} \|f\|_{p,\lambda,\alpha} ,$$

where $1/p + 1/p' = 1$.

4. Main result

Now we define the Dunkl-type maximal function (see [1, 10, 16]) by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu_\alpha B(0,r)} \int_{B(0,r)} \tau_x |f|(y)\, d\mu_\alpha(y).$$

In [1, 10, 16] was proved the following theorem (see also [6, 7]).

Theorem 1.*1. If* $f \in L_{1,\alpha}(\mathbb{R})$, *then* $Mf \in WL_{1,\alpha}(\mathbb{R})$ *and*

$$\|Mf\|_{WL_{1,\alpha}} \le C_1 \|f\|_{1,\alpha},$$

where $C_1 > 0$ *is independent of* f.

2. If $f \in L_{p,\alpha}(\mathbb{R})$, $1 < p \le \infty$, *then* $Mf \in L_{p,\alpha}(\mathbb{R})$ *and*

$$\|Mf\|_{p,\alpha} \le C_2 \|f\|_{p,\alpha},$$

where $C_2 > 0$ *is independent of* f.

Corollary 1. *If* $f \in L^{loc}_{1,\alpha}(\mathbb{R})$, *then*

$$\lim_{r\to 0} \frac{1}{\mu_\alpha B(0,r)} \int_{B(0,r)} \left|\tau_x f(y) - f(x)\right| d\mu_\alpha(y) = 0$$

for a. e. $x \in \mathbb{R}$.

Corollary 2. *If* $f \in L^{loc}_{1,\alpha}(\mathbb{R})$, *then*

$$\lim_{r\to 0} \frac{1}{\mu_\alpha B(0,r)} \int_{B(0,r)} \tau_x f(y) d\mu_\alpha(y) = f(x)$$

for a. e. $x \in \mathbb{R}$.

The following theorem is our main result in which we obtain the boundedness of the Dunkl-type maximal operator M in the Dunkl-type Morrey spaces.

Theorem 2. *Let* $0 \le \lambda < 2\alpha + 2$.

1. If $f \in L_{1,\lambda,\alpha}(\mathbb{R})$, *then* $Mf \in WL_{1,\lambda,\alpha}(\mathbb{R})$ *and*

$$\|Mf\|_{WL_{1,\lambda,\alpha}} \le C_3 \|f\|_{p,\lambda,\alpha},$$

where $C_3 > 0$ *is independent of* f.

2. If $f \in L_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p \le \infty$, *then* $Mf \in L_{p,\lambda,\alpha}(\mathbb{R})$ *and*

$$\|Mf\|_{p,\lambda,\alpha} \le C_4 \|f\|_{p,\lambda,\alpha},$$

where $C_4 > 0$ *is independent of* f.

Proof. The maximal function $Mf(x)$ may be interpreted as a maximal function defined on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying

$$\mu(E(x,2r)) \le C_0 \mu(E(x,r)) \qquad (1)$$

with a constant C_0 independent of x and $r > 0$. Here $E(x,r) = \{y \in X : \rho(x,y) < r\}$, $\rho(x,y) = |x-y|$. Let (X,ρ,μ) be a space of homogeneous type. Define

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(E(x,r))} \int_{E(x,r)} |f(y)| d\mu(y).$$

It is well known that the maximal operator M_μ is bounded from $L_1(X,\lambda,\mu)$ to $WL_1(X,\lambda,\mu)$ for $0 \le \lambda < 2\alpha + 2$ and is bounded on $L_p(X,\lambda,\mu)$ for $1 < p <$ and $0 \le \lambda < 2\alpha + 2$ (see [4, 13, 14]). We shall use this result in the case in which $X = \mathbb{R}$, $\rho(x,y) = |x-y|$, $d\mu(x) = d\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (**??**).

We will shall show that

$$Mf(x) \le C_5 M_\mu f(x), \qquad (2)$$

where $C_5 > 0$ is independent of f.

From the definition of the generalized shift operator it follows that $\tau_x \chi_{B(0,r)}(y)$ is supported in $B(x,r)$.

Moreover

$$0 \le \tau_x \chi_{B(0,r)}(y) \le \min\left\{1, \frac{2c_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{2\alpha+1}\right\}, \qquad \forall y \in B(x,r). \qquad (3)$$

In the case $|x| \le r$ this follows from the simple inequality $0 \le \tau_x \chi_{B(0,r)}(y) \le 1$.

To prove (**??**) in the case $|x| > r$, we proceed as follows:

$$\tau_x \chi_{B(0,r)}(y) = c_\alpha \int_{\left\{\theta \in (0,\pi) : \frac{x^2+y^2-r^2}{2|xy|} \le \cos\theta\right\}} (\sin\theta)^{2\alpha} d\theta = c_\alpha \int_{\frac{x^2+y^2-r^2}{2|xy|}}^{1} (1-t^2)^{\alpha-1/2} dt \le$$

$$\le 2^{(\alpha-1/2)_+} c_\alpha \int_{\frac{x^2+y^2-r^2}{2|xy|}}^{1} (1-t)^{\alpha-1/2} dt = \frac{2^{(\alpha-1/2)_+} c_\alpha}{\alpha+1/2}\left(1 - \frac{x^2+y^2-r^2}{2|xy|}\right)^{\alpha+1/2} \le$$

$$\leq \frac{2c_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{\alpha+1/2}\left(\frac{r-|x-y|}{|y|}\right)^{\alpha+1/2},$$

where $a_+ = a$ if $a \geq 0$ and $a_+ = 0$ if $a < 0$.

In the case $|y| > |x|$

$$\tau_x \chi_{B(0,r)}(y) \leq \frac{2c_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{2\alpha+1}$$

and in the case $|y| < |x|$ the inequality $\dfrac{r-|x-y|}{|y|} < \dfrac{r}{|x|}$ is equivalent to $r < |x|$.

Therefore we have

$$\tau_x \chi_{B(0,r)}(y) \leq \frac{2c_\alpha}{2\alpha+1}\left(\frac{r}{|x|}\right)^{2\alpha},$$

which proves (**??**) in the case $|y| < |x|$ as well.

Also

$$\mu_\alpha B(x,r) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}\int_{B(x,r)} |y|^{2\alpha+1}\, dy \leq$$

$$\leq \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}\begin{cases} 2\int_{|x|-r}^{|x|+r} y^{2\alpha+1}dy, & r < |x| \\ 2\int_0^{|x|+r} y^{2\alpha+1}dy, & r \geq |x| \end{cases} \leq$$

$$\leq \frac{2^{\alpha+1}}{\Gamma(\alpha+1)}\begin{cases} r|x|^{2\alpha+1}, & r < |x| \\ r^{2\alpha+2}, & r \geq |x| \end{cases} = \frac{2^{\alpha+1}}{\Gamma(\alpha+1)} r^{2\alpha+2}\begin{cases} (|x|/r)^{2\alpha+2}, & r < |x| \\ 1, & r \geq |x|. \end{cases}$$

Then

$$Mf(x) = \sup_{r>0}\frac{1}{\mu_\alpha B(0,r)}\int_{\mathbb{R}} \tau_x |f(y)|\, \chi_{B(0,r)}(y) d\mu_\alpha(y) =$$

$$= \sup_{r>0}\frac{1}{\mu_\alpha B(0,r)}\int_{\mathbb{R}} |f(y)|\, \tau_x \chi_{B(0,r)}(y) d\mu_\alpha(y) =$$

$$= \sup_{r>0}\frac{1}{\mu_\alpha B(0,r)}\int_{B(x,r)} |f(y)|\, \tau_x \chi_{B(0,r)}(y) d\mu_\alpha(y).$$

Thus

$$Mf(x) \leq M_1 f(x) + M_2 f(x),$$

where

$$M_1 f(x) = \sup_{r\geq|x|}\frac{1}{\mu_\alpha B(0,r)}\int_{B(0,r)} \tau_x |f(y)|\, d\mu_\alpha(y),$$

$$M_2 f(x) = \sup_{r<|x|}\frac{1}{\mu_\alpha B(0,r)}\int_{B(0,r)} \tau_x |f(y)|\, d\mu_\alpha(y).$$

If $r \geq |x|$, then $\mu_\alpha B(x,r) \leq \dfrac{2^{\alpha+1}}{\Gamma(\alpha+1)} r^{2\alpha+2}$, also $\mu_\alpha B(0,r) = b_\alpha r^{2\alpha+2}$ and $\tau_x \chi_{B(0,r)}(y) \leq 1$ for all $y \in B(x,r)$. Thus yields

$$M_1 f(x) = \sup_{r\geq|x|}\frac{1}{\mu_\alpha B(0,r)}\int_{B(x,r)} |f(y)|\, \tau_x \chi_{B(0,r)}(y) d\mu_\alpha(y) \leq$$

$$\leq 2^{2\alpha+2}\,(\alpha+1)\sup_{r>0}\frac{1}{\mu_\alpha B(x,r)}\int_{B(x,r)}|f(y)|d\mu(y)\leq C_6\,M_\mu f(x).$$

If $r<|x|$, then by (**??**) $\mu_\alpha B(x,r)\leq\frac{2^{\alpha+1}}{\Gamma(\alpha+1)}\,r|x|^{2\alpha+1}$ and

$$\tau_x\chi_{B(0,r)}(y)\leq\frac{2c_\alpha}{2\alpha+1}\Big(\frac{r}{|x|}\Big)^{2\alpha+1}$$

for all $y\in B(x,r)$. Thus we have

$$M_2 f(x)=\sup_{r<|x|}\frac{1}{\mu_\alpha B(0,r)}\int_{B(x,r)}|f(y)|\tau_x\chi_{B(0,r)}(y)d\mu_\alpha(y)\leq$$

$$\leq C_7\sup_{r>0}\frac{1}{\mu_\alpha B(x,r)}\int_{B(x,r)}|f(y)|d\mu(y)\leq C_8 M_\mu f(x).$$

Therefore we get (**??**), which completes the proof 1) and 2).

Theorem 2 has been proved.

For $1\leq p,q\leq\infty$, $0\leq\lambda<2\alpha+2$ and $0<s<2$, the Dunkl-type Besov-Morrey $BD^s_{pq,\lambda,\alpha}(\mathbb{R})$ consists of all functions f in $L_{p,\lambda,\alpha}(\mathbb{R})$ so that

$$\|f\|_{BD^s_{pq,\lambda,\alpha}}=\|f\|_{L_{p,\lambda,\alpha}}+\left(\int_{\mathbb{R}}\frac{\|\tau_x f(\cdot)-f(\cdot)\|^q_{L_{p,\lambda,\alpha}}}{|x|^{2\alpha+2+sq}}dm_\alpha(x)\right)^{1/q}<\infty.$$

Besov spaces in the setting of the Dunkl operators studied by C. Abdelkefi and M. Sifi [2, 3], R. Bouguila, M.N. Lazhari and M. Assal [5], V.S. Guliyev and Y.Y. Mammadov [7] and L. Kamoun [9]. In the following theorem we prove the boundedness of the Dunkl-type maximal operator in the Dunkl-type Besov-Morrey spaces.

Theorem 3. *For $1<p<\infty$, $1\leq q\leq\infty$, $0\leq\lambda<2\alpha+2$ and $0<s<2$ the Dunkl-type maximal operator is bounded on $BD^s_{pq,\alpha}(\mathbb{R})$. More precisely, there is a constant $C>0$ such that*

$$\|Mf\|_{BD^s_{pq,\lambda,\alpha}}\leq C\|f\|_{BD^s_{pq,\lambda,\alpha}}$$

hold for all $f\in BD^s_{pq,\lambda,\alpha}(\mathbb{R})$.

Remark Note that Theorem 3 in the case $\lambda=0$ was proved in [7].

References.

[1]. Abdelkefi C. and Sifi M. *Dunkl Translation and Uncentered Maximal Operator on the Real Line*, JIPAM. J. Inequal. Pure Appl. Math. 2007, 8 No 3, Article 73, 11 p.

[2]. Abdelkefi C. and Sifi M. *On the uniform convergence of partial Dunkl integrals in Besov- Dunkl spaces*, Fractional Calculus and Applied Analysis, 2006, 9, 1, pp. 43-56.

[3]. Abdelkefi C. and Sifi M. *Characterization of Besov spaces for the Dunkl operator on the real line*, Journal of Inequalities in Pure and Applied Mathematics, 2007, vol. 8, issue 3, Article 73, 11 p.

[4]. Arai H., Mizuhara T. *Morrey spaces on spaces of homogeneous type and estimates for* $\Box_b$ *and the Cauchy-Szego projection*, Math. Nachr. 1977, 185 (1), pp. 5-20.

[5]. Bouguila R., Lazhari M.N. and Assal M., *Besov spaces associated with Dunkl's operator*, Integral Transforms and Special Functions, 2007, 18 (8), pp. 545-557.

[6]. Guliyev V.S., Mammadov Y.Y. *Function spaces and integral operators for the Dunkl operator on the real line*, Khazar Journal of Mathematics, 2006, 2 (4), pp. 17-42.

[7]. Guliyev V.S., Mammadov Y.Y. *On fractional maximal function and fractional integral associated with the Dunkl operator on the real line.* Journal of Mathematical Analysis and Applications, 2009, 353, issue 1, pp.449-459.

[8]. Dunkl C.F. *Differential-difference operators associated with reflections groups*, Trans. Amer. Math. Soc. 1989, 311, pp. 167-183.

[9]. Kamoun L. *Besov-type spaces for the Dunkl operator on the real line*, Journal of Computational and Applied Mathematics, 2007, 199, pp. 56-67.

[10]. Mammadov Y.Y. *On maximal operator associated with the Dunkl operator on* $\mathbb{R}$, Khazar Journal of Mathematics, 2006, 2 (4), pp. 59-70.

[11]. Mammadov Y.Y. *Some embeddings into the Morrey spaces associated with the Dunkl operator on R.* Proc. of NASA, Embedding theorems. Harmonic analysis. 2007, issue XIII, pp. 258-269.

[12]. Morrey C.B. *On the solutions of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc. 1938, 43, pp. 126-166.

[13]. Pradolini G., Salinas O. *Maximal operators on spaces of homogeneous type*, Proc. Amer. Math. Soc. 2004, 132, pp.435-441.

[14]. Samko N. *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl. 2009, 350, pp. 56-72.

[15]. Soltani F. *Lp-Fourier multipliers for the Dunkl operator on the real line*, J. Funct. Anal. 2004, 209, pp. 16-35.

[16]. Soltani F. *Littlewood-Paley operators associated with the Dunkl operator on* $\mathbb{R}$, J. Funct. Anal. 2005, 221, pp. 205-225.

[17]. Rösler M. *Bessel-type signed hypergroups on* $\mathbb{R}$, in Probability measures on groups and related structures, XI (Oberwolfach, 1994), H.Heyer and A.Mukherjea, Eds., 1995, pp. 292-304, World Scientific, River edge, NJ, USA.

[18]. Trimeche K. *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Int. Trans. Spec. Funct. 2002, 13, pp. 17-38.

Yagub Y. Mammadov
Nakhcivan State University, Nakhcivan, Azerbaijan,
Nakhchivan Teacher-Training Institute, Azerbaijan,
E-mail: yagubmammadov@yahoo.com

Received September 02, 2009; Revised November 27, 2009.