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ACCUMULATION PROCESSES CONNECTED WITH WAITING TIME OF CUSTOMERS IN MULTITERMINAL QUEUING SYSTEMS

Abstract

Multiterminal queueing systems with cyclic service are considered, asymptotic expansions of expectation and variance, asymptotic distributions of waiting time and waiting expense of customers are found at each terminal. Similar results are obtained for the cases when control is conducted in one terminal and when customers arrive with mixed Poisson's flow.

Introduction. At present, mathematical models of queuing systems are widely used in transport systems, computer networks and etc. In such systems, estimation of waiting time and waiting expense of customers for investigation of optimal control problem of a server is one of the important problems.

The estimation methods of the average waiting time and average waiting expense are given in the one-terminal and two-terminal queuing systems for the cases when conducted control depends on the number of arriving customers and departing time between terminals are defined as constants [1, 2]. Numerical methods for calculation of average waiting time and average waiting expense are given in the case when the control is conducted in two terminals and the departing times are taken as random variables[3].

In [4, 5], service problem of an infinite capacity server and two-terminal queuing system is researched when a control is determined depending on the departing time between the terminals. The expression and moments of average waiting time are found for these systems.

Fig 1.

When a control is conducted in one terminal of three terminal queuing systems and when a control is in several terminals of terminal queuing systems calculation formulas for average waiting time and average waiting expense of customers are given in [6, 7].

Moments, asymptotic distributions and characteristics of accumulation processes in simple type renewal processes are investigated in [8].

Some problems related with characteristics of accumulated waiting times and waiting expense arriving at multiterminal queuing systems remain open up today. In this paper, n terminal queuing system and motion of infinite capacity server is researched (fig. 1). Asymptotic expansions of numerical characteristics and distributions of accumulated waiting time and waiting expense under different conditions are found.

1. Problem statement. n **terminal queuing systems.** A cyclic motion of a server successively delivering customers between n terminals is considered (fig. 1). The server has an infinite capacity, i.e. it can deliver all customers waiting for service at the terminal. It is assumed that the customers arrive at the terminal according to Poisson process with parameter $\lambda_i\,(i = 1, 2, ..., n)$. The j-th travel time of a server from i-th terminal to the next terminal is denoted by $X_j^i\,(i = 1, 2, ..., n;\ j = 1, 2, ...)$. For the fixed i, the random variables $X_j^i\,(i = 1, 2, ..., n;\ j = 1, 2, ...)$ are bounded, independent and identically distributed. At the each terminal a cost of waiting time of customers $c_i\,(i = 1, 2, ..., n)$ is given. The total waiting expense of customers in the system is taken as efficiency index.

The j-th departure moment of a server of the i-th terminal are denoted by t_j^i and $t_1^1 = 0$ is taken. Then

For terminal one

$$t_1^1 = 0, t_2^1 = t_1^1 + X_1^1 + X_1^2 + ... + X_1^n, ..., t_{m+1}^1 = t_m^1 + X_m^1 + X_m^2 + ... + X_m^n.$$

For terminal two

$$t_2^1 = t_1^1 + X_1^1,\ \ t_2^2 = t_2^1 + X_2^1,\ ...\ ,\ t_{m+1}^2 = t_{m+1}^1 + X_{m+1}^1$$

....

For terminal n

$$t_1^n = t_1^{n-1} + X_1^{n-1},\ \ t_2^n = t_2^{n-1} + X_2^{n-1},\ ...\ ,\ t_{m+1}^n = t_{m+1}^{n-1} + X_{m+1}^{n-1}.$$

The following denotation is made:

$$\eta_j^i = t_{j+1}^i - t_j^i = \sum_{l=i}^{n} X_j^l + \sum_{l=1}^{i-1} X_{j+1}^l.$$

It is clear that $\eta_j^i\,(j = 1, 2, ...)$ are times between service moments at the i-th terminal and they are independent, identically distributed random variables, i.e. they make renewal processes. Then by [6] the average waiting time at the i-th terminal as follows:

$$W_i = \frac{E\left(\eta_j^i\right)^2}{2E\eta_j^j} = \frac{E\left(\sum\limits_{k=1}^{n} X_1^k\right)^2}{2E\left(\sum\limits_{k=1}^{n} X_1^k\right)}, i = 1, 2, ..., n$$

and average waiting expense of any customer in this system

$$C_0 = \frac{\sum_{k=1}^{n} \lambda_k c_k W_k}{\sum_{k=1}^{n} \lambda_k} = \frac{\sum_{k=1}^{n} \lambda_k c_k}{\sum_{k=1}^{n} \lambda_k} W_1$$

are defined. We take in the consideration that, $W_1 = W_2 = ... = W_n$.

We denote by N_t^i the number of customers arriving at any i-th terminal in a period $(0, t)$ and by V_j^i the waiting time of the j-th customer. Then the waiting time of all customers accumulated in the period $(0, t)$ at the i-th terminal will be in the form:

$$Z_t^i = \sum_{j=1}^{N_t^i} V_j^i.$$

Following denotations are made:

$$a_i = \lambda_i \frac{E\left(\eta_1^i\right)^2}{2E\left(\eta_1^i\right)}, \quad \delta_i^2 = \lambda_i \frac{E\left(\eta_1^i\right)^3}{3E\left(\eta_1^i\right)} - \lambda_i^2 \frac{\left(E\left(\eta_1^i\right)^2\right)^2}{2E\left(\eta_1^i\right)} + \lambda_i^2 \frac{\left(E\left(\eta_1^i\right)^2\right)^3}{4\left(E\left(\eta_1^i\right)\right)^3}.$$

Theorem 1. *The asymptotical expansions of the mean value and variance of the random process Z_t^i are defined as follows:*

$$E\left(Z_t^i\right) \sim \lambda_i t W_i = a_i t,$$

$$Var\left(Z_t^i\right) \sim \delta_i^2 t.$$

Proof. The number of services at the i-th terminal in the period $(0, t)$ is denoted by M_t. The time periods between service moments in the period $(0, t)$ at the i-th terminal will be $\eta_1^i, \eta_2^i, ..., \eta_{M_t}^i$. It is clear that

$$\sum_{k=1}^{M_t} \eta_k^i \le t < \sum_{k=1}^{M_t+1} \eta_k^i.$$

We denote by $U_k^i\,(k = 1, 2, ..., M_t)$ - the sum of waiting times of customers until the service moment t_{k+1}^i arriving at the time interval $\left(t_k^i, t_{k+1}^i\right)$ (in time period $\eta_k^i = t_{k+1}^i - t_k^i$), by $\overline{U}(t)$ - the sum of waiting times until the service moment t arriving at the time interval $\left(t_{M_t}^i, t\right)$ at the i-th terminal. Then the accumulated waiting time Z_t^i of customers at the i-th terminal can be written in the form:

$$Z_t^i = \sum_{k=1}^{M_t} U_k^i + \overline{U}(t). \qquad (1).$$

As the customers arrive at the terminals by the Poisson arrivals and $\eta_1^i, \eta_2^i, ..., \eta_{M_t}^i$ are independent and identically distributed random variables, we get that

$U_k^i\,(k=1,2,...,M_t)$ are also independent and identically distributed random variables. Let s_j be an arrival moment of the j-th customer at the i-th terminal, $\psi(s_j)$ be the number of customers arriving at this moment. The denotation

$$A_k=\left\{s:\psi(s)>0,s\in\left[t_k^i,t_{k+1}^i\right)\right\}$$

is done. Then by Campbell's formula, we can write:

$$E\left(U_k^i/\eta_k^i\right)=E\left\{\sum_{s\in A_k}\left(t_{k+1}^i-s\right)\psi(s)/\eta_k^i\right\}=\int_{t_k^i}^{t_{k+1}^i}\left(t_{k+1}^i-s\right)\lambda_i ds=\frac{\lambda_i\left(\eta_k^i\right)^2}{2}.$$

Using the result, we can write

$$E\left(Z_t^i/M_t=r\right)E\left(EZ_t^i/\eta_1^i,\eta_2^i,...,\eta_r^i\right)/M_t=r=$$

$$=E\left(\sum_{k=1}^r E\left(U_k^i/\eta_1^i,\eta_2^i,...,\eta_r^i\right)+\overline{U}(t)/M_t=r\right)=$$

$$=E\left(\frac{\lambda_i}{2}\sum_{k=1}^r\left(\eta_k^i\right)^2+\overline{U}(t)/M_t=r\right)=$$

$$=\frac{\lambda_i}{2}\sum_{k=1}^r E\left(\eta_k^i\right)^2+G(t)=\frac{\lambda_i rE\left(\eta_k^i\right)^2}{2}+G(t). \tag{2}$$

It is known that average value and variance of the random variable M_t is defined by the following formula [8]:

$$EM_t=\frac{t}{E\eta_1^i},\quad Var(M_t)=\frac{tVar\left(\eta_1^i\right)}{\left[E\left(\eta_1^i\right)\right]^3}.$$

On the other hand, as $\overline{U}(t)\le U_{M_t+1}^i$ the quantity $G(t)$ is restricted:

$$G(t)=E\left(\overline{U}(t)/M_t=r\right)\le E\left(U_{M_t+1}^i/M_t=r\right)=\frac{\lambda_i E\left(\eta_1^i\right)^2}{2}.$$

So, taking these relations into account, the first statement of the theorem was proved.

In order to prove the second statement of the theorem we show in the same way

$$E\left(\left(U_k^i\right)^2/\eta_k^i\right)=\frac{\lambda_i\left(\eta_k^i\right)^3}{3}$$

and

$$Var\left(Z_t^i/M_t=r\right)=\sum_{k=1}^r Var\left(U_k^i\right)+W(t)=rVar\left(U_1^i\right)+W(t). \tag{3}$$

Hence, we get

$$Var\left(Z_t^i\right)=EM_tVar\left(U_1^i\right)+Var(M_t)\left(EU_1^i\right)^2+W(t).$$

In the same way, using the relation $W(t) \leq M < \infty$ we prove the theorem.

Theorem 2. *The random variable Z_t asymptotically has the normal distribution*

$$\frac{Z_t^i - a_i t}{\delta_i \sqrt{t}} \xrightarrow{d} N(0,1).$$

Proof. It is seen from relation (1) for the large values of r the random process Z_t^i is approximately distributed by a normal distribution with mean value (2) and variance (3). It's conditional characteristic function will be in the form

$$\exp\left(ip\left(\frac{\lambda_i r E\left(\eta_1^i\right)^2}{2} + G(t)\right) - \frac{1}{2}p^2\left(rVar\left(U_1^i\right) + W(t)\right)\right). \tag{4}$$

The unconditional characteristic function will equal mathematical expectation of expression (4) with respect to distribution of service number M_t. The random variable M_t has a normal distribution with approximate mean value $\frac{t}{E\eta_1^i}$ and variance $\frac{tVar\left(\eta_1^i\right)}{\left[E\left(\eta_1^i\right)\right]^3}$ [8]. Then the mathematical expectation of expression (4) with respect to asymptotic normal distribution is

$$\exp\left(\left(ip\frac{\lambda_i E\left(\eta_1^i\right)^2}{2} - \frac{1}{2}p^2 Var\left(U_1^i\right)\right)\frac{1}{E\eta_1^i} + \right.$$

$$\left. +\frac{1}{2}\left(ip\frac{\lambda_i E\left(\eta_1^i\right)^2}{2} - \frac{1}{2}\,p^2 Var\left(U_1^i\right)^2\right)^2 \frac{tVar\left(\eta_1^i\right)}{\left[E\left(\eta_1^i\right)\right]^3}\right)\exp\left(ipG(t) - \frac{1}{2}\,p^2 W(t)\right).$$

If we normalize the random variable

$$\frac{Z_t^i - a_i t}{\sqrt{\delta_i^2 t}}$$

and write the expression for the characteristic function for it, then as $t \to \infty$ we get

$$e^{-p^2/2}\left(1 + 0(1)\right).$$

This shows that the random variable Z_t^i is distributed by a normal law with asymptotic expansions of the mean value and variance

$$\lambda_i t \frac{E\left(\eta_1^i\right)^2}{2E\left(\eta_1^i\right)}, \quad \lambda_i t \frac{E\left(\eta_1^i\right)^3}{3E\left(\eta_1^i\right)} - \lambda_i^2 t \frac{\left(E\left(\eta_1^i\right)^2\right)^2}{2E\left(\eta_1^i\right)} + \lambda_i^2 t \frac{\left(E\left(\eta_1^i\right)^2\right)^3}{4\left(E\left(\eta_1^i\right)\right)^3}.$$

Remark. We denote by N_t the number of arrivals in the system in the period $(0,t)$ and by p_j the waiting expense of the j-th customer. Then the accumulated waiting expense of customers in the period $(0,t)$ at the i-th terminal will be in the form

$$C_t^i = \sum_{j=1}^{N_t^i} p_j.$$

We can easily show that

$$C_t^i = \sum_{i=1}^{n} c_i Z_t^i.$$

Then we get from theorem 1 that the asymptotic expansions of the mean value of accumulated waiting expense of the customers in the system is

$$EC_t^i \sim \sum_{i=1}^{n} c_i \lambda_i t \frac{E\left(\eta_1^i\right)^2}{2E\left(\eta_1^i\right)}.$$

However, since the accumulated waiting times $Z_t^i\,(i = 1, 2, ..., n)$ at the terminals are dependent random variables, there is a difficulty in finding variance and asymptotic distribution of the accumulated waiting expense.

2. Accumulation process for the queuing systems with delays in a single terminal. The control is conducted as a delay of a server at the first terminal and is defined as a non-negative, measurable function of departing times in the previous period. Then the time periods between service moments are

$$\xi_j^i = \eta_j^i + g\left(X_j\right), \qquad i = 1, 2, ..., n,$$

here $X_j\left(X_j^1, X_j^2, ..., X_j^n\right)$, $j = 1, 2, ...$. We get the time periods between service moments at the terminal are dependent, but identically distributed random variables. In this similar way, we can show the formula

$$EZ_t^i \sim \lambda_i t \frac{E\left(\eta_1^i + g\left(X_j\right)\right)^2}{2E\left(\eta_1^i + g\left(X_j\right)\right)}$$

for average value of the random variables $Z_t^i\,(i = 1, 2, ..., n)$ and

$$EC_t^i \sim \sum_{i=1}^{n} c_i \lambda_i t \frac{E\left(\eta_1^i + g\left(X_j\right)\right)^2}{2E\left(\eta_1^i + g\left(X_j\right)\right)}$$

for average value of the random variable C_t^i are true.

3. The case when the customers arrive with mixed Poisson arrivals. It is assumed that the customers arrive at the terminals by the mixed Poisson arrivals with parameters $\lambda_i\,(i = 1, 2, ..., n)$, i.e. the customers are in the form of group, the groups arrive by the Poisson arrivals with parameters $\lambda_i\,(i = 1, 2, ..., n)$ and the number of customers in these groups $D_1^i, D_2^i, ...$ are non-negative, integer, bounded, identically distributed and independent random variables. The following denotations are made

$$\overline{a}_i = \lambda_i ED_1^i \frac{E\left(\eta_1^i\right)^2}{2E\left(\eta_1^i\right)}, \quad \overline{\delta}_i^2 = \lambda_i \left(ED_1^i\right)^2 \frac{E\left(\eta_1^i\right)^3}{3E\left(\eta_1^i\right)} -$$

$$-\lambda_i^2 \left(ED_1^i\right)^2 \frac{\left(E\left(\eta_1^i\right)^2\right)^2}{2E\left(\eta_1^i\right)} + \lambda_i^2 \left(ED_1^i\right)^2 \frac{\left(E\left(\eta_1^i\right)^2\right)^3}{4\left(E\left(\eta_1^i\right)\right)^3}.$$

Theorem 3. *For the considered problem, at the first terminal the accumulated waiting time Z_t^i of customers has asymptotic normal distribution*

$$\frac{Z_t^i - \bar{a}_i t}{\bar{\delta}_i \sqrt{t}} \xrightarrow{d} N(0,1)$$

and the formula

$$EC_t^i \sim \sum_{i=1}^{n} c_i \lambda_i E D_1^i t \frac{E\left(\eta_1^i\right)^2}{2E\left(\eta_1^i\right)}$$

is true for the average value of the accumulated waiting expense of customers in the system.

The proof of the theorem 2 is obtained similar to the proof of the above cited way using the following relations:

$$E\left(U_k^i/\eta_k^i\right) = E\left\{\sum_{s\in A_k}\left(t_{k+1}^i - s\right)\psi(s)/\eta_k^i\right\} =$$

$$= \int_{t_k^i}^{t_{k+1}^i}\left(t_{k+1}^i - s\right)\lambda_i E\left(D_1^i\right) ds = \frac{\lambda_i E\left(D_1^i\right)\left(\eta_k^i\right)^2}{2}$$

$$E\left(\left(U_k^i\right)^2/\eta_k^i\right) = \frac{\lambda_i E\left(D_1^i\right)^2\left(\eta_k^i\right)^3}{3}.$$

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