Kamran A. DJALILOV

ON THE SOLUTION OF A FRACTIONAL DERIVATIVE BOUNDARY VALUE PROBLEM

Abstract

A boundary value problem is considered for an equation containing a fractional derivative of order $1 < \alpha < 2$ in the Riemann-Liouville sense. The problem is reduced to Fredholm integral equation of second kind and is solved by the approximate method.

Let a fractional derivative differential equation

$$\frac{d^4u(x)}{dx^4} = \lambda\Gamma(3-\alpha) D_{0x}^{\alpha} u(x) + f(x), \quad x \in (0,1), \tag{1}$$

be given. Here $\Gamma(3-\alpha)$ is Euler's gamma function, λ is a real parameter, $f(x)$ is a given continuous on [0,1] function and $D_{0x}^{\alpha}u(x)$ is a derivative from $u(x)$ of fractional order α in the Riemann-Liouville sense determined by the formula [1]:

$$D_{0x}^{\alpha} u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_0^x u(t)(x-t)^{1-\alpha} dt, \quad 1 < \alpha < 2. \tag{2}$$

We give the boundary conditions:

$$u(0) = u'(0) = u''(1) = u'''(1) = 0. \tag{3}$$

Under the solution of problem (1)-(3) we mean the function $u(x)$ possessing the following properties:

1^0. it has a continuous derivative to fourth order inclusively in (0,1);

2^0. it satisfies boundary conditions (3);

3^0. it possesses a partial derivative $D_{0x}^{\alpha}u(x)$, $1 < \alpha < 2$ in (0,1);

4^0. it reduces equation (1) to an identity.

We write the desired solution of the stated problem by means of the Green function of the equation $\frac{d^4u(x)}{dx^4} = 0$ with boundary conditions (3) that is represented by the formula

$$G(x,\xi) = \begin{cases} \frac{1}{6}\left(3x^2\xi - x^3\right) & \text{for} \quad x \le \xi, \\ \frac{1}{6}\left(3x^2\xi - \xi^3\right) & \text{for} \quad \xi \le x. \end{cases}$$

Thus, for the solution of problem (1), (3) we'll have the representation:

$$u(x) = \lambda\Gamma(3-\alpha)\left(\int_0^x \frac{1}{6}\left(3x\xi^2 - \xi^3\right) D_{0\xi}^{\alpha} u(\xi)\, d\xi + \int_0^x \frac{1}{6}\left(3x\xi^2 - x^3\right) D_{0\xi}^{\alpha} u(\xi)\, d\xi\right) +$$

$$+\int_0^1 G(x,\xi) f(\xi)\, d\xi. \tag{4}$$

Now, by integration by parts we represent formula (2) for fractional derivative, in the form:

$$D_{0\xi}^{\alpha}u(\xi)=\sum_{k=0}^{1}\frac{u^{(k)}(0)\,\xi k-\alpha}{\Gamma(1+k-\alpha)}+\frac{1}{\Gamma(2-\alpha)}\int_{0}^{\xi}(\xi-t)^{1-\alpha}u''(t)\,dt.$$

Considering boundary conditions (3) and the last formula we can represent the solution of (4) as follows:

$$u(x)=\frac{\lambda\Gamma(3-\alpha)}{\Gamma(2-\alpha)}\left(\int_{0}^{1}\frac{1}{6}\left(3x\xi^{2}-\xi^{3}\right)\left(\int_{0}^{\xi}(\xi-t)^{1-\alpha}u''(t)\,dt\right)d\xi+\right.$$

$$\left.+\int_{0}^{x}\frac{1}{6}(x-\xi)^{3}\left(\int_{0}^{\xi}(\xi-t)^{1-\alpha}u''(t)\,dt\right)d\xi\right)+\int_{0}^{1}G(x,\xi)\,f(\xi)\,d\xi. \tag{5}$$

Further, for the solution of equation (5) we'll use the small kernel method. Therefore, along with equation (5) we consider the following equation:

$$\overline{u}(x)=\frac{\lambda\Gamma(3-\alpha)}{\Gamma(2-\alpha)}\int_{0}^{1}\frac{1}{6}\left(3x^{2}\xi-x^{3}\right)\times$$

$$\times\left(\int_{0}^{\xi}(\xi-t)^{1-\alpha}\overline{u}''(t)\,dt\right)d\xi+\int_{0}^{1}G(x,\xi)\,f(\xi)\,d\xi. \tag{6}$$

Having differentiated the last equality twice with respect to x and denoted $\overline{u}''(x)=\overline{z}(x)$, we get

$$\overline{z}(x)=\frac{\lambda\Gamma(3-\alpha)}{\Gamma(2-\alpha)}\int_{0}^{1}(\xi-x)\left(\int_{0}^{\xi}(\xi-t)^{1-\alpha}\overline{z}''(t)\,dt\right)d\xi+F(x),$$

where

$$F(x)=-\int_{x}^{1}(x-\xi)\,f(\xi)\,d\xi.$$

If we integrate with respect to the variable ξ, the last equality is reduced to the most convenient form:

$$\overline{z}(x)=\lambda\int_{0}^{1}\left(\frac{(1-t)^{3-\alpha}}{3-\alpha}+(1-x)(1-t)^{2-\alpha}\right)\overline{z}(t)\,dt+F(x). \tag{7}$$

Thus, we get a Fredholm equation of second kind with degenerate kernel:

$$K(x,t)=-\frac{(1-t)^{3-\alpha}}{3-\alpha}+(1-x)(1-t)^{2-\alpha}.$$

The resolvent that corresponds to equation (7) is calculated by the known method [2]:

$$R(x,t,\lambda)=\frac{\Delta(x,t,\lambda)}{\Delta(\lambda)},$$

where

$$\Delta(\lambda)=\lambda^2-\lambda(2-\alpha)(3-\alpha)(4-\alpha)(5-\alpha)+(3-\alpha)^2(4-\alpha)^2(5-\alpha),$$

$$\Delta(x,t,\lambda)=\frac{(1-t)^{3-\alpha}}{3-\alpha}\left(1-\frac{\lambda}{4-\alpha}+\frac{(1-x)\lambda}{3-\alpha}\right)+$$

$$+(1-t)^{2-\alpha}\left(\frac{\lambda}{(3-\alpha)(5-\alpha)}-(1-x)\left(1+\frac{\lambda}{(3-\alpha)(4-\alpha)}\right)\right).$$

By direct calculation we get that for $1<\alpha<2$, the discriminant corresponding to the square trinomial $\Delta(\lambda)$ is:

$$D=(3-\alpha)^2(4-\alpha)^4(5-\alpha)(1-\alpha)<0.$$

Thus, we get that the resolvent $R(x,t,\lambda)$ has no real eigen values. So, equation (7) has a unique solution represented by the formula:

$$\overline{z}(x)=F(x)+\lambda\int_0^1 R(x,t,\lambda)F(t)\,dt.$$

Further, using the denotation $\overline{u}''(x)=\overline{z}(x)$ and conditions (3), we get that the solution of equation (6) has the representation:

$$\overline{u}(x)=\int_0^x (x-s)F(s)\,ds+\lambda\int_0^x (x-s)\int_0^1 R(s,t,\lambda)F(t)\,dtds.$$

Repeating the above mentioned reasonings, we can compose a relation corresponding to equation (5):

$$z(x)=\lambda\int_0^1\left(-\frac{(1-t)^{3-\alpha}}{3-\alpha}+(1-x)(1-t)^{2-\alpha}+G_0(x,t)\right)z(t)\,dt+F(x), \quad (8)$$

here

$$G_0(x,t)=\begin{cases}\dfrac{(x-t)^{3-\alpha}}{3-\alpha}, & \text{for } t\le x,\\ 0 & \text{for } x\le t.\end{cases}$$

If by $L(x,t)$ we denote a kernel corresponding to equation (8), we easily get the difference estimate:

$$\sup_{x\in[0,1]}\int_0^1 |L(x,t)-K(x,t)|\,dt=\sup_{x\in[0,1]}\int_0^x\frac{(x-t)^{3-\alpha}}{3-\alpha}dt\le\frac{1}{(3-\alpha)(4-\alpha)}.$$

Now, let's estimate the resolvent $R(x,t,\lambda)$ of equation (7):

$$\int_0^1 |R(x,t,\lambda)|\,dt < R,$$

where

$$R = \frac{(3-\alpha)(4-\alpha)(5-\alpha)+\left(a^2-9\alpha+21\right)}{\lambda^2-\lambda(2-\alpha)(3-\alpha)(4-\alpha)(5-\alpha)+(3-\alpha)^2(4-\alpha)^2(5-\alpha)}. \qquad (9)$$

It is known that [2] if there exists a number λ for which

$$1-\frac{\lambda}{(3-\alpha)(4-\alpha)}(1+\lambda R)>0, \qquad (10)$$

equation (8) has a unique solution such that

$$|z(x)-\overline{z}(x)| < \frac{M\lambda(1+\lambda R)^2}{(3-\alpha)(4-\alpha)-\lambda(1+\lambda R)}, \qquad (11)$$

where

$$M = \sup_{x\in[0,1]} F(x).$$

It is easy to verify that any number $0<\lambda<3-\alpha$, $1<\alpha<2$ satisfies relation (10), where R is determined by formula (9). So, equation (8) has a unique solution and its difference with the solution of equation (7) is estimated by formula (11). Then, the difference of the solution $u(x)$ of initial equation (5) and $\overline{u}(x)$ of equation (6) will have the estimate:

$$|u(x)-\overline{u}(x)| = \left|\int_0^x (x-s)(z(s)-\overline{z}(s))\,ds\right| \leq \frac{M\lambda(1+\lambda R)^2}{2((3-\alpha)(4-\alpha)-\lambda(1+\lambda R))}. \qquad (12)$$

Thus, we get the following

Theorem. *For* $0<\lambda<3-\alpha$, $1<\alpha<2$*, equation (5) has a unique solution* $u(x)$ *and its difference with exact solution of equation (6) is estimated by inequality (12).*

Notice that the multiplier $\Gamma(3-\alpha)$ on the right hand side of equation (1) was taken for convenience of calculations.

References

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Kamran A. Djalolov
Baku State University.
23, Z.I.Khalilov str., AZ 1148, Baku, Azerbaijan.
Tel:. (99412) 439 47 20 (off).
e-mail: calilov kamran@rambler.ru.

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