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ESTIMATIONS FOR ROOT FUCNTIONS OF DIRAC OPERATOR

Abstract

In the paper, Dirac's one-dimensional operator is considered. A shift formula is derived for its root vector-function, and a priori estimations, the estimations between different L_p norms of the root vector-functions of the given operator are established.

1. Formulation of the results

In the paper we establish bilateral estimations of the root vector-functions responding to the Dirac one-dimensional operator

$$Du = B\frac{du}{dx} + P(x)\ u, \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ P(x) = (P_{ij}(x))_{i,j=1}^2,\ u(x) = (u^1(x), u^2(x))^T.$$

The coefficient $P(x)$ is a complex-valued matrix-function determined on an arbitrary interval G of a real line.

Let $L_p^2(G),\ p \geq 1$ be a space of two component vector-functions with the norm

$$\|f\|_{p,2} = \left(\int_G |f(x)|^p\, dx \right)^{\frac{1}{p}}, \quad \left(\|f\|_{\infty,2} = \sup_{x\in G} vrai\, |f(x)| \right).$$

Denote by $\|P(x)\|$ the norm of the matrix $P(x)$, i.e.

$$\|P(x)\| = \sum_{i=1}^{2} \sum_{j=1}^{2} |P_{ij}(x)|\,.$$

The estimations of the root functions of ordinary differential operators were earlier established in the papers [1]-[6]. These estimations found their wide applications in investigation of the problems of basicity and equiconvergence with trigonometric series of spectral expansions [7]-[9].

In the present paper, based on the shift formula obtained in [10], a new formula for the vector-functions is derived and different estimation for the root vector-functions of Dirac operator are established with its help.

Following V.A. Il'in [1], introduce the notion of the root vector-functions of the operator in the generalized sense.

Definition. *Under the eigen vector-function or adjoint zero order vector-function responding to the complex eigen value λ, we'll understand any vector-function* $\overset{0}{u}(x)$

that differs from identity zero and that absolutely is continuous on each compact of the interval G and almost everywhere in G satisfies the equation $D\overset{0}{u} = \lambda\overset{0}{u}$.

If the adjoint function $\overset{\ell-1}{u}(x)$ *of order* $\ell - 1 \geq 0$ *is determined, then under the adjoint vector-function of order* ℓ *responding to the eigen value* λ *and the eigen function* $\overset{\ell}{u}(x)$, *we'll understand the vector-function* $\overset{\ell}{u}(x)$ *that is absolutely continuous on any compact of interval G and almost everywhere in G satisfies the equation* $D\overset{\ell}{u} = \lambda\overset{\ell}{u} - \overset{\ell-1}{u}$.

The following theorems are proved in the paper.

Theorem 1 (Shift formula). *Let* $P_{ij}(x) \in L_1^{loc}(G)$, $i, j = 1, 2$ *and* (c, d) *be inside of the interval G. Then there exists a positive number* R^* *such that for any* $t \in (0, R^*]$ *and each* $[c + t, d - t]$, *the shift formula*

$$\overset{\ell}{u}(x \pm t) = \sum_{j=0}^{\ell} F_j^{\pm}(t)\overset{\ell-j}{u}(x), \tag{2}$$

is valid and the following estimations are fulfilled for the matrix $F_j^{\pm}(t)$

$$\left\| F_0^{\pm}(t) - \cos\lambda t I \pm \sin\lambda t B \right\| \leq \omega(t) ch(tJm\lambda); \tag{3}$$

$$\left\| F_j^{\pm}(t) \right\| \leq 5(8t)^j ch(tJm\lambda),\ j = \overline{0, \ell}, \tag{4}$$

where $\omega(t)$ *is a non-negative function that monotonically tends to zero as* $t \to 0+0$, I *is a unit matrix in* E^2.

Fix an arbitrary segment $K = [a, b] \subset G$ and such a segment $K_R = [a + R,\ b - R]$ that it contains, that $R = dist(K_R, \partial K) < \dfrac{mesK}{2}$.

Theorem 2. *Let* $P_{11}(x) = p(x)$, $P_{22}(x) = q(x)$, $P_{12}(x) = P_{21}(x) \equiv 0$ *and* $p(x), q(x) \in L_1^{loc}(G)$. *Then for the segments K and* K_R *there exits positive constants* $C_i(K, \ell), i = \overline{1, 3}$; $C_i(K, K_R, \ell)$, $i = 4, 5$, *independent of* λ *such that the following estimations are true:*

$$C_1 \left\| \overset{\ell}{u} \right\|_{p,2,K} \leq [1 + |Jm\lambda|]^{\frac{1}{s} - \frac{1}{p}} \left\| \overset{\ell}{u} \right\|_{s,2,K} \leq C_2 \left\| \overset{\ell}{u} \right\|_{p,2,K},\ 1 \leq p < s \leq \infty; \tag{5}$$

$$\left\| \overset{\ell-1}{u} \right\|_{p,2,K} \leq C_3 \left[1 + |Jm\lambda|\right] \left\| \overset{\ell}{u} \right\|_{p,2,K},\ p \geq 1; \tag{6}$$

$$\begin{gathered} C_4 \left[1 + |Jm\lambda|\right]^{-\ell} \left\| \overset{\ell}{u} \right\|_{p,2,K} \leq \left\| \overset{\ell}{u} \right\|_{p,2,K_R} \exp(R\,|Jm\lambda|) \leq \\ \leq C_5 \left[1 + |Jm\lambda|\right]^{\ell} \left\| \overset{\ell}{u} \right\|_{p,2,K},\ p \geq 1,\ \ell \geq 0, \end{gathered} \tag{7}$$

where

$$\overset{-1}{u}(x) \equiv 0,\ \ \|\cdot\|_{p,2K} = \|\cdot\|_{L_p^2(K)}$$

Remark. Under the condition $p(x),\ q(x) \in L_1(G),\ mesG < \infty$, in estimations (5)-(7) the segment K may be replaced by the domain $\overline{G}$.

2. Shift formula

Derivation of the shit formula (2) for the root functions $\overset{\ell}{u}(x)$ is based on the following lemma.

Lemma [10]. *If the functions $P_{ij}(x) \in L_1^{loc}(G)$, $i,j = 1,2$ and the points $x-t, x, x+t$ are in the domain G, the following formulae are true:*

$$\overset{\ell}{u}(x \pm t) = (\cos\lambda t \cdot I \mp \sin\lambda t \cdot B)\overset{\ell}{u}(x) + \int\limits_0^t (\sin\lambda(t-r)\cdot I \pm \\ \pm \cos\lambda(t-r)\cdot B)\left[P(x\pm r)\overset{\ell}{u}(x\pm r) + \overset{\ell-1}{u}(x\pm r)\right] dr, \tag{8}$$

$$\overset{\ell}{u}(x-t) + \overset{\ell}{u}(x+t) = 2\overset{\ell}{u}(x)\cos\lambda t + \int\limits_{x-t}^{x+t} (\sin\lambda(t-|x-\xi|)\cdot I + \\ + sgn(\xi - x)\cos\lambda(t-|x-\xi|\cdot B)\left(P(\xi)\overset{\ell}{u}(\xi) + \overset{\ell-1}{u}(\xi)\right) d\xi. \tag{9}$$

Notice that for deriving formula (8), it suffices to act on the equation

$$D\overset{\ell}{u}(x\pm r) = \lambda\overset{\ell}{u}(x\pm r) + \overset{\ell-1}{u}(x\pm r)$$

by the operator $\lambda(t-r)\cdot I \pm \cos\lambda(t-r)\cdot B$, integrate with respect to the parameter r from 0 to t, and carry out integration by parts in the expression of the form

$$\int\limits_0^t (\sin\lambda(t-r)\cdot I \pm \cos\lambda(t-r)\cdot B)\cdot Bd\overset{\ell}{u}(x\pm r).$$

Now, derive the shift formula (2) by the mathematical induction method. Solve equation (8) for $\ell = 0$ with respect to $\overset{0}{u}(x)$ by successive iterations. Denoting

$$A^{\pm}(t,\lambda) = \cos\lambda t\cdot I \mp \sin\lambda t \cdot B;$$

$$T_{\pm}\psi(t) = \int\limits_0^t A^{\mp}(t-r,\lambda)P(x\pm r)\psi(r)\, dr, \tag{10}$$

where $\psi(t)$ is a matrix function of dimension 2×2, as a result of the first iteration we get the equality

$$\overset{\ell}{u}(x\pm r) = \left\{A^{\pm}(t,\lambda) + \int\limits_0^t A^{\pm}(t-t_1,\lambda)P(x\pm t_1)A^{\pm}(t_1,\lambda)dt_1\right\}\overset{0}{u}(x) +$$

$$+\int\limits_0^t A^{\mp}(t-t_1,\lambda)P(x\pm t_1)\int\limits_0^{t_1} A^{\mp}(t-t_2,\lambda)P(x\pm t_2)\overset{0}{u}(x\pm t_2)dt_2 dt_1 =$$

$$= (I+T_{\pm})A^{\pm}(t,\lambda)\overset{0}{u}(x) + \int\limits_0^t A^{\mp}(t-t_1,\lambda)P(x\pm t_1)\times$$

$$\times \int_0^{t_1} A^{\mp}(t_1 - t_2, \lambda) P(x \pm t_2) \overset{0}{u}(x \pm t_2) dt_2 dt_1.$$

Continuing the unrestricted iteration process, we arrive at the formal equality

$$\overset{0}{u}(x \pm r) = \left\{ A^{\pm}(t, \lambda) + \int_0^t A^{\mp}(t - t_1, \lambda) P(x \pm t_1) A^{\pm}(t_1, \lambda) dt_1 + \right.$$

$$+ \int_0^t A^{\pm}(t - t_1, \lambda) P(x \pm t_1) \int_0^{t_1} A^{\mp}(t_1 - t_2, \lambda) P(x \pm t_2) A^{\pm}(t_2, \lambda) dt_2 dt_1 + ... +$$

$$+ \int_0^t A^{\mp}(t - t_1, \lambda) P(x \pm t_1) \int_0^{t_1} A^{\mp}(t_1 - t_2, \lambda) P(x \pm t_2) ...$$

$$\left. ... \int_0^{t_{k-1}} A^{\mp}(t_{k-1} - t_k, \lambda) P(x \pm t_k) A^{\pm}(t_k, \lambda) dt_k dt_{k-1} ... dt_1 + ... \right\} \overset{0}{u}(x) = F_0^{\pm}(t) \overset{0}{u}(x),$$

where

$$F_0^{\pm}(t) = \left(I + \sum_{k=1}^{\infty} T_{\pm}^k \right) A^{\pm}(t, \lambda). \tag{11}$$

Consequently, for $\ell = 0$, the formula (2) is valid (formally).

Now, assume that formula (2) is valid for some $(l-1), l > 1$.

Then by formula (8)

$$\overset{i}{u}(x \pm t) = A^{\pm}(t, \lambda) \overset{i}{u}(x) + \int_0^t A^{\mp}(t - r, \lambda) \left\{ P(x \pm r) \overset{i}{u}(x \pm t) + \right.$$

$$\left. + \sum_{j=0}^{i-1} F_j^{\pm}(r) \overset{i-1-j}{u}(x) \right\} dr = A^{\pm}(t, \lambda) \overset{i}{u}(x) + \int_0^t A^{\mp}(t - r, \lambda) P(x \pm r) \overset{i}{u}(x \pm r) dr +$$

$$+ \sum_{j=0}^{i-1} \int_0^t A^{\mp}(t - r, \lambda) F_j^{\pm}(r) \overset{i-1-j}{u}(x).$$

Carrying out successive iteration with respect to $\overset{i}{u}(x; \lambda)$, from the last equality we find

$$\overset{i}{u}(x \pm r) = \left(I + \sum_{k=1}^{\infty} T_{\pm}^k \right) A^{\pm}(t, \lambda) \overset{i}{u}(x) + \sum_{j=0}^{i-1} \times$$

$$\times \left[\left(I + \sum_{k=1}^{\infty} T_{\pm}^k \right) \int_0^t A^{\mp}(t - r, \lambda) F_j^{\pm}(r) dr \right] \times$$

$$\times \overset{i-j-1}{u}(x) = F_0^{\pm}(t)\overset{i}{u}(x,t) + \sum_{j=1}^{i} F_j^{\pm}(t)\overset{i-j}{u}(x) = \sum_{j=0}^{i} F_j^{\pm}(t)\overset{i-j}{u}(x),$$

where the functions $F_0^{\pm}(t)$ are determined by (11), and the functions $F_j^{\pm}(t)$ for $j \geq 1$ by the formulae

$$F_j^{\pm}(t) = \left(I + \sum_{k=1}^{\infty} T_{\pm}^k \right) \int_0^t A^{\mp}(t-r,\lambda) F_{j-1}^{\pm}(r) dr, \tag{12}$$

The formal derivation of formula (2) is completed.

For grounding formula (2) it suffices to show that series (11) and (12) converge absolutely and uniformly for $0 < t \leq R^*$.

Denote

$$\omega(t) = 32 \max \left\{ \sup_{c<x<d-t} \int_x^{x+t} \|P(\tau)\| \, d\tau, \sup_{c+t \leq x<d} \int_{x-t}^{x} \|P(\tau)\| \, d\tau \right\}, \ t \leq \frac{d-c}{4}.$$

Choose the number $R^* \leq \dfrac{d-c}{4}$ so that $\omega(R^*) < 1$. Then for an arbitrary $t \in (0, R^*]$ the inequality $\omega(t) < 1$ will be fulfilled.

By definition of the quantity $\omega(t)$ we have:

$$\|T_{\pm}\psi(t)\| \leq \int_0^t \left\| A^{\mp}(t-r,\lambda) \right\| \|P(x \pm r)\| \|\psi(r)\| \, dr \leq$$

$$\leq \int_0^t \|P(x \pm r)\| \left\{ 4ch(Jm\lambda(t-r)) \|\psi(r)\| \right\} dr \leq$$

$$\leq 4 \sup_{0 \leq r \leq t} \left\{ ch\left(Jm\lambda(t-r)\right) \|\psi(r)\| \right\} \int_0^t \|P(x \pm r)\| \, dr \leq$$

$$\leq \frac{1}{8}\omega(t) \sup_{0 \leq r \leq t} \left\{ ch\left(Jm\lambda(t-r)\right) \|\psi(r)\| \right\}.$$

Therefore

$$\left\| T_{\pm}A^{\pm}(t,\lambda) \right\| \leq \frac{1}{8}\omega(t) \sup_{0 \leq r \leq t} \left\{ ch\left(Jm\lambda(t-r)\right) \left\| A^{\pm}(r,\lambda) \right\| \right\} \leq$$

$$\leq \frac{1}{8}\omega(t) \sup_{0 \leq r \leq t} \left\{ ch\left(Jm\lambda(t-r)\right) 4ch(Jm\lambda t) \right\} \leq \frac{1}{2}\omega(t) ch(Jm\lambda t),$$

$$\left\| T_{\pm}^2 A^{\pm}(t,\lambda) \right\| = \left\| T_{\pm}\left(T_{\pm}A^{\pm}(t,\lambda) \right) \right\| \leq$$

$$\leq \frac{1}{8}\omega(t) \sup_{0 \leq r \leq t} \left\{ ch\left(Jm\lambda(t-r)\right) \left\| T_{\pm}A^{\pm}(r,\lambda) \right\| \right\} \leq$$

$$\leq \frac{1}{8}\omega(t)\sup_{0\leq r\leq t}\left\{ch\left(Jm\lambda(t-r)\right)\frac{1}{2}\omega(r)ch(Jm\lambda r)\right\}\leq$$

$$\leq \frac{1}{16}\omega^2(t)ch(Jm\lambda t)\leq\left(\frac{1}{2}\omega(t)\right)^2 ch(Jm\lambda t).$$

Continuing this process, we get

$$\left\|T_{\pm}^k A^{\pm}(t,\lambda)\right\|\leq\left(\frac{1}{2}\omega(t)\right)^k ch(Jm\lambda t), \tag{13}$$

where $k=1,2,...$

Applying these estimations, from (11) we find that for $t\in(0,R^*]$,

$$\left\|F_0^{\pm}(t)-\cos\lambda tI\pm\sin\lambda tB\right\|\leq\sum_{k=1}^{\infty}\left(\frac{1}{2}\omega(R^*)\right)^k\exp\left(|Jm\lambda|\,R^*\right)\leq\exp\left(|Jm\lambda|\,R^*\right);$$

$$\left\|F_0^{\pm}(t)-\cos\lambda tI\pm\sin\lambda tB\right\|=\left\|F_0^{\pm}(t)-A^{\pm}(t,\lambda)\right\|=\left\|\sum_{k=1}^{\infty}T_{\pm}^k A^{\pm}(t,\lambda)\right\|\leq$$

$$\leq\sum_{k=1}^{\infty}\left(\frac{1}{2}\omega(t)\right)^k ch(Jm\lambda t)\leq\omega(t)ch(Jm\lambda t).$$

Estimation (3) is proved.

Now, investigate series (12) by the mathematical induction method and prove estimation (4). Estimation (4) for $j=0$ directly follows from estimation (3). Let estimation (4) be valid for $j-1$, prove it for j.

$$\left\|F_j^{\pm}(t)\right\|\leq\left\|\int_0^t A^{\mp}(t-r,\lambda)F_{j-1}^{\pm}(r)dr\right\|+\sum_{k=1}^{\infty}\left\|T_{\pm}^k\int_0^t A^{\mp}(t-r,\lambda)F_{j-1}^{\pm}(r)dr\right\|\leq$$

$$\leq\int_0^t 4ch(Jm\lambda(t-r)5(8t)^{j-1}ch(Jm\lambda r)dr+$$

$$+\sum_{k=1}^{\infty}\left\|T_{\pm}^k\left[\int_0^t A^{\mp}(t-r,\lambda)F_{j-1}^{\pm}(r)dr\right]\right\|\leq$$

$$\leq\frac{5}{2}(8t)^j ch(Jm\lambda t)+\sum_{k=1}^{\infty}\left\|T_{\pm}^k\left[\int_0^t A^{\mp}(t-r,\lambda)F_{j-1}^{\pm}(r)dr\right]\right\|. \tag{14}$$

By definition of the operators $T_{\pm}$ and induction supposition

$$\left\|T_{\pm}\left[\int_0^t A^{\mp}(t-r)F_{j-1}^{\pm}(r)dr\right]\right\|\leq\frac{1}{8}\omega(t)\sup_{0\leq r\leq t}\left\{ch\left(Jm\lambda(t-r)\right)\times\right.$$

$$\times \left\| \int_0^r A^{\mp}(r-\tau)F_{j-1}^{\pm}(r)dr \right\| \leq \frac{1}{8}\omega(t) \sup_{0\leq r\leq t} \{ch\left(Jm\lambda(t-r)\right) \times$$

$$\left. \times \int_0^r 4ch\left(Jm\lambda(r-\tau)\right) 5(8\tau)^{j-1} chJm\lambda\tau) d\tau \right\} \leq \frac{1}{8}\omega(t) \sup_{0\leq r\leq t} \times$$

$$\times \left\{ ch\left(Jm\lambda(t-r)\right) 20rchJm\lambda r)(8r)^{j-1} \right\} \leq \frac{1}{8}\omega(t) 20tch(Jm\lambda t)(8t)^{j-1} \leq$$

$$\leq \frac{5}{2}(8t)^j \left(\frac{\omega(t)}{2}\right) ch(Jm\lambda t),$$

$$\left\| T_{\pm}^k \left[\int_0^t A^{\mp}(t-r,\lambda) F_{j-1}^{\pm}(r)dr \right] \right\| \leq \frac{5}{2}(8t)^j \left(\frac{\omega(t)}{2}\right)^k ch(Jm\lambda t).$$

Allowing for these inequalities, from (14) we get

$$\left\| F_j^{\pm}(t) \right\| \leq \frac{5}{2}(8t)^j ch(Jm\lambda t) \left[1 + \frac{1}{2}\omega(t) + (\frac{1}{2}\omega(t))^2 + ... \right] \leq$$
$$\leq 5(8t)^j ch(Jm\lambda t). \tag{15}$$

Estimation (4) is proved. Uniformity of convergence of series (12) follows from (15) for $t = R^*$ and $\omega(R^*) < 1$. So, derivation of shift formula (2) is completed.

It follows from the obtained estimations (3), (4) and shift formula (2) that all the solutions of the equation $Du - \lambda u = f$ with absolute ℓ_y continuous on $[a,b]$ right hand side and summable on (a,b) coefficient $P(x)$ have finite limits as $x \to a+0$ and as $x \to b-0$. Therefore, all of them will be absolutely continuous on the closed interval $[a,b]$. Consequently, the eigen and adjoint functions of the operator D will be absolutely continuous on $\overline{G} = [a,b]$, $mesG < \infty$. If $P(x) \in L_p(G)$, $p \geq 1$, then the belonging of the components of the root functions in $W_p^1(G)$, $G = (a,b)$, $mesG < \infty$ follows from the equation $D\overset{\ell}{u} - \lambda\overset{\ell}{u} = \overset{\ell-1}{u}$.

3. Proof of theorem 2

At first we establish the right hand side of estimation (5). It is known that (see [10], estimation (12)) that for $p \geq 1$ it is valid

$$\left\| \overset{\ell}{u} \right\|_{\infty,2,K} \leq C(K,\ell)(1+|Jm\lambda|)^{\frac{1}{p}} \left\| \overset{\ell}{u} \right\|_{p,2,K}. \tag{16}$$

Using estimation (16) and considering $p < s$, we find

$$\left\| \overset{\ell}{u} \right\|_{s,2,K} = \left(\int_K \left| \overset{\ell}{u}(x) \right|^{s-p} \left| \overset{\ell}{u}(x) \right|^p dx \right)^{\frac{1}{s}} \leq \left\| \overset{\ell}{u} \right\|_{\infty,2,K}^{\frac{s-p}{s}} \left(\int_K \left| \overset{\ell}{u}(x) \right|^p dx \right)^{\frac{1}{s}} \leq$$

$$\leq (C(K,\ell))^{\frac{s-p}{s}} [1+|Jm\lambda|]^{\frac{s-p}{ps}} \left\|\overset{\ell}{u}\right\|^{\frac{s-p}{s}}_{p,2,K} \left\|\overset{\ell}{u}\right\|^{\frac{p}{s}}_{p,2,K} =$$

$$= (C(K,\ell))^{\frac{s-p}{s}} [1+|Jm\lambda|]^{\frac{1}{p}-\frac{1}{s}} \left\|\overset{\ell}{u}\right\|_{p,2,K}.$$

Hence, it follows the right hand side of estimation (5), i.e.

$$[1+|Jm\lambda|]^{\frac{1}{s}-\frac{1}{p}} \left\|\overset{\ell}{u}\right\|_{s,2,K} \leq C_2(K,\ell) \left\|\overset{\ell}{u}\right\|_{p,2,K}, \tag{17}$$

where $C_2(K,\ell) = (C(K,\ell))^{\frac{s-p}{s}}$.

Now, establish the left hand side of estimation (5). For that we prove the following statement.

Statement. *For any $\ell = 0,1,...$ on any segment $[\alpha,\beta] \subset [a,b] = K$, for which $\|\|P(x)\|\|_{L_1[\alpha,\beta]} \leq \frac{1}{2^{\ell+2}}$, it is valid the inequality*

$$m_{\ell,\alpha,\beta} \leq M_{\ell,\alpha,\beta} \left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}, \tag{18}$$

where

$$m_{\ell,\alpha,\beta} \equiv \max_{x\in[\alpha,\beta]} \left\{ \left|\overset{\ell}{u}(x)\right| (1+|Jm\lambda|\, d_{\alpha,\beta}(x))^{-\ell} \exp\left(|Jm\lambda|\, d_{\alpha,\beta}(x)\right) \right\},$$

$$d_{\alpha,\beta}(x) \equiv \min\left\{|x-\alpha|, |x-\beta|\right\}.$$

Proof of the statement. Apply the mathematical induction method. Let $\ell = 0$, and $[\alpha,\beta]$ be an arbitrary segment contained in $K = [a,b]$, moreover $\|\|P(x)\|\|_{L_1[\alpha,\beta]} \leq \frac{1}{4}$. Let the maximum of the left hand side of (18) be attained at the point $y \in [\alpha,\beta]$, and $t = d_{\alpha,\beta}(y)$. Using the inequality

$$\exp\left(|Jm\lambda|\right) - 1 \leq |2\cos\lambda t|$$

and the mean value formula (9), we find

$$m_{\ell,\alpha,\beta}(1+|Jm\lambda|\, t)^{\ell} - \left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]} \leq \left|\overset{\ell}{u}(y)\right| (\exp(|Jm\lambda|\, t) - 1) \leq$$

$$\leq |2\cos\lambda t| \left|\overset{\ell}{u}(y)\right| = \left|2\overset{\ell}{u}(y)\cos\lambda t\right| \leq \left|\overset{\ell}{u}(y-t) + \overset{\ell}{u}(y+t)\right| +$$

$$+ \left| \int\limits_{y-t}^{y+t} (\sin\lambda(t-|y-\xi|)\cdot I + sgn(\xi-y)\cos\lambda(t-|y-\xi|\cdot B)\times \right.$$

$$\left. \times \left(P(\xi)\overset{\ell}{u}(\xi) + \overset{\ell-1}{u}(\xi)\right) d\xi \right| \leq 2 \left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]} +$$

$$+2\,\|\|P(x)\|\|_{L_1[\alpha,\beta]} \max_{|y-\xi|<t} \left\{ \left|\overset{\ell}{u}(\xi)\right| (1+\exp(|Jm\lambda|\,(t-|y-\xi|))) \right\} +$$

$$+4t\max_{|y-\xi|<t}\left\{\left|\overset{\ell-1}{u}(\xi)\right|(1+\exp(|Jm\lambda|(t-|y-\xi|)))\right\}\leq 2\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+$$
$$+2\,\|\|P(x)\|\|_{L_1[\alpha,\beta]}\left\{\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+m_{\ell,\alpha,\beta}(1+2\,|Jm\lambda|\,t)^{\ell}\right\}+$$
$$+4t\left\{\left\|\overset{\ell-1}{u}\right\|_{\infty,2,[\alpha,\beta]}+m_{\ell-1,\alpha,\beta}(1+2\,|Jm\lambda|\,t)^{\ell-1}\right\},\tag{19}$$

For $\ell=0$, (19) yields (allowing for $\overset{-1}{u}(x)\equiv 0$)

$$m_{0,\alpha,\beta}-\left\|\overset{0}{u}\right\|_{\infty,2,[\alpha,\beta]}\leq 2\left\|\overset{0}{u}\right\|_{\infty,2,[\alpha,\beta]}+\frac{1}{2}\left\{\left\|\overset{0}{u}\right\|_{\infty,2,[\alpha,\beta]}+m_{0,\alpha,\beta}\right\}.$$

Hence we get

$$m_{0,\alpha,\beta}\leq 7\left\|\overset{0}{u}\right\|_{\infty,2,[\alpha,\beta]}.$$

Now, let inequality (18) be valid for some $\ell-1,\ \ell\geq 2$. Establish inequality (18) for the case ℓ. For that we fix an arbitrary segment $[\alpha,\beta]\subset[a,b]$ for which $\|\|P(x)\|\|_{L_1[\alpha,\beta]}\leq\dfrac{1}{2^{\ell+2}}$. Then from (19) we find

$$\begin{gathered}m_{\ell,\alpha,\beta}\,(1+|Jm\lambda|\,t)^{\ell}-\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}\leq 2\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+\\+\frac{1}{2^{\ell+2}}\left\{\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+m_{\ell,\alpha,\beta}\,(1+2\,|Jm\lambda|\,t)^{\ell}\right\}+\\+4t\left\{\left\|\overset{\ell-1}{u}\right\|_{\infty,2,[\alpha,\beta]}+m_{\ell-1,\alpha,\beta}\,(1+2\,|Jm\lambda|\,t)^{\ell-1}\right\}.\end{gathered}\tag{20}$$

Since $\dfrac{1}{2^{\ell+1}}\geq\dfrac{1}{2^{\ell+2}}\geq\|\|P(x)\|\|_{L_1[\alpha,\beta]}$, then the inequality

$$m_{\ell-1,\alpha,\beta}\leq M_{\ell-1,\alpha,\beta}\left\|\overset{\ell-1}{u}\right\|_{\infty,2,[\alpha,\beta]}$$

will be valid for $m_{\ell-1,\alpha,\beta}$. On the other hand, it is known that (see [10])

$$\left\|\overset{\ell-1}{u}\right\|_{\infty,2,[\alpha,\beta]}\leq C(\ell,\alpha,\beta)(1+|Jm\lambda|)\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}.\tag{21}$$

Therefore, from (20) we get

$$m_{\ell,\alpha,\beta}\,(1+|Jm\lambda|\,t)^{\ell}\leq 3\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+$$
$$+\frac{1}{2^{\ell+1}}\left\{\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+m_{\ell,\alpha,\beta}\,(1+2\,|Jm\lambda|\,t)^{\ell}\right\}+$$
$$+4tC(\ell,\alpha,\beta)(1+|Jm\lambda|)\left\{1+M_{\ell-1,\alpha,\beta}(1+2\,|Jm\lambda|\,t)^{\ell-1}\right\}\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}\leq$$
$$\leq\left(3+\frac{1}{2^{\ell+2}}\right)\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+\frac{1}{2}m_{\ell,\alpha,\beta}\,(1+|Jm\lambda|\,t)^{\ell}+$$
$$+2^{\ell+3}C(\ell,\alpha,\beta)(t+|Jm\lambda|\,t)(1+M_{\ell-1,\alpha,\beta})(1+|Jm\lambda|\,t)^{\ell-1}\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}\leq$$

$$\leq \left\{3+\frac{1}{2^{\ell+2}}+2^{\ell+3}C(\ell,\alpha,\beta)(1+\beta-\alpha)(1+M_{\ell-1,\alpha,\beta})\right\}\times$$

$$\times\left(1+|Jm\lambda|\,t\right)^{\ell}\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}+\frac{1}{2}m_{\ell,\alpha,\beta}\left(1+|Jm\lambda|\,t\right)^{\ell}$$

Having cancelled each side of the last relation by $(1+|Jm\lambda|\,t)^{\ell}$, we find

$$m_{\ell,\alpha,\beta}\leq\frac{1}{2}m_{\ell,\alpha,\beta}+$$

$$+\left\{3+\frac{1}{2^{\ell+1}}+2^{\ell+3}C(\ell,\alpha,\beta)(1+\beta-\alpha)(1+M_{\ell-1,\alpha,\beta})\right\}\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}.$$

Consequently,

$$m_{\ell,\alpha,\beta}\leq M_{\ell,\alpha,\beta}\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]},$$

where

$$M_{\ell,\alpha,\beta}=6+\frac{1}{2^{\ell}}+2^{\ell+4}C(\ell,\alpha,\beta)(1+\beta-\alpha)(1+M_{\ell-1,\alpha,\beta}).$$

The statement is proved.

It follows from the proved statement that there exists a constant $M^{1}_{\ell,\alpha,\beta}$ such that for any $x\in[\alpha,\beta]$.

$$\left|\overset{\ell}{u}(x)\right|\leq M^{1}_{\ell,\alpha,\beta}\exp\left\{-\frac{1}{2}\,|Jm\lambda|\,d_{\alpha,\beta}(x)\right\}\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha,\beta]}. \qquad (22)$$

Partition the segment $[a,b]$ into the finite number $n(\ell)$ equal segments $[\alpha_i,\beta_i]$ so that the inequality $\|\|P(x)\|\|_{L_1[\alpha_i,\beta_i]}\leq\frac{1}{2^{\ell+2}}$ be fulfilled for each of them.

Then, applying inequality (22), we find

$$\left\|\overset{\ell}{u}\right\|^{p}_{p,2,[a,b]}=\sum_{i=1}^{n(\ell)}\left\|\overset{\ell}{u}\right\|^{p}_{p,2,[\alpha_i,\beta_i]}\leq\sum_{i=1}^{n(\ell)}\left(\left(M^{1}_{\ell,\alpha_i,\beta_i}\right)\left\|\overset{\ell}{u}\right\|_{\infty,2,[\alpha_i,\beta_i]}\right)^{p}$$

$$\int_{\alpha_i}^{\beta_i}\exp\left(-\frac{p}{2}\,|Jm\lambda|\,d_{\alpha_i,\beta_i}(x)\right)dx\leq\frac{4}{p\,|Jm\lambda|}\left\|\overset{\ell}{u}\right\|^{p}_{\infty,2,[a,b]}\sum_{i=1}^{n(\ell)}\left(M^{1}_{\ell,\alpha_i,\beta_i}\right).$$

Hence for $|Jm\lambda|\geq1$, we get

$$\left\|\overset{\ell}{u}\right\|_{p,2,[a,b]}\leq\left(\sum_{i=1}^{n(\ell)}\left(M^{1}_{\ell,\alpha_i,\beta_i}\right)^{p}\right)^{\frac{1}{p}}4^{\frac{1}{p}}p^{\frac{1}{p}}\,|Jm\lambda|^{\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{\infty,2,[a,b]}\leq$$

$$\leq 8(1+|Jm\lambda|)^{\frac{1}{p}}\left(\sum_{i=1}^{n(\ell)}\left(M_{\ell,\alpha_i,\beta_i}\right)^{p}\right)^{\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{\infty,2,[a,b]}.$$

Consequently, for $|Jm\lambda|\geq1$, the estimation

$$\left\|\overset{\ell}{u}\right\|_{p,2,K}\leq M(p,\ell)(1+|Jm\lambda|)^{-\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{\infty,2,K}$$

is fulfilled.

And for $|Jm\lambda| < 1$,

$$\left\|\overset{\ell}{u}\right\|_{p,2,K} \leq (b-a)^{\frac{1}{p}} \left\|\overset{\ell}{u}\right\|_{p,2,K} \leq 2(1+b-a)(1+|Jm\lambda|)^{-\frac{1}{p}} \left\|\overset{\ell}{u}\right\|_{\infty,2,K}.$$

is valid.

It follows from the last two inequalities that

$$\left\|\overset{\ell}{u}\right\|_{p,2,K} \leq M_1(p,\ell)(1+|Jm\lambda|)^{-\frac{1}{p}} \left\|\overset{\ell}{u}\right\|_{\infty,2,K}.$$

Hence, allowing for (17), for $s=\infty, p=s$, we get the estimation

$$\left\|\overset{\ell}{u}\right\|_{p,2,K} \leq M_2(p,\ell)(1+|Jm\lambda|)^{\frac{1}{s}-\frac{1}{p}} \left\|\overset{\ell}{u}\right\|_{\infty,2,K}.$$

for $p < s$. The left hand side of estimation (5) is established.

Estimation (6) follows from estimations (5) and (21). Really,

$$\left\|\overset{\ell-1}{u}\right\|_{p,2,K} \leq C_1^{-1}\left[1+|Jm\lambda|\right]^{-\frac{1}{p}} \left\|\overset{\ell-1}{u}\right\|_{\infty,2,K} \leq$$

$$\leq C_1^{-1}C(\ell,K)\left(1+|Jm\lambda|\right)^{-\frac{1}{p}+1} \left\|\overset{\ell}{u}\right\|_{\infty,2,K} \leq C_1^{-1}C(\ell,K)\left(1+|Jm\lambda|\right)^{-\frac{1}{p}+1} \times$$

$$\times C_2\left(1+|Jm\lambda|\right)^{\frac{1}{p}} \left\|\overset{\ell}{u}\right\|_{p,2,K} \leq C_3\left(1+|Jm\lambda|\right) \left\|\overset{\ell}{u}\right\|_{p,2,K}.$$

Now, prove estimation (7). Let R^* be the number chosen in theorem 1. Fix an arbitrary number r from the interval $(0,R^*)$ and consider the segments $K=[a,b]$, $K_r=[a+r,b-r]$, $r \leq \min\left\{\frac{b-a}{4}; \frac{1}{16}\right\}$. Applying the shift formula (2) and estimation (4), we have:

$$\left\|\overset{\ell}{u}\right\|_{p,2,K} \leq \left\|\overset{\ell}{u}(\cdot+r)\right\|_{p,2,K_r} + \left\|\overset{\ell}{u}(\cdot-r)\right\|_{p,2,K_r} \leq$$

$$\leq \sum_{i=0}^{\ell}\left\{\left\|F_j^+(r)\right\| + \left\|F_j^-(r)\right\|\right\} \left\|\overset{\ell-j}{u}\right\|_{p,2,K_r} \leq$$

$$\leq \sum_{i=0}^{\ell} 10(8r)^j ch(rJm\lambda) \left\|\overset{\ell-j}{u}\right\|_{p,2,K_r} \leq 10\exp(r\,|Jm\lambda|)\sum_{i=0}^{\ell} 2^{-j} \left\|\overset{\ell-j}{u}\right\|_{p,2,K_r}.$$

Consequently, it is valid the inequality

$$\left\|\overset{\ell}{u}\right\|_{p,2,K} \leq 10\exp(r\,|Jm\lambda|)\sum_{i=0}^{\ell} 2^{-j} \left\|\overset{\ell-j}{u}\right\|_{p,2,K_r}, \tag{23}$$

where $\ell = 0,1,...; p \geq 1$.

Let $\ell = 0$. Then it follows from (23) that

$$\left\|\overset{0}{u}\right\|_{p,2,K} \leq 10\exp(r\,|Jm\lambda|) \left\|\overset{0}{u}\right\|_{p,2,K_r}. \tag{24}$$

For $\ell = 1$, using (24) and (6), from (23) we get

$$\left\|\overset{1}{u}\right\|_{p,2,K} \leq 10\exp(r\,|Jm\lambda|)\sum_{i=0}^{1}2^{-j}\left\|\overset{1-j}{u}\right\|_{p,2,K_r} \leq$$

$$\leq 10\exp(r\,|Jm\lambda|)\left\{\left\|\overset{1}{u}\right\|_{p,2,K_r}+\frac{1}{2}\left\|\overset{0}{u}\right\|_{p,2,K_r}\right\} \leq$$

$$\leq 10\exp(r\,|Jm\lambda|)\left\{1+\frac{1}{2}C_3[1+|Jm\lambda|]\right\}\left\|\overset{1}{u}\right\|_{p,2,K_r} \leq$$

$$\leq 10(1+\frac{1}{2}C_3)(1+|Jm\lambda|)\exp(r\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_r}. \tag{25}$$

Continuing this process, we get the left hand side of estimation (7) for the segments K and K_r, i.e.

$$\left\|\overset{\ell}{u}\right\|_{p,2,K} \leq C(K,K_r)[1+|Jm\lambda|)]^{\ell}\exp(r\,|Jm\lambda|)\left\|\overset{\ell}{u}\right\|_{p,2,K_r}, \tag{26}$$

Now, let R be an arbitrary fixed number satisfying the condition $0 < R < < \frac{b-a}{2}$. Choose a natural number k so that $r = \frac{R}{k} \in (0, R^*]$ and $r \leq \min\left\{\frac{b-a}{4}; \frac{1}{16}\right\}$. Consider the segments $K = [a,b]$, $K_r, K_{2r}, \ldots, K_{kr} = K_R$. Applying k times the inequality (23) for $\ell = 0$, we get

$$\left\|\overset{0}{u}\right\|_{p,2,K} \leq 10^k\exp(kr\,|Jm\lambda|)\left\|\overset{0}{u}\right\|_{p,2,K_{kr}} = 10^k\exp(R\,|Jm\lambda|)\left\|\overset{0}{u}\right\|_{p,2,K_R}. \tag{27}$$

Applying two times repeatedly inequality (23) for $\ell = 1$ and considering (24), (25), for K_r, K_{2r} we find

$$\left\|\overset{1}{u}\right\|_{p,2,K} \leq 10\exp(r\,|Jm\lambda|)\sum_{i=0}^{1}2^{-j}\left\|\overset{1-j}{u}\right\|_{p,2,K_r} \leq$$

$$\leq 10\exp(r\,|Jm\lambda|)\left\{\left\|\overset{1}{u}\right\|_{p,2,K_r}+\frac{1}{2}\left\|\overset{0}{u}\right\|_{p,2,K_r}\right\} \leq$$

$$\leq 10\exp(r\,|Jm\lambda|)\left\{10\left(1+\frac{1}{2}C_3\right)(1+|Jm\lambda|)\exp(r\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_{2r}}+\right.$$

$$\left.+\frac{1}{2}10\exp(r\,|Jm\lambda|)\left\|\overset{0}{u}\right\|_{p,2,K_{2r}}\right\} \leq$$

$$\leq 10^2\exp(r\,|Jm\lambda|)\left(1+\frac{C_3(K_r)}{2}+\frac{C_3(K_{2r})}{2}\right)(1+|Jm\lambda|)\exp(r\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_{2r}} \leq$$

$$\leq C(K,K_{2r})(1+|Jm\lambda|)\exp(2r\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_{2r}}.$$

Thus,

$$\left\|\overset{1}{u}\right\|_{p,2,K} \leq C(K,K_{2r})(1+|Jm\lambda|)\exp(2r\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_{2r}}, \tag{28}$$

In order to establish the estimation between the norms $\left\|\overset{1}{u}\right\|_{p,2,K}$ and $\left\|\overset{1}{u}\right\|_{p,2,K_{3r}}$, we must write inequality (23) for $\ell=1$, and estimate the norms $\left\|\overset{0}{u}\right\|_{p,2,K_r}$ in it by means of inequality (27) by $\left\|\overset{0}{u}\right\|_{p,2,K_{3r}}$, estimate $\left\|\overset{1}{u}\right\|_{p,2,K_r}$ with the help of inequality (28) by $\left\|\overset{1}{u}\right\|_{p,2,K_{3r}}$, and then apply anti a priori estimation (6) to $\left\|\overset{0}{u}\right\|_{p,2,K_{3r}}$. As a result of these operations we get the estimation

$$\left\|\overset{1}{u}\right\|_{p,2,K}\leq C(K,K_{3r})(1+|Jm\lambda|)\exp(3r\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_{3r}}.$$

Continuing this process to the segment K_{kr}, we get the estimation

$$\left\|\overset{1}{u}\right\|_{p,2,K}\leq C(K,K_r)(1+|Jm\lambda|)\exp(R\,|Jm\lambda|)\left\|\overset{1}{u}\right\|_{p,2,K_R}$$

Applying sequentially the above mentioned scheme for $\ell=2,\ \ell=3,...$ we get validity of the left estimation of (7) for the segments K and $K_R=K_{kr}$.

Now, establish the right hand side of estimation (7). Write the mean value formula obtained from shift formula (2):

$$\overset{\ell}{u}(x+r)+u(x-r)=2\cos\lambda r\overset{\ell}{u}(x)+\left[F_0^+(r)-\cos\lambda r\cdot I+\sin\lambda r\cdot B\right]\times$$
$$\times\overset{\ell}{u}(x)+\left[F_0^-(r)-\cos\lambda r\cdot I-\sin\lambda r\cdot B\right]\overset{\ell}{u}(x)+$$
$$+\sum_{j=1}^{\ell}\left[F_j^+(r)+F_j^-(r)\right]\overset{\ell-j}{u}(x),\ x\in K_r,\ 0<r\leq R^*,\ K=[a,b]$$

Fixing r, applying the inequality $chJmz\leq 2\left|\cos z\right|,\ \ |Jmz|\geq 1$ and estimations (3), (4), from the mean value formula we find that for $|Jm\lambda|\geq\dfrac{1}{r}$

$$\frac{ch\,(rJm\lambda)}{2}\left|\overset{\ell}{u}(x)\right|\leq\frac{\left|\overset{\ell}{u}(x+r)\right|+\left|\overset{\ell}{u}(x-r)\right|}{2}+ch\,(rJm\lambda)\,\omega(r)\left|\overset{\ell}{u}(x)\right|+$$
$$+5\sum_{j=1}^{\ell}(8r)^j ch\,(rJm\lambda)\left|\overset{\ell-1}{u}(x)\right|.$$

Hence, in its turn we find

$$ch\,(rJm\lambda)\left(\frac{1}{2}-\omega(r)\right)\left|\overset{\ell}{u}(x)\right|\leq\frac{1}{2}\left\{\left|\overset{\ell}{u}(x+r)\right|+\left|\overset{\ell}{u}(x-r)\right|\right\}+$$
$$+5ch\,(rJm\lambda)\sum_{j=1}^{\ell}(8r)^j\left|\overset{\ell-1}{u}(x)\right|.$$

Choose the fixed r so that $\omega(r)\leq\dfrac{1}{4}$ be fulfilled. Then it follows from the last relation that

$$ch\,(rJm\lambda)\left|\overset{\ell}{u}(x)\right|\leq\left|\overset{\ell}{u}(x-r)\right|+\left|\overset{\ell}{u}(x+r)\right|+10ch\,(rJm\lambda)\sum_{j=1}^{\ell}(8r)^j\left|\overset{\ell-1}{u}(x)\right|$$

Hence we find

$$ch\,(rJm\lambda)\left|\overset{\ell}{u}(x)\right|_{p,2,K_r}\leq 2\left|\overset{\ell}{u}(x)\right|_{p,2,K}+10ch\,(rJm\lambda)\sum_{j=1}^{\ell}(8r)^j\left\|\overset{\ell-1}{u}(x)\right\|_{p,2,K_r}.$$

Here, applying the inequality $\frac{1}{2}\exp\left(|Jm\lambda|\,r\right)\leq ch\,(rJmz)\leq\exp\left(|Jm\lambda|\,r\right)$ and requiring $r\leq\frac{1}{320}$, we get

$$\left\|\overset{\ell}{u}(x)\right\|_{p,2,K_r}\leq 4\left\|\overset{\ell}{u}(x)\right\|_{p,2,K}\exp(-|Jm\lambda|\,r)+\sum_{j=1}^{\ell}2^{-j}\left\|\overset{\ell-j}{u}(x)\right\|_{p,2,K_r}. \tag{29}$$

From (29) for $\ell=0$ it follows the right hand side of estimation [7] for the eigen function $\overset{0}{u}(x)$ for K and K_r, i.e.

$$\left\|\overset{0}{u}(x)\right\|_{p,2,K_r}\leq 4\left\|\overset{0}{u}\right\|_{p,2,K}\exp(-|Jm\lambda|\,r), \tag{30}$$

For $\ell=1,$ applying (29), (30) and a priori estimation (6), we get

$$\left\|\overset{1}{u}\right\|_{p,2,K_r}\leq 4\left\|\overset{1}{u}\right\|_{p,2,K}\exp(-|Jm\lambda|\,r)+\frac{1}{2}\left\|\overset{0}{u}\right\|_{p,2,K_r}\leq 4\exp(-|Jm\lambda|\,r)\left\|\overset{1}{u}\right\|_{p,2,K}+$$

$$+2\left\|\overset{0}{u}\right\|_{p,2,K}\exp(-|Jm\lambda|\,r)\leq 4\exp(-|Jm\lambda|\,r)\left\|\overset{1}{u}\right\|_{p,2,K}+$$

$$+2C_3(K)(1+|Jm\lambda|)\exp(-|Jm\lambda|\,r)\left\|\overset{1}{u}\right\|_{p,2,K}\leq$$

$$\leq C(K,K_r)(1+|Jm\lambda|)\exp(-|Jm\lambda|\,r)\left\|\overset{1}{u}\right\|_{p,2,K}.$$

Continuing this process for $\ell=2,\ \ell=3,...,$ we get the estimation

$$\left\|\overset{\ell}{u}\right\|_{p,2,K_r}\leq C(K,K_r,\ell)(1+|Jm\lambda|)^{\ell}\exp(-|Jm\lambda|\,r)\left\|\overset{1}{u}\right\|_{p,2,K}.$$

Consequently, the right hand side of estimation (7) is valid in the case of the segments K and K_r (in the case $|Jm\lambda|<\frac{1}{r}$ this estimation follows from the inequality $\left\|\overset{\ell}{u}\right\|_{p,2,K_r}\leq\left\|\overset{\ell}{u}\right\|_{p,2,K}$ and boundedness of the quantity $\exp\left(|Jm\lambda|\,r\right)/(1+|Jm\lambda|^{\ell})$.

To prove the right hand side of estimation (7) for the segments K and K_R, $R<\frac{b-a}{2}$, we choose a natural number k so that $r=\frac{R}{k}\in(0,R^*)\,,\ \omega(r)\leq\frac{1}{4},\ r\leq\frac{1}{320}$.

Consider the case $|Jm\lambda|\geq\frac{1}{r}$ (for $|Jm\lambda|<\frac{1}{r}$ the right hand side of estimation (7) is fulfilled trivially). Fixing k (the same number r), again we consider the segments $K=[a,b],\ [K_{j,r}=[a+jr,\ b-jr],\ j=\overline{1,k}$.

For the function $\overset{0}{u}(x)$, the right hand side of estimation (7) is obtained by k times successive application of estimation (30). Derivation of the right hand side of estimation (7) in the cases $\ell=1,2,...$ is the same as in derivation of the left hand side of estimation (7). Thereby, inequality (29) and anti a priori estimations (6) are used.

Theorem 2 is proved.

Remark. Notice that estimations (5)-(7) are valid also in the case when the coefficients $P_{12}(x)$, $P_{21}(x)$, are not identically equal to zero, and $P_{ij}(x) \in L_1^{loc}(G)$, $i, j = 1, 2$.

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References

[1]. Il'in V.A. *Necessary and sufficient conditions on basicity and equiconvergence with trigonometric series of spectral expansions.* I // Diff. uravn. 1980. vol. 16, No 5, pp. 771-794 (Russian).

[2]. Tikhomirov V.V. *Exact estimations of regular solutions of Schrodinger one-dimensional equation with spectral parameter.* // DAN SSSR. 1983. vol. 273, No 4, pp. 807-810 (Russian).

[3]. Joo I. *Upper estimates for the eigenfunctions of the Schrodinger operator* // Acta Sci. Math. 1982. vol.44, No1-2, pp. 87-93.

[4]. Komornik V. *Lower estimates for the eigenfunctions of the Schrodinger operator* // Acta sci. Math. 1982. vol. 44, No 1, pp. 95-98.

[5]. Kritskov L.V. *Uniform estimation of order of adjoint functions and distribution of eigen values of Schrodinger's one-dimensional operator* // Diff. Uravn. 1989. vol. 25, No 7, pp. 1121-1129 (Russian).

[6]. Kerimov N.B. *Some properties of eigen and adjoint functions of ordinary differential operators* // DAN SSSR. 1986. vol. 201, No 5, pp. 1054-1056 (Russian).

[7]. Il'in V.A. *Necessary and sufficient conditions on basicity and equiconvergence with trigonometric series of spectral expansions II* // Diff. Uravn. 1980. vol. 16, No 6, pp. 980-1009 (Russian).

[8]. Kurbanov V.M. *Equiconvergence of biorthogonal expansions by the root functions of differential operators I* // Diff. Uravn. 1999. vol. 35, No 12, pp. 1597-1609 (Russian).

[9]. Kurbanov V.M. *Equiconvergence of biorthogonal expansions by the root functions of differential operators II* // Diff. Uravn. 2000. vol.36, No 3, pp. 319-335 (Russian).

[10]. Kurbanov V.M. *On Bessel property and unconditional basicity of the system of the root vector-functions of Dirac operator* // Diff. Uravn. 1996, vol. 32, No 12, pp. 1608-1617 (Russian).

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