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THE UNIQUE STRONG SOLVABILITY OF THE MIXED BOUNDARY VALUE PROBLEM FOR LINEAR NON-DIVERGENT PARABOLIC EQUATIONS OF THE SECOND ORDER IN THE SPACE SOBOLEV

Abstract

The mixed boundary value problem is considered for linear non-divergent parabolic equations of the second order with generally speaking, discontinuous coefficients satisfying Cordes conditions. The one-valued,strongly (almost everywhere) solvability of this problem is proved in the space $\hat{W}_p^{2,1}$, where p belongs to same segment containing the point 2.

Introduction. Let E_n and R_{n+1} be n and $(n+1)$-dimensional Euclidean spaces of the points $x=(x_1,x_2,...,x_n)$ and $(t,x)=(t,x_1,x_2,...,x_n)$ respectively, $\Omega\subset E_n$- be bounded domain with boundary $\partial\Omega\in C^2$, $B_R^{x^0}$-n - dimensional open sphere of the radius R with the centre at the point $x^0=(x_1^0,x_2^0,...,x_n^0)$, $Q_R^{x^0}\times(0,T)$, $Q_T=\{(t,x)|\,0<t<T<\infty, x\in\Omega\}$, $S_T=\{(t,x)|\,0<t<T, x\in\partial\Omega\}$, $\mathcal{A}(Q_R^T)$- be the set of all functions $u(t,x)$ from $C^\infty(\bar{Q}_R^T)$ with support in $B_\rho^{x^0}\times[0,T]$, $\rho<R$, for which $u(0,x)=0$.

Consider in the domain Q_T the mixed boundary value problem for linear parabolic equations of the form

$$\mathcal{L}u=\sum_{i,j=1}^{n}a_{ij}\left(t,x\right)u_{ij}+\sum_{i=1}^{n}b_i\left(t,x\right)u_i-u_t=f\left(t,x\right), \tag{1}$$

$$u|_{t=0}=0,\ \frac{\partial u}{\partial n}|_{S_T}=0, \tag{2}$$

under the assumptions that $\|a_{ij}(t,x)\|$- is a real symmetrical matrix, moreover for all $(t,x)\in Q_T$ and $\xi\in E_n$ the conditions

$$\gamma\left|\xi\right|^2\le\sum_{i,j=1}^{n}a_{ij}(t,x)\xi_i\xi_j\le\gamma^{-1}\left|\xi\right|^2,\gamma\in(0,1]-const, \tag{3}$$

is fulfilled.

Besides we'll suppose that all coefficients of the operator $\mathcal{L}$ are real and measurable in Q_T functions.

The aim of the present paper is finding the conditions on coefficients of equations (1) by fulfilling of which the mixed boundary value problem (1) -(2)on identically strong (almost everywhere) solvable in the space $\hat{W}_p^{2,1}$ for any $f(t,x)\in L_p(Q_T)$, $p\in[p_1,p_2]$, where $p_1\in(1,2)$, $p_2\in(2,\infty)$.

In case, when the leading coefficients of linear operator are uniformly continuous in the cylindrical domain, and minor coefficients are elements of corresponding Lebesgue spaces then uniform strong (almost everywhere) solvability of the Dirichlet and the mixed problems for the parabolic and elliptic equations in the respective space Sobolev is proved in [1,2]. The example indicating the exactness of Cordes conditions is in [3]. In [4],[5] the indicated fact is transported on the class of non-linear parabolic equations the second order, under the more rigid condition than the Cordes condition. Denote that the Dirichlet problem for linear and quasilinear parabolic and elliptic equations the second order non-divergent structure with discontinuous coefficients are studied in [6-12].

1. Some auxiliary assertions. Let agree at first in some notations and definitions.We'll denote by u_t , u_i, and u_{ij} the derivatives $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial x_i}$ and $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$; $i,j=1,...,n$, respectively. Let $W_p^{1,0}(Q_T)$ and $W_p^{2,1}(Q_T)$ be Banach space of the measurable functions $u(t,x)$ given on Q_T with finite norms

$$\|u\|_{W_p^{1,0}(Q_T)} = \left(\int\limits_{Q_T} \left(|u|^p + \sum_{i=1}^{n} |u_i|^p \right) dtdx \right)^{\frac{1}{p}}$$

and

$$\|u\|_{W_p^{2,1}(Q_T)} = \left(\int\limits_{Q_T} \left(|u|^p + \sum_{i=1}^{n} |u_i|^p + \sum_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p \right) dtdx \right)^{\frac{1}{p}},$$

respectively. Denote by $\hat{W}_p^{2,1}(Q_T)$ the subspace $W_p^{2,1}(Q_T)$, in which dense set is collection of all functions from $C^\infty(\bar{Q_T})$ vanishing on $t=0$ and $\frac{\partial u}{\partial n}|_{S_T}=0$. The functions $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ is called strong solvability of the mixed boundary value problem (1)-(2) if it satisfies equation (1) almost everywhere in Q_T.

Further everywhere the note $C(...)$ means that the positive constant C depends only on the contest of parenthesis.

Lemma 1. *If* $u\,(t,x) \in \mathcal{A}\left(Q_R^T\right)$, *then*

$$\int\limits_{Q_R^T} \left(\sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 \right) dtdx \le \int\limits_{Q_R^T} (\mathcal{M}_0 u)^2 dtdx,$$

where $\mathcal{M}_0 = \Delta - \dfrac{\partial}{\partial t}$.

Proof. We have

$$\int\limits_{Q_R^T} (\mathcal{M}_0 u)^2\, dtdx = \int\limits_{Q_R^T} \left((\Delta u)^2 - 2\Delta u\; u_t + u_t^2 \right) dtdx =$$

$$= \int\limits_{Q_R^T} \left(\sum_{i,j=1}^{n} u_{ii}u_{jj} - 2\sum_{i=1}^{n} u_{ii}u_t + u_t^2 \right) dtdx = -\int\limits_{Q_R^T} \sum_{i,j=1}^{n} u_i u_{jji} dtdx +$$

$$+2\int\limits_{Q_R^T} \sum_{i=1}^{n} u_i u_{it} dtdx + \int\limits_{Q_R^T} u_t^2 dtdx = \int\limits_{Q_R^T} \left(\sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 \right) dtdx +$$

$$+\int\limits_{Q_R^T} \sum_{i=1}^{n} \left(u_i^2\right)_t dtdx = \int\limits_{Q_R^T} \left(\sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 \right) dtdx +$$

$$+\int\limits_{B_R^{x^0}} \sum_{i=1}^{n} \left(u_i^2 (T,x) - u_i^2 (0,x)\right) dx.$$

Since $u(0,x) = 0$, then hence it follows the required inequality.

Lemma 2. *If* $u(t,x) \in \mathcal{A}(Q_R^T)$ *and* $p \in (1,\infty)$,*then*

$$\int\limits_{Q_R^T} \left(\sum_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p \right) dtdx \le C_1 (p,n) \int\limits_{Q_R^T} |\mathcal{M}_0 u|^p \, dtdx.$$

Proof. Let

$$F(t,x) = \Delta u(t,x) - u_t(t,x),$$

$$G(t,x) = \begin{cases} a_0 t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t} \right), & \text{at } t > 0, \\ 0, \quad \text{at} \quad t \le 0, & (\text{except for } t = |x| = 0), \end{cases}$$

where $a_0 = 2^{-n}\pi^{-\frac{n}{2}}$. Then

$$u(t,x) = \int\limits_{Q_R^T} G(t-\tau, x-y) F(\tau,y) d\tau dy.$$

For $i = 1,...,n$ we have

$$u_i(t,x) = \int\limits_{Q_R^T} G_i(t-\tau, x-y) F(\tau,y) d\tau dy = \int\limits_{Q_R^T} G_i(t-\tau, y-x) F(\tau,y) d\tau dy =$$

$$= \int\limits_{\mathbb{R}_{n+1}} G_i(t-\tau, \vartheta) F(\tau, \vartheta + x) d\tau d\vartheta.$$

Further acting as at differentiation of integrals with weak singularity [12], we obtain

$$u_{ij}(t,x) = -G_{ij} * F + F(t,x) \lim_{\rho\to 0} \int\limits_{\partial B_{0,1/\rho}^{(t,x)}} G_i(t-\tau, x-y) \cos(\overline{n}, y_j) ds_{\tau,y},$$

where

$$G_{ij}*F=\lim_{\rho\to 0}\int\limits_{B^{(t,x)}_{0,1/\rho}} G_{ij}\left(t-\tau,x-y\right)F\left(\tau,x\right)d\tau dy,$$

$$B^{(t,x)}_{0,1/\rho}=\left\{(\tau,y):0<\frac{G\left(t-\tau,x-y\right)}{t-\tau}<\frac{1}{\rho}\right\},$$

and $\partial B^{(t,x)}_{0,1/p}$− its boundary.

Let's calculate

$$J_{ij}\left(\rho\right)=\int\limits_{\partial B^{(x,t)}_{0,1/\rho}} G_i\left(t-\tau,x-y\right)\cos\left(\bar{n},y_j\right)ds_{\tau,y}=$$

$$=\int\limits_{\partial B^{(0,0)}_{0,1/\rho}} G_i\left(-\tau,-y\right)\cos\left(\bar{n},y_j\right)ds_{\tau,y}=\frac{1}{\rho}\int\limits_{\partial B^{(0,0)}_{0,1/\rho}}\frac{y_i}{2}\cos\left(\bar{n},y_j\right)ds_{\tau,y}.$$

If $i\neq j$, then $J_{ij}=0$. Let now $i=j$. Consider for example the case $i=j=n$ since in all remaining cases the proof is analogous. Denote by S_ρ that part $\partial B^{(t,x)}_{0,1/\rho}$ on which $y_n>0$ and by Π_ρ the projection S_ρ on hyperline $y_n=0$.

Then

$$J_{nn}\left(\rho\right)=\frac{2}{\rho}\int\limits_{S_\rho}\frac{y_n}{2}\cos\left(\bar{n},y_n\right)ds_{\tau,y}=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\frac{y_n}{2}\cos\left(\bar{n},y_n\right)\frac{1}{cos(\overline{n},y_n)}d\tau dy_1...dy_{n-1}=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\frac{y_n}{2}d\tau dy_1...dy_{n-1}=\frac{2}{\rho}\int\limits_{\Pi_\rho}\sum_{i=1}^{n}\frac{y_i^2}{4}d\tau dy_1...dy_{n-1}=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\sqrt{\frac{n+2}{2}\left(-\tau\right)\ln\frac{\left(a_0\rho\right)^{\frac{2}{n+2}}}{-\tau}-\sum_{i=1}^{n-1}\frac{y_i^2}{4}}d\tau dy_1...dy_{n-1}.$$

Let's make change of the variables $u=-\tau\left(a_0\rho\right)^{-\frac{2}{n+2}}$, $\vartheta_i=y_i\left(a_0\rho\right)^{-\frac{1}{n+2}}$; $i=1,...,n-1$. Let Π^+ be image Π_ρ at such transformation. We have

$$J_{nn}\left(\rho\right)=2a_0\int\limits_{\Pi^+}\sqrt{\frac{n+2}{2}u\ln\frac{1}{u}-\sum_{i=1}^{n-1}\frac{\vartheta_i^2}{4}}dud\vartheta_1...d\vartheta_{n-1}=$$

$$=\frac{2^{n+1}}{n+2}\int\limits_0^1\sqrt{\ln\frac{1}{r}}dr\int\limits_{\mathbb{E}_{n-1}}\exp\left[-\sum_{i=1}^{n-1}\xi_i^2\right]d\xi_1...d\xi_{n-1},$$

where $r=\exp\left[\sum\limits_{i=1}^{n-1}\frac{\vartheta_i^2}{4u}-\frac{n+2}{2}\ln\frac{1}{u}\right]$, $\xi_i=\dfrac{\vartheta_i}{2\sqrt{u}}$; $i=1,...,n-1$.

It is easy to see that the last integral is equal to $\frac{1}{n+2}$. Subject to these calculations for u_{ij} we have

$$u_{ij}(t,x) = -G_{ij} * F + \frac{\delta_{ij}}{n+2} F(t,x), \quad i,j = 1,...,n, \tag{4}$$

where δ_{ij} is Cronecker symbol and $G_{ij} * F$ is a parabolic singular integral with the kernel in G_{ij}. By Jones theorem [13] for $p \in (1, \infty)$, $i,j = 1,...,n$

$$\|G_{ij} * F\|_{L_p(Q_R^T)} \le C_{ij}(p,n) \|F\|_{L_p(Q_R^T)}.$$

Subject this inequality in (4) we'll obtain

$$\sum_{i,j=1}^{n} \|u_{ij}\|_{L_p(Q_R^T)} \le C_1(p,n) \|F\|_{L_p(Q_R^T)}. \tag{5}$$

Now let's show that $\|u_t\|_{L_p(Q_R^T)} \le C_2(p,n) \|F\|_{L_p(Q_R^T)}$. Really from the relations $u_t = \Delta u - F$ and (5) we have

$$\|u_t\|_{L_p(Q_R^T)} \le \|\Delta u\|_{L_p(Q_R^T)} + \|F\|_{L_p(Q_R^T)} \le \sum_{i=1}^{n} \|u_{ii}\|_{L_p(Q_R^T)} +$$
$$+ \|F\|_{L_p(Q_R^T)} \le C_2(p,n) \|F\|_{L_p(Q_R^T)}$$

Then

$$\left(\int_{Q_R^T} \left(\sum_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p \right) dtdx \right)^{\frac{1}{p}} \le \left(\int_{Q_R^T} \sum_{i,j=1}^{n} |u_{ij}|^p \, dtdx \right)^{\frac{1}{p}} +$$
$$+ \left(\int_{Q_R^T} |u_t|^p \, dtdx \right)^{\frac{1}{p}} \le \sum_{i,j=1}^{n} \|u_{ij}\|_{L_p(Q_R^T)} + \|u_t\|_{L_p(Q_R^T)} \le$$
$$\le C_3(p,n) \left(\int_{Q_R^T} |\mathcal{M}_0 u|^p \, dtdx \right)^{\frac{1}{p}}.$$

The lemma is proved.

Denote now by $\mathring{W}_p^{2,1}(Q_R^T)$ and $\mathring{V}_p^{2,1}(Q_R^T)$ closures $\mathcal{A}(Q_R^T)$ by the norms

$$\|u\|_{\mathring{W}_p^{2,1}(Q_R^T)} = \left(\int_{Q_R^T} \left(\sum_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p \right) dtdx \right)^{\frac{1}{p}}$$

and

$$\|u\|_{\overset{\circ}{V}{}_p^{2,1}(Q_R^T)} = \left(\int\limits_{Q_R^T} |\mathcal{M}_0 u|^p \, dtdx \right)^{\frac{1}{p}},$$

respectively, $p \in (1, \infty)$. According to the Friedrichs type inequality and lemma2 functionals determined above are really norms. Denote by $T(p)$ the operator, associating to each functions $u(x,t) \in \overset{\circ}{V}{}_p^{2,1}(Q_R^T)$ itself as element of the space $\overset{\circ}{W}{}_p^{2,1}(Q_R^T)$. By lemma 2 the operator $T(p)$ is bounded. Denote by $K(p)$ its norm. By lemma1 $K(2) \leq 1$. Let p_0- be an arbitrary number from the interval (1,2). According to Riez-Thorin theorem on convexity [14] for any $p \in [p_0, 2]$

$$K(p) \leq (K(p_0))^{1-\theta} (K(2))^{\theta} \leq (K(p_0))^{1-\theta},$$

where $\theta = \dfrac{2(p-p_0)}{p(2-p_0)}$.

Thus

$$K(p) \leq K(p_0)^{\dfrac{p_0(2-p)}{p(2-p_0)}}.$$

Let's fix $p_0 = \dfrac{5}{3}$ and denote $a = \max\left\{ \left(\dfrac{5}{3}\right)^3, \left(K\left(\dfrac{5}{3}\right)\right)^3 \right\}$. Since for $p \in \left[\dfrac{5}{3}, 2\right]$ $\dfrac{p_0(2-p)}{p(2-p_0)} \leq \dfrac{2-p}{2-p_0} = 3(2-p)$, then we finally obtain

$$K(p) \leq a^{2-p}.$$

And so we proved the following assertions

Lemma 3. *If* $u(t,x) \in \overset{\circ}{W}{}_p^{2,1}(Q_R^T)$, *then for any* $p \in \left[\dfrac{5}{3}, 2\right]$

$$\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \leq a^{2-p} \|u\|_{\overset{\circ}{V}{}_p^{2,1}(Q_R^T)}.$$

Note that at this the constant $a > 1$ depends only on n. For $p \in \left[\dfrac{5}{3}, 2\right]$ $\sup\limits_{Q_R^T} \left(\sum\limits_{i,j=1}^{n} |a_{ij}(t,x) - \delta_{ij}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}$ denote by δ_p (for brevity we write sup instead $ess\sup$) and let $\delta_2 = \delta$, $h = \max\left\{ \dfrac{1-\gamma^2}{\gamma}, 1 \right\}$.

Lemma 4. *For* $p \in \left[\dfrac{5}{3}, 2\right]$ *it holds the estimation*

$$\delta_p \leq h^{\frac{2-p}{p}} \delta^{\frac{2(p-1)}{p}}.$$

Proof. From the condition (3) it follows that for $i = 1, ..., n$

$$\gamma - 1 \leq a_{ii}(t,x) - 1 \leq \gamma^{-1} - 1,$$

and since $\gamma - 1 \geq 1 - \gamma^{-1}$ that

$$|a_{ii}(t,x) - 1| \leq \frac{1-\gamma}{\gamma}. \tag{6}$$

If $i \neq j$ then

$$2\gamma \leq a_{ii}(t,x) + a_{jj}(t,x) + 2a_{ij}(t,x) \leq 2\gamma^{-1}.$$

Therefore

$$|a_{ij}(t,x)| \leq \frac{1-\gamma^2}{\gamma}. \tag{7}$$

From (6)and (7)we conclude that for $i, j = 1, ..., n$

$$|a_{ij}(t,x) - \delta_{ij}| \leq h \tag{8}$$

On the other hand allowing for (8), we obtain

$$\delta_p = \sup_{Q_R^T} \left(\sum_{i,j=1}^{n} (a_{ij}(t,x) - \delta_{ij})^2 |a_{ij}(t,x) - \delta_{ij}|^{\frac{2-p}{p-1}} \right)^{\frac{p-1}{p}} \leq h^{\frac{2-p}{p}} \delta^{\frac{2(p-1)}{p}}$$

and lemma is proved.

Lemma 5. *Let* $\delta < 1$. *Then there exists* $p_1(\gamma, \delta, n) \in \left[\frac{5}{3}, 2\right]$, *such that for allp* $\in [p_1, 2]$.

Proof. According to the previous lemma

$$a^{2-p}\delta_p \leq \delta^{1/3}.$$

But $h^{\frac{1}{p}} \leq h^{\frac{3}{5}} = h_1$, $\dfrac{p-1}{p} \geq \dfrac{1}{3}$. Therefore

$$a^{2-p}\delta_p \leq (ah_1)^{2-p}\,\delta^{\frac{2}{3}}. \tag{9}$$

Let now $p_1 = \max\left\{\frac{5}{3}, 2 - \frac{\ln\frac{1}{\delta}}{3\ln(ah_1)}\right\}$. Then at $p \in [p_1, 2]$ $(ah_1)^{2-p} \leq \delta^{-\frac{1}{3}}$ and from (9) it follows the assertions of the lemma.

2. Internal priory estimation. Consider the operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t},$$

together with the operator $\mathcal{L}$.

Lemma 6. *If relative to the coefficients of the operator $\mathcal{L}_0$ the condition (3) and $\delta<1$ is fulfilled, then at all $p\in[p_1,2]$ for any function $u(t,x)\in\mathring{W}_p^{2,1}\left(Q_R^T\right)$ the estimation*

$$\|u\|_{\mathring{W}_p^{2,1}\left(Q_R^T\right)}\le C_4(\gamma,\delta,n)\,\|\mathcal{L}_0u\|_{L_p\left(Q_R^T\right)}$$

is true.

Proof. According to lemma 3

$$\|u\|_{\mathring{W}_p^{2,1}\left(Q_R^T\right)}\le a^{2-p}\,\|M_0u\|_{L_p\left(Q_R^T\right)}\le a^{2-p}\,\|\mathcal{L}_0u\|_{L_p\left(Q_R^T\right)}+$$

$$+a^{2-p}\left\|\sum_{i,j=1}^n\left(a_{ij}(t,x)-\delta_{ij}\right)u_{ij}\right\|_{L_p\left(Q_R^T\right)}\le a^{\frac{2}{5}}\,\|\mathcal{L}_0u\|_{L_p\left(Q_R^T\right)}+$$

$$+a^{2-p}\left\|\sum_{i,j=1}^n\left(a_{ij}(t,x)-\delta_{ij}\right)u_{ij}\right\|_{L_p\left(Q_R^T\right)}. \qquad (10)$$

But other hand

$$\left\|\sum_{i,j=1}^n\left(a_{ij}(t,x)-\delta_{ij}\right)u_{ij}\right\|_{L_p\left(Q_R^T\right)}\le$$

$$\left(\int\limits_{Q_R^T}\left(\sum_{i,j=1}^n|u_{ij}|^p\right)\left(\sum_{i,j=1}^n\left|a_{ij}(t,x)-\delta_{ij}\right|^{\frac{p}{p-1}}\right)^{p-1}dtdx\right)^{\frac{1}{p}}\le\delta_p\,\|u\|_{\mathring{W}_p^{2,1}\left(Q_R^T\right)}.$$

Therefore from (10) and lemma 5 we conclude

$$\|u\|_{\mathring{W}_p^{2,1}\left(Q_R^T\right)}\le a^{\frac{2}{5}}\,\|\mathcal{L}_0u\|_{L_p\left(Q_R^T\right)}+a^{2-p}\delta_p\,\|u\|_{\mathring{W}_p^{2,1}\left(Q_R^T\right)}\le$$

$$\le a^{\frac{2}{5}}\,\|\mathcal{L}_0u\|_{L_p\left(Q_R^T\right)}+\delta^{\frac{1}{3}}\,\|u\|_{\mathring{W}_p^{2,1}\left(Q_R^T\right)}$$

and and the assertion of the lemma is proved.

Further everywhere not specifying it we will suppose that the radius R of the sphere $B_R^{x^0}$ ($B_R^{x^0}$ is foundation of the cylinder Q_R^T)) doesn't exceed 1.

Lemma 7. *If the conditions of previous lemma are proved then at all $p\in[p_1,2]$ for any functions $u(t,x)\in\mathcal{A}\left(Q_R^T\right)$ the inequality*

$$\|u\|_{W_p^{2,1}\left(Q_R^T\right)}\le C_5(\gamma,\delta,n)\,\|\mathcal{L}_0u\|_{L_p\left(Q_R^T\right)}$$

is true.

It is enough to apply the Friedrichs inequality and lemma 6 for proving.

Now assume the following Cordes condition on leading coefficients of the operator $\mathcal{L}$

$$\sigma_0=\frac{\sup\limits_{Q_T}\sum\limits_{i,j=1}^n a_{ij}^2(t,x)}{\left[\inf\limits_{Q_T}\sum\limits_{i=1}^n a_{ii}(t,x)\right]^2}<\frac{1}{n-1}. \qquad (11)$$

At this we'll suppose that condition (11) is fulfilled to within non-singular linear transformation,i.e.we can cover the domain Q_T with finite number of the subdomains $Q_1, ..., Q_m$ so in every Q_i there exists non-singular linear transformation at which the image of the operator $\mathcal{L}$ satisfies condition (11) in the image of subdomain $Q_i,\ i = 1, ..., m$.

Lemma 8. *Conditions $\delta < 1$ to within non-singular linear transformation coincides with the conditions (11).*

Proof. Let's make the transformation $\tau = k^2 t, y_i = kx_i;\ i = 1, ..., n$, where $k = \left(\frac{\sup\limits_{Q_T} \sum\limits_{i,j=1}^{n} a_{ij}^2 (t, x)}{\inf\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii} (t, x)} \right)^{-\frac{1}{2}}$. Then if $\|\mathcal{A}_{ij} (\tau, y)\|$ is matrix of leading part of image of the operator $\mathcal{L}$ then $\mathcal{A}_{ij} (\tau, y) = k^2 a_{ij} (t, x);\ i, j = 1, ..., n$. Condition $\delta < 1$ in new variables will take the form

$$\sup_{\tilde{Q}_T} \sum_{i,j=1}^{n} \mathcal{A}_{ij}^2 (\tau, y) - 2\inf_{\tilde{Q}_T} \sum_{i=1}^{n} \mathcal{A}_{ii} (\tau, y) + n < 1, \tag{12}$$

where $\tilde{Q}_T$–is the image of the domain Q_T. It is clear, that coincides with the conditions

$$\frac{\sup\limits_{Q_T} \sum\limits_{i,j=1}^{n} a_{ij}^2 (t, x)}{\left[\inf\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii} (t, x)\right]^2} < \frac{1}{n-1}.$$

Lemma 9. *Let relative to the coefficients of the operator $\mathcal{L}_0$ the conditions (3) and (11) be fulfilled. Then there exists the constant $C_6 (\gamma, \sigma, n)$ such that for any function $u (t, x) \in C^\infty \left(\bar{Q}_R^T\right),\ u|_{t=0} = 0$ at every $p \in [p_1, 2]$ and $R_1 \in (0, R)$ the estimation*

$$\|u\|_{W_p^{2,1}\left(Q_{R_1}^T\right)} \leq C_5 \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{C_6}{(R - R_1)^2} \|u\|_{L_p(Q_R^T)} +$$

$$\frac{C_6}{R - R_1} \|u\|_{W_p^{1,0}(Q_R^T)}$$

is true.

Proof. Let the functions $\eta (x) \in C_0^\infty \left(B_R^{x^0}\right)$ be such that $\eta (x) = 1$ in $B_{R_1}^{x^0}$, $0 \leq \eta (x) \leq 1$, moreover

$$|\eta_i| \leq \frac{C_7}{R - R_1},\quad |\eta_{ij}| \leq \frac{C_7}{(R - R_1)^2};\quad i, j = 1, ..., n, \tag{13}$$

where $C_7 = C_7 (n)$.

Applying to the functions $u\eta$ lemma 7 we'll obtain

$$\|u\|_{W_p^{2,1}\left(Q_{R_1}^T\right)} \leq C_5 \|\mathcal{L}_0 (u\eta)\|_{L_p(Q_R^T)}. \tag{14}$$

But on the other hand

$$\left|\mathcal{L}_0(u\eta)\right| \le \left|\mathcal{L}_0 u\right| + |u| \left|\sum_{i=1}^{n} a_{ij}(t,x)\,\eta_{ij}\right| + 2\left|\sum_{i,j=1}^{n} a_{ij}(t,x)\,u_i\eta_i\right|, \tag{15}$$

and further allowing for (13)

$$\left|\sum_{i,j=1}^{n} a_{ij}(t,x)\,\eta_{ij}\right| \le \frac{C_8(\gamma,n)}{(R-R_1)^2},$$

$$2\left|\sum_{i,j=1}^{n} a_{ij}(t,x)\,u_i\eta_j\right| \le 2\left(\sum_{i,j=1}^{n} a_{ij}(t,x)\,u_iu_j\right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{n} a_{ij}(t,x)\,\eta_i\eta_j\right)^{\frac{1}{2}} \le$$

$$\le 2\gamma^{-1}\left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} \eta_i^2\right)^{\frac{1}{2}} \le 2\gamma^{-1}\sum_{i=1}^{n}|u_i|\sum_{i=1}^{n}|\eta_i| \le \frac{2n\gamma^{-1}C_7}{R-R_1}\sum_{i=1}^{n}|u_i|.$$

Thus from (15) we conclude

$$\|\mathcal{L}_0(u\eta)\|_{L_p(Q_R^T)} \le \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{C_8}{(R-R_1)^2}\|u\|_{L_p(Q_R^T)} +$$

$$+\frac{C_9(\gamma,n)}{R-R_1}\sum_{i=1}^{n}\|u_i\|_{L_p(Q_R^T)} \le \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{C_8}{(R-R_1)^2}\|u\|_{L_p(Q_R^T)} +$$

$$+\frac{C_9}{R-R_1}\|u\|_{W_p^{1,0}(Q_R^T)}. \tag{16}$$

Subject to (16) in (14) and denoting by $\max\{C_5C_8, C_5C_9\}$ the C_{10} we arrive at the required estimation (13).

Lemma 10. *Let relative to the coefficients of the operator $\mathcal{L}_0$ conditions of the previous lemma be fulfilled. Then there exists the constant $C_{11}(\gamma,\sigma,n)$ such that for any functions $u(t,x) \in C^\infty\left(\bar{Q}_R^T\right)$, $u|_{t=0} = 0$ at any $\varepsilon > 0$ and $p \in [p_1, 2]$ the estimation*

$$\|u\|_{W_p^{2,1}\left(Q_{\frac{R}{2}}^T\right)} \le C_5\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \varepsilon\|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{11}}{\varepsilon R^2}\|u\|_{L_p(Q_R^T)}$$

is true.

Proof. We'll use the following interpolation inequality ([1]): let $p \in (1,\infty)$ then for any functions$u(t,x) \in W_p^{2,1}\left(Q_R^T\right)$; at any $\varepsilon > 0$ and $p \in [p_1, 2]$ the estimation

$$\|u\|_{W_p^{1,0}(Q_R^T)} \le \varepsilon\|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{12}(p,n)}{\varepsilon}\|u\|_{L_p(Q_R^T)} \tag{17}$$

is true.

Let's fix an arbitrary $\varepsilon > 0$ and let $\varepsilon_1 > 0$ be a number which will be choosen later. According to lemma 9 and the inequality (17)

$$\|u\|_{W_p^{2,1}\left(Q_{\frac{R}{2}}^T\right)} \le C_5\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{4C_6}{R^2}\|u\|_{L_p(Q_R^T)} + \frac{2C_6}{R}\|u\|_{W_p^{1,0}(Q_R^T)} \le$$

$$\leq C_5 \|L_0 u\|_{L_p(Q_R^T)} + \frac{4C_6}{R^2} \|u\|_{L_p(Q_R^T)} + \frac{2C_6\varepsilon_1}{R} \|u\|_{W_p^{2,1}(Q_R^T)} +$$

$$+\frac{2C_6 C_{13}}{R\varepsilon_1} \|u\|_{L_p(Q_R^T)},$$

where $C_{13} = \sup\limits_{p\in[p_1,2]} C_{12}(p,n)$.

Now it is enough to choose $\varepsilon_1 = \dfrac{\varepsilon R}{2C_6}$, the lemma is proved.

Remark. If the minor coefficients of the operator $\mathcal{L}$ are bounded ,then there exists such $R_0(\gamma, \sigma_0, n, \mathbb{B}, c)$, that at $R \leq R_0$ the assertion of lemma 10 is also true for the operator $\mathcal{L}$. Here $\mathbb{B} = (b_1(t,x), ..., b_n(t,x))$. For $\rho > 0$ the set $\{x : x \in \Omega, dist(x, \partial\Omega) > \rho\}$ denote by Ω_ρ.

Lemma 11. *Let relative to the coefficients of the operator $\mathcal{L}_0$ the conditions (3) and (11) be fulfilled. Then for any function $u(t,x) \in C^\infty\left(\bar{Q}_R^T\right)$, , $u|_{t=0} = 0$ at any $\varepsilon > 0$, $\rho > 0$ and $p \in [p_1, 2]$ the estimation*

$$\|u\|_{W_p^{2,1}(\Omega_\rho\times(0,T))} \leq C_{14}(\gamma, \sigma, n, \rho, \Omega) \|\mathcal{L}_0 u\|_{L_p(Q_T)} +$$

$$+\varepsilon \|u\|_{W_p^{2,1}(Q_T)} + \frac{C_{15}(\gamma, \sigma, n, \rho, \Omega)}{\varepsilon} \|u\|_{L_p(Q_T)}$$

is true.

Proof. Let's fix an arbitrary $\varepsilon > 0$, $\rho > 0$ and $\varepsilon_2 > 0$ be a number which will be choosen later. Let cover $\bar{\Omega}_\rho$ by the system of spheres $\left\{B_{\frac{\rho}{2}}^{x^i}\right\}$ and choose from this cover the finite subcovering $B^1, ..., B^N$. It is evident that the number N depends only on ρ, n and $diam\Omega$. Applying for every $i = 1, ..., N$ lemma 10 we obtain

$$\|u\|^p_{W_p^{2,1}(B^i\times(0,T))} \leq 3^{p-1} \left(C_5^p \|\mathcal{L}_0 u\|^p_{L_p(Q_T)} + \varepsilon_2^p \|u\|^p_{W_p^{2,1}(Q_T)} + \right.$$

$$\left. + \frac{C_{11}^p}{\varepsilon_2^p \rho^{2p}} \|u\|^p_{L_p(Q_T)} \right).$$

Summarising this inequality by i from 1 to N we conclude

$$\|u\|_{W_p^{2,1}(\Omega_\rho\times(0,T))} \leq$$

$$\leq 3^{p-1} N \left(C_5^p \|\mathcal{L}_0 u\|^p_{L_p(Q_T)} + \varepsilon_2^p \|u\|^p_{W_p^{2,1}(Q_T)} + \frac{C_{11}^p}{\varepsilon_2^p \rho^{2p}} \|u\|^p_{L_p(Q_T)} \right).$$

Now it is suffisients to choose $\varepsilon_2 = \dfrac{\varepsilon}{3N}$ and the lemma is proved.

3. Basic coercive estimation. The assertion of lemma 11 is true without any demands relative to the domain $\partial\Omega$. All next assertion of the present paper hold under the conditions $\partial\Omega \in C^2$ which we'll always suppose as fulfilled one.

Lemma 12. *Let relative to the coefficients of the operator $\mathcal{L}_0$ the conditions (3) and (11) be fulfilled. Then there exist positive constant p_1, $C16$ and C_{17} depending* *on γ, σ_0, n and the domain Ω such that for any function on $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ at every $\varepsilon > 0$ and $p \in [p_1, 2]$ the estimation*

$$\|u\|_{W_p^{2,1}((\Omega\backslash\Omega_{\rho_1})\times(0,T))} \leq C_{16} \|\mathcal{L}_0 u\|_{L_p(Q_T)} + \varepsilon \|u\|_{W_p^{2,1}(Q_T)} + \frac{C_{17}}{\varepsilon} \|u\|_{L_p(Q_T)}$$

is true.

Proof. It is sufficient to prove the lemma for the functions $u(t,x) \in C^\infty\left(\bar{Q}_T\right)$, $u|_{t=0} = 0, \frac{\partial u}{\partial n}|_{S_T} = 0$. Besides non losing generality we'll suppose that the coefficients of the operator $\mathcal{L}_0$ are infinite differentiable $\bar{Q}_T$. Let's fix an arbitrary $\varepsilon > 0$ and the point $x^0 \in \partial\Omega$. Make orthogonal transformation of the coordinate $x \to y$ such that the tangent hyperline to $\partial\tilde{\Omega}$ at the point y^0 will be perpendicular to the axis Oy_n. Here $\tilde{\Omega}$ and y^0 are images of the domain Ω and the point x^0 respectively at such transformation. Denote by $\tilde{u}(t,y)$ the image of the function $u(t,x)$. We'll suppose for simplicity that the domain $\partial\tilde{\Omega}$ at intersection $\partial\tilde{\Omega}$ with some neighbourhood O_h of the point y^0 is given by the equation $y_n = \varphi(y_1, ..., y_{n-1})$ with twice continuously differentiable function φ and the part $\tilde{\Omega}$ adjacent to $\partial\tilde{\Omega} \cap O_h$ belongs to the set $\{y : y_n > \varphi(y_1 ... y_{n-1})\}$. Let $\mathcal{A}(t,x) = \|a_{ij}(t,x)\|$- be a matrix of leading coefficients of the operator $\mathcal{L}_0$, $\widetilde{\mathcal{A}}(t,y) = \|\tilde{a}_{ij}(t,y)\|$, where $\tilde{a}_{ij}(t,y)$ are leading coefficients of the image $\tilde{\mathcal{L}}_0$ operator $\mathcal{L}_0$ at our transformation; $i,j = 1,...,n$. Show now that eigen numbers of the matrices $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ coincide. Really, fix an arbitrary point $(t,x) \in Q_T$ and λ is an arbitrary eigen number of the matrix $\mathcal{A}$ and x^λ be corresponding to it eigen vector. By virtue of orthogonality of our transformation there exists a non-degenerated matrix T such that $\widetilde{\mathcal{A}} = T^{-1}\mathcal{A}T$. Denote by the $T^{-1}x^\lambda$. We have

$$\widetilde{\mathcal{A}}y^\lambda = T^{-1}\mathcal{A}x^\lambda = \lambda T^{-1}x^\lambda = \lambda y^\lambda.$$

On the other hand we can write condition (11) in the following form

$$\sigma = \sup_{Q_T} \frac{\sum\limits_{i=1}^{n} \lambda_i^2(t,x)}{\left[\sum\limits_{i=1}^{n} \lambda_i(t,x)\right]^2} < \frac{1}{n-1},$$

where $\lambda_i(t,x)$ are eigen numbers of the matrix$\mathcal{A}(t,x)$; $i = 1,...,n$. Thus the condition (11) is fulfilled also for the operator $\tilde{\mathcal{L}}_0$, moreover with the same constant σ. Analogously it is shown that for the operator $\tilde{\mathcal{L}}_0$ the conditions (3) are fulfilled (with the same constant γ). Let's make one more transformation $z_i = y_i$; $i = 1,...,n-1$, $z_n = y_n - \varphi(y_1,...,y_{n-1})$. Let $\mathcal{L}_0'$, Ω' and z^0 be images of the operator $\tilde{\mathcal{L}}_0$, of the domain $\tilde{\Omega}$ and the point y^0 respectively at our transformation, and $a'_{ij}(t,z)$ be leading coefficients of the operator $\mathcal{L}_0'$; $i,j = 1,...,n$. It is easy to see that

$$a'_{ij}(t,z) = \sum_{k,l=1}^{n} \tilde{a}_{kl}(t,y) \frac{\partial z_i}{\partial y_k} \frac{\partial z_j}{\partial y_l}; \quad i,j = 1,...,n.$$

Therefore

$$a'_{ij}(t,z) = \tilde{a}_{ij}(t,y) \quad \text{if} \quad 1 \leq i,j \leq n-1,$$

$$a'_{nj}(t,z) = -\sum_{k=1}^{n-1} \tilde{a}_{kj}(t,y)\frac{\partial\varphi}{\partial y_k} + \tilde{a}_{nj}(t,y) \quad \text{if} \quad 1 \le j \le n-1,$$

$$a'_{nn}(z,t) = \sum_{k,l=1}^{n} \tilde{a}_{kl}(t,y)\frac{\partial\varphi}{\partial y_k}\frac{\partial\varphi}{\partial y_l} - 2\sum_{k=1}^{n-1} \tilde{a}_{nk}(t,y)\frac{\partial\varphi}{\partial y_k} + \tilde{a}_{nn}(t,y).$$

Since $\frac{\partial\varphi}{\partial y_i}\left(y^0\right) = 0$ for $i = 1, ..., n-1$, then there exists $h_1\left(y^0,\varphi\right)$ such that at $h \le h_1$ at intersection $\Omega' \cap \left(B_h^{z^0} \times (0,T)\right)$ the condition (11) (with the same constant $\sigma' = \frac{\sigma + \frac{1}{n-1}}{2}$) is fulfilled. Besides for the operator $\mathcal{L}'_0$ in indicated intersection the conditions (3) are fulfilled (with the constant $\frac{\gamma}{2}$). Assume that $r = r\left(z^0\right) = h_1(y_0,\varphi)$ and let $u'(t,z)$ be image of the function $\tilde{u}(t,y)$ at our transformation. It is clear that in variables z the intersection $\Omega' \cap B_r^{z^0}$ represent hemisphere $B_r^+ = \left\{z : \left|z - z^0\right| < r, z_n > 0\right\}$. Continue the function $u'(t,z)$ and coefficients of the operator$\mathcal{L}'_0$ by the even form by the hyperlane $z_n = 0$ in $B_r^{z^0} \backslash B_r^+$ and denote by $u'(t,z)$ and $\mathcal{L}'_0$ the obtained in this function and the operator respectively. Since $u'(t,z) \in W_p^{2,1}\left(B_r^{z^0} \times (0,T)\right)$ then according to lemma 10

$$\left\|u'\right\|_{W_p^{2,1}\left(B_{\frac{r}{2}}^{z^0} \times (0,T)\right)} \le C_5\left\|\mathcal{L}'_0 u'\right\|_{L_p\left(B_r^{z^0} \times (0,T)\right)} + \varepsilon_3\left\|u'\right\|_{W_p^{2,1}\left(B_r^{z^0} \times (0,T)\right)} +$$

$$+\frac{C_{11}}{\varepsilon_3 r^2}\left\|u'\right\|_{L_p\left(B_r^{z^0} \times (0,T)\right)}, \tag{18}$$

where $\varepsilon_3 > 0$ will be choosen later. But on the other hand each of norms at the right-hand side (18) represent the corresponding norm taken by semi-cylinder $Q_r^+ = B_r^+ \times (0,T)$ and multiplied by $2^{\frac{1}{p}}$. Therefore from (18) we conclude

$$\left\|u'\right\|_{W_p^{2,1}\left(Q_{\frac{r}{2}}^+\right)} \le C_5\left\|\mathcal{L}'_0 u'\right\|_{L_p(Q_r^+)} + \varepsilon_3\left\|u'\right\|_{W_p^{2,1}(Q_r^+)} + \frac{C_{11}}{\varepsilon_3 r^2}\left\|u'\right\|_{L_p(Q_r^+)}. \tag{19}$$

Cover $\partial\Omega'$ by the system of spheres $\left\{B_{\frac{r}{2}}^{z^i}\right\}$ and choose from this cover finite subcovering $B^1, ..., B^M$. At this the number M is determined only by the quantities $\gamma,\ \sigma_0,\ h$ and the domain Ω. Writing out the inequality of the form (19) for every semi-cylinder $B_r^+\left(z^i\right) \times (0,T);\ \ i = 1, ..., M$ raising both sides of obtained inequalities to power p and summarising by i from 1 to M, we obtain

$$\left\|u'\right\|_{W_p^{2,1}(\mathcal{B}\times(0,T))}^p \le 3^{p-1}M\left(C_5\left\|\mathcal{L}'_0 u'\right\|_{L_p(\Omega'\times(0,T))}^p + \varepsilon_3^p\left\|u'\right\|_{W_p^{2,1}(\Omega'\times(0,T))} +\right.$$

$$\left.+\frac{C_{11}^p}{\varepsilon_3^p r_0^{2p}}\left\|u'\right\|_{L_p(\Omega'\times(0,T))}\right),$$

where $\mathcal{B} = \bigcup_{i=1}^{M} B_{\frac{r}{2}}^+\left(z^i\right)$, and $r_0 = \min\left\{r(z_1), ..., r(z_M)\right\}$. Returning to the variables x and noting that pre-image $\mathcal{B}$ contains the set $\Omega\backslash\Omega_{\rho_1}$ with some $\rho_1(\gamma,\sigma,n,\Omega)$, we conclude

$$\|u\|_{W_p^{2,1}((\Omega\backslash\Omega_{\rho_1})\times(0,T))} \le C_{18}\|\mathcal{L}_0 u\|_{L_p(Q_T)} +$$

$$+C_{19}\varepsilon_3 \|u\|_{W_p^{2,1}(Q_T)} + \frac{C_{20}}{\varepsilon_3} \|u\|_{L_p(Q_T)},$$

where the constants C_{18}, C_{19} and C_{20} depend only on γ, σ, n and the domain Ω. Now it is sufficient to choose $\varepsilon_3 = \frac{\varepsilon}{C_{19}}$, and the lemma is proved.

It follows the following from lemmas 11 and 12

Lemma 13. *Let relative to coefficients of the operator $\mathcal{L}_0$ the conditions (3) and (11) be fulfilled. Then for any function $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ at any $p \in [p_1, 2]$ the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{21}(\gamma, \sigma, n, \Omega) \left(\|\mathcal{L}_0 u\|_{L_p(Q_T)} + \|u\|_{L_p(Q_T)} \right)$$

is true.

Now impose the following conditions on minor coefficient of the operator $\mathcal{L}$. For $p \in [p_1, 2]$

$$b_i(t,x) \in L_{n+2}(Q_T); \quad i = 1, ..., n, \tag{20}$$

Let $\psi(t,x) \in L_p(Q_T)$, $1 < p < \infty$. The quantity

$$\omega_{\psi;p}(\delta) = \sup_{e \subset Q_T, mes\ e \leq \delta} \left(\int_e |\psi|^p \, dtdx \right)^{\frac{1}{p}},$$

is called $\mathcal{A}C$ modulus of the function $\psi(t,x)$. Denote by $\omega_{B;p}(\delta)$ the $\max_{1 \leq i \leq n} \{\omega_{b_i;p}(\delta)\}$.

Further everywhere the symbol $C(\mathcal{L})$ means that the positive constant depends only on $\gamma, \sigma$and $\omega_{B;n+2}(\delta)$.

Lemma 14. *Let relative to the coefficients of the operator $\mathcal{L}$ the conditions (3), (11) and (20) be fulfilled. Then there exist the constants $C_{22}(\mathcal{L}, n, \Omega)$, $T_0(\mathcal{L}, n)$, such that if $T \leq T_0$, then for any function $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ at every $p \in [p_1, 2]$ the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{22} \|\mathcal{L}u\|_{L_p(Q_T)}.$$

is true.

Proof. We'll use the following embedding theorems [1]: for any function $u(t,x) \in \dot{W}_q^{2,1}(Q_T)$ it holds the estimation

$$\|u_i\|_{L_{\frac{q(n+2)}{n+2-q}}(Q_T)} \leq C_{23}(q,n) \|u\|_{W_q^{2,1}(Q_T)}, \quad \text{if } 1 \leq q < n+2, \tag{21}$$

According to lemma 13

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{21} \|\mathcal{L}u\|_{L_p(Q_T)} + C_{21} \|(\mathcal{L} - \mathcal{L}_0) u\|_{L_p(Q_T)} + C_{21} \|u\|_{L_p(Q_T)} \leq$$

$$\leq C_{21} \|\mathcal{L}u\|_{L_p(Q_T)} + C_{21} \sum_{i=1}^{n} \|b_i u_i\|_{L_p(Q_T)} + C_{21} \|u\|_{L_p(Q_T)}. \tag{23}$$

Let's fix an arbitrary i, $1 \leq i \leq n$ and assume in (21) $q = p$. We obtain

$$\|b_i u_i\|_{L_p(Q_T)} \leq \|b_i\|_{L_{n+2}(Q_T)} \|u_i\|_{L_{\frac{(n+2)p}{n+2-p}}(Q_T)} \leq C_{23} \|b_i\|_{L_{n+2}(Q_T)} \|u\|_{W_p^{2,1}(Q_T)}.$$

Thus

$$\sum_{i=1}^{n}\|b_iu_i\|_{L_p(Q_T)}\leq C_{23}\sum_{i=1}^{n}\|b_i\|_{L_{n+2}(Q_T)}\|u\|_{W_p^{2,1}(Q_T)}\leq$$

$$\leq C_{25}(n)\,\omega_{\mathbb{B};n+2}(\delta)\,\|u\|_{W_p^{2,1}(Q_T)}, \tag{24}$$

where $\delta=T\ mes\Omega,\ C_{25}=\sup\limits_{p\in[p_1,2]}C_{23}(p,n)$.

Let now $t\in(0,T)$. We have

$$u(t,x)=\int_0^t u_t(\tau,x)\,d\tau.$$

Thus using the Holder inequality we obtain

$$u(t,x)\leq T^{\frac{p-1}{p}}\left(\int_0^T|u_t(\tau,x)|^p\,d\tau\right)^{\frac{1}{p}},$$

and consequently

$$|u(t,x)|^p\leq T^{p-1}\int_0^T|u_t(\tau,x)|^p\,d\tau.$$

Integrating the both sides of this inequality by Q_T and raising to power $\dfrac{1}{p}$ we have

$$\|u\|_{L_p(Q_T)}\leq T\,\|u_t\|_{L_p(Q_T)}. \tag{25}$$

Subject to (25), (26) and (27) in (24) we come to the estimation

$$\|u\|_{W_p^{2,1}(Q_T)}\leq C_{21}\,\|\mathcal{L}u\|_{L_p(Q_T)}+C_{21}\left(C_{24}\omega_{\mathbb{B};n+2}(\delta)+T\right)\times$$

$$\times\|u\|_{W_p^{2,1}(Q_T)}.$$

Then there exists the constant $T_0(\mathcal{L},n)$ such that at $T\leq T_0$

$$C_{24}\omega_{\mathbb{B};n+2}(\delta)+T<\frac{1}{2C_{21}}.$$

The lemma is proved.

4. Case p > 2. Let $p\in\left[2,\dfrac{7}{3}\right]$, and $K(p)$ have the same meaning as in lemma 3. By Riez-Theorin theorem for any $p\in\left[2,\dfrac{7}{3}\right]$,

$$K(p)\leq(K(2))^{1-\theta}\left(K\left(\frac{7}{3}\right)\right)^{\theta}\leq\left(K\left(\frac{7}{3}\right)\right)^{\theta},$$

where $\theta = \dfrac{2(p-2)}{p\left(\frac{7}{3}-2\right)}$. Denoting by $a_1(n)$ the $max\left\{\left(\frac{7}{3}\right)^3, \left(K\left(\frac{7}{3}\right)\right)^3\right\}$ we obtain

$$K(p) \leq a_1^{p-2}.$$

Thus the following analogue of lemma 3 is true.

Lemma 15. *If* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ *then for any* $p \in \left[2, \frac{7}{3}\right]$

$$\|u\|_{\mathring{W}_p^{2,1}(Q_T^T)} \leq a_1^{p-2} \|u\|_{\mathring{V}_p^{2,1}(Q_R^T)}$$

is true. The analogy of lemmas 4 and 5 is proved absolutely analogously.

Lemma 16. *For* $p \in \left[2, \frac{7}{3}\right]$ *it holds the estimation*

$$\delta_p \leq h^{\frac{p-2}{p}} \delta.$$

Lemma 17. *Let* $\delta < 1$. *Then there exists* $p_2(\gamma, \delta, n) \in \left(2, \frac{7}{3}\right]$ *such that for all* $p \in [2, p_2]$

$$a_1^{2-p} \delta_p \leq \delta^{\frac{1}{3}}.$$

Impose now the following restrictions on minor coefficients of the operator $\mathcal{L}$ for $p \in (2, p_2]$

$$b_i(t,x) \in L_{n+2}(Q_T); \quad i = 1, ..., n. \tag{26}$$

Using the scheme conducted in lemmas 6-13, and subject to lemmas 15-17 we are sure in validity of lemma 14 for $p \in (2, p_2]$ and $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ if only relative to the coefficients of the operator $\mathcal{L}$ the conditions (3), (11) and (26) are fulfilled.

Theorem 1. *Let relative to coefficients of the operator* $\mathcal{L}$ *the conditions (3), (11) and (26) be fulfilled. Then there exists the positive constants* $T_0(\mathcal{L}, n)$ *and* $C_{25}(\gamma, \sigma, n, \Omega)$ *such that for any functions* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ *at* $T \leq T_0$ *and at every* $p \in [p_1, p_2]$ *the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{25} \|\mathcal{L}u\|_{L_p(Q_T)}$$

is true.

5. Solvability of the mixed boundary value problem. Now consider the mixed boundary value problem (1)-(2).

Theorem 2. *Let in domain* Q_T *be given the coefficients of the operator* $\mathcal{L}$ *satisfying the conditions (3), (11) and (26). Then if* $T \leq T_0$ *and* $\partial\Omega \in C^2$ *then the fixed boundary value problem is identically strongly solvable in the space* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ *at every* $f(t,x) \in L_p(Q_T)$, $p \in [p_1, p_2]$. *At this for solution* $u(t,x)$ *the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{25} \|f\|_{L_p(Q_T)} \tag{27}$$

is true.

Remark. In case $p = 2$ and the operator $\mathcal{L}$, theorem 2 is correct and without the assumption $T \leq T_0$ (see: [11]).

Proof. Let's prove the theorem by the method of continuation by parameter introduce for $s \in [0, 1]$ the family of the operator $\mathcal{L}_s = s\mathcal{L} + (1 - s) M_0$.

It is easy to see that the conditions (3) and (11) are fulfilled for the operator $\mathcal{L}_s$ with the constant γ and σ respectively. Show this on the example of condition (11). According to lemma 8 the last to within non singular linear transformation considers with the condition $\delta < 1$. Let $a_{ij}^s (t, x)$ be leading coefficients of the operator $\mathcal{L}_s$; $i, j = 1, ..., n$ and

$$\delta^s = \sup_{Q_T} \left(\sum_{i,j=1}^{n} \left(a_{ij}^s (t, x) - \delta_{ij} \right)^2 \right)^{\frac{1}{2}}.$$

We have

$$\delta^s = \sup_{Q_T} \left(\sum_{i,j=1}^{n} \left(s a_{ij} (t, x) + (1 - s) \delta_{ij} - \delta_{ij} \right)^2 \right)^{\frac{1}{2}} =$$

$$= s \sup_{Q_T} \left(\sum_{i,j=1}^{n} \left(a_{ij} (t, x) - \delta_{ij} \right)^2 \right)^{\frac{1}{2}} = s\delta \leq \delta.$$

Besides if $b_i^s (t, x)$; $i = 1, ..., n$ are minor coefficients of the operator $\mathcal{L}_s$, then the quantity $\sum\limits_{i=1}^{n} \|b_i^s (t, x)\|_{L_{n+2}(Q_T)} + \|c^s (t, x)\|_{L_m(Q_T)}$ is by majorized by the constant, depending only on $\sum\limits_{i=1}^{n} \|b_i\|_{L_{n+2}(Q_T)}$. Hence it follows that the assertion of theorem 1 is true for the operator $\mathcal{L}_s$ with the constant C'_{25} not depending on s. Denote by E the problem $[0, 1]$ has solution. Note that by virtue of theorem 2 this solution is unique. Now show that the set E is nonempty and it is open and closed simultaneously relative to $[0, 1]$. Then

$$\mathcal{L}_s u = f (t, x); \quad (t, x) \in Q_T, \quad u \in \hat{W}_p^{2,1} (Q_T), \tag{28}$$

coincides with the segment $[0, 1]$ and in particular the problem (28) is identically solvable at $s = 1$ when $\mathcal{L}_1 = \mathcal{L}$. At this the estimation (27) follows from theorem 2. Nonemptiness of the E follows form that problem (28) is solvable at $s = 0$ (see:[1]). Show that the set E is open relative to $[0, 1]$. Let $s^0 \in E$, $s \in [0, 1]$ be such that $\left| s - s^0 \right| < \alpha$ where $\alpha > 0$ will be choosen later. Represent the problem (28) in the form

$$\mathcal{L}_{s^0} u = f (t, x) + (\mathcal{L}_{s^0} - \mathcal{L}_s) u; \quad (t, x) \in Q_T, \quad u \in \hat{W}_p^{2,1} (Q_T). \tag{29}$$

It is easy to see that $\mathcal{L}_{s^0} - \mathcal{L}_s = \left(s^0 - s \right) (\mathcal{L} - M_0)$. Consider auxiliary problem

$$\mathcal{L}_{s^0} u = f (t, x) + \left(s^0 - s \right) (\mathcal{L} - M_0) \vartheta; \quad (t, x) \in Q_T, \quad u \in \hat{W}_p^{2,1} (Q_T), \tag{30}$$

where $\vartheta(t,x) \in \dot{W}_p^{2,1}(Q_T)$. Acting as in theorem 1 we can show that

$$\|(\mathcal{L}-\mathcal{M}_0)\,\vartheta\|_{L_p(Q_T)} \le C_{31}(\mathcal{L},n)\,\|\vartheta\|_{W_p^{2,1}(Q_T)}.$$

Thus the operator $\mathcal{M}$ associating to every function $\vartheta(t,x) \in \hat{W}_p^{2,1}(Q_T)$ the solution $u(t,x)$ of the problem (30) is determined, i.e. $u = \mathcal{M}\vartheta$. Show that at corresponding way chosen by α the operator $\mathcal{M}$ is contractive. Let $u^1 = \mathcal{M}\vartheta^1,\ u^2 = \mathcal{M}\vartheta^2$. We have

$$\mathcal{L}_{s^0}\left(u^1-u^2\right) = \left(s^0-s\right)(\mathcal{L}-M_0)\left(\vartheta^1-\vartheta^2\right); \quad u^1-u^2 \in \hat{W}_p^{2,1}(Q_T).$$

Then according to theorem 1

$$\left\|u^1-u^2\right\|_{W_p^{2,1}(Q_T)} \le C_{25}\alpha C_{26}\left\|\vartheta^1-\vartheta^2\right\|_{W_p^{2,1}(Q_T)},$$

and it is sufficient to choose $\alpha = \dfrac{1}{2C_{25}C_{26}}$. Then the operator $\mathcal{M}$ has a fixed point $u = \mathcal{M}u$. But at $\vartheta = u$ the problem (30) coincides with the problem (29), i.e. with (28). The openness of the set E is proved. Now prove its closure. Let $s^m \in E;\ m = 1,2,...,\ s^0 = \lim\limits_{m\to\infty} s^m$. Show that $s^0 \in E$. Denote by $u^m(t,x)$ the solution of the boundary value problem

$$\mathcal{L}_{s^m}u^m = f(t,x); \quad (t,x) \in Q_T, \quad u^m \in \hat{W}_p^{2,1}(Q_T).$$

According to theorem 1

$$\|u^m\|_{W_p^{2,1}(Q_T)} \le C_{25}\,\|f\|_{L_p(Q_T)}.$$

Thus the sequence $\{u^m(t,x)\}$ is bounded by the norm $W_p^{2,1}(Q_T)$. Hence it follows that it is wearily compact, i.e. there exist subsequence $m_k \to \infty$ at $k \to \infty$ and the function $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ such that $u(t,x)$ is weak limit in $\hat{W}_p^{2,1}(Q_T)$ of the subsequence $\{u^{m_k}(t,x)\}$ at $k \to \infty$ Hence in particular it follows that for any function $\hat{W}_p^{2,1}(Q_T)$

$$\langle \mathcal{L}_{s^0}u^{m_k},\varphi\rangle \to \langle \mathcal{L}_{s^0},\varphi\rangle; \qquad k \to \infty$$

where $\langle u,\vartheta\rangle = \int\limits_{Q_T} u\vartheta dtdx$. But

$$\langle L_{s_0}u^{m_k},\varphi\rangle = \langle(\mathcal{L}_{s^0}-\mathcal{L}_{s^{m_k}})\,u^{m_k},\varphi\rangle + \langle \mathcal{L}_{s^{m_k}}u^{m_k},\varphi\rangle = i_1 + i_2.$$

We have

$$|i_1| \le \left|s^0-s^{m_k}\right|\left|\langle(\mathcal{L}-\mathcal{M}_0)\,u^{m_k},\varphi\rangle\right| \le$$
$$\le \left|s^0-s^{m_k}\right| C_{27}(\varphi,p)\,C_{26}\,\|u^{m_k}\|_{W_p^{2,1}(Q_T)} \le$$
$$\le C_{25}C_{27}C_{26}\left|s^0-s^{m_k}\right|\|f\|_{L_p(Q_T)}.$$

Thus $i_1 \to 0$ at $k \to \infty$. On the other hand $i_2 = \langle f,\varphi\rangle$. So for any function $\varphi(t,x) \in \mathring{W}_0^{2,1}\left(\bar{Q}_T\right)$

$$\langle \mathcal{L}_{s^0}u,\varphi\rangle = \langle f,\varphi\rangle.$$

It means that $\mathcal{L}_{s^0}u = f(t,x)$ almost everywhere in Q_T, i.e. $s^0 \in E$. The theorem is proved.

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