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SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS FOR LAPLACE EQUATION

Abstract

One of the important solution methods of boundary value problems for elliptic equations is the method of potentials theory used by many authors [1]-[3].

When boundary conditions of the considered problems contain higher derivatives, the use of classic potentials of simple or double layers doesn't lead to the aim.

In the given paper, we introduce special potentials [4]-[7] that allow to solve boundary value problems for elliptic equations when a boundary condition may contain higher derivatives. Here, consideration of Laplace equation is caused by simplification of notation. As a matter of fact, these problems are solved for second order elliptic equations.

Denote $m-$dimensional real Euclidean space by $R^m, (m \geq 2)$, its points by $x = (x_1, x_2, ..., x_m) y, z, \xi$ and $|x| = (x_1^2 + x_2^2 + ... + x_m^2)^{1/2}$. Let G be a bounded, simply-connected, open domain in R^m and S its surface.

Problem Statement. Find the solution of the equation

$$\Delta u(x) = 0, \quad x \in G, \tag{1}$$

satisfying the boundary condition

$$\left(\frac{\partial u}{\partial \alpha}\right)_{-}(z) = \varphi_0(z), \quad z \in S, \tag{2}$$

or

$$\left(\frac{\partial^2 u}{\partial \alpha \partial n}\right)_{-}(z) = \varphi_1(z), \quad z \in S \tag{3}$$

where $n \equiv n_z$ is an external normal to S at the point $z \in S$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$, is some direction, $|\alpha| = 1$, $\left(\frac{\partial u}{\partial \alpha}\right)_{-}(z)$ and $\left(\frac{\partial^2 u}{\partial \alpha \partial n}\right)_{-}(z)$ are the tame derivatives on S from within S, $\varphi_i(z)$, $(i = \overline{0,1})$ are some given functions on S, Δ is Laplace operator, i.e. $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_m^2}$; $u(x)$ is a desired solution.

It is assumed that the following restrictions are fulfilled:

1^0. Let S be a Lyapunov surface.

2^0. Domain G is convex in the direction of α.

3^0. The functions $\varphi_0(z)$ and $\varphi_1(z)$ are continuous on S.

It follows from restriction 2^0 that surface S may be divided into two disjoint parts S_1 and S_2 so that the half-lines $z + t\alpha,\ z \in S_1,\ 0 \leq t < \infty$ and $z - t\alpha,\ z \in$

S_2, $0 \le t < \infty$ don't intersect with domain G. By $P(x)$ we denote a fundamental solution of Laplace operator. As is known [2]

$$P(x) = \frac{1}{2\pi} \ell n\,|x| \quad for \quad m = 2,$$

$$P(x) = -\frac{1}{(m-2)\sigma_m\,|x|^{m-2}} \quad for \quad m \ge 3,$$

where σ_m is surface area of a unit sphere in R^m.

We'll look for the solution $u_0(x)$ of problem (1), (2) in the form

$$\begin{gathered} u_0(x) = -\int\limits_{S_1} \mu_0(z) ds_z \int\limits_0^{a(z)} \frac{\partial_\xi P(x-\xi)}{\partial n_z}\,|_{\xi = z + t\alpha}\, dt + \\ + \int\limits_{S_2} \mu_0(z) ds_z \int\limits_0^{b(z)} \frac{\partial_\xi P(x-\xi)}{\partial n_z}\,|_{\xi = z - t\alpha}\, dt, \quad x \in G, \end{gathered} \tag{4}$$

the solution $u_1(x)$ of problem (1), (3) in the form

$$\begin{gathered} u_1(x) = -\int\limits_{S_1} \mu_1(z) ds_z \int\limits_0^{a(z)} P(x-\xi)\,|_{\xi = z + t\alpha}\, dt + \\ + \int\limits_{S_2} \mu_1(z) ds_z \int\limits_0^{b(z)} P(x-\xi)\,|_{\xi = z - t\alpha}\, dt, \quad x \in G, \end{gathered} \tag{5}$$

(see [4]-[7], where $\mu_i(z)$, $(i = 0,1)$ are the unknown desired densities, $a(z)$ and $b(z)$ are some continuous functions satisfying the inequalities

$$a(z) \ge R, \quad z \in S_1, \quad b(z) \ge R, \quad z \in S_2, \tag{5_1}$$

R is some number satisfying the inequality

$$R > d_m + \left(\frac{2\delta_m}{\sigma_m}\right)^{\frac{1}{m-1}}, \tag{5_2}$$

d_m is diameter of the surface S, δ_m is area of surface S. Here and in the sequel, under the notation $\dfrac{\partial_\xi P(x-\xi)}{\partial \beta}$ we'll understand the derivative of $P(x-\xi)$, as a function of ξ in the direction of β. A priori we assume that the unknown densities $\mu_0(z)$ and $\mu_1(z)$ are continuous in S_1 and S_2 respectively and bounded. Then, from (4) we have

$$\frac{\partial u_0(x)}{\partial \alpha} = -\int\limits_{S_1} \mu_0(z) ds_z \int\limits_0^{a(z)} \frac{\partial_x}{\partial \alpha} \frac{\partial_\xi P(x-\xi)}{\partial n_z}\,|_{\xi = z + t\alpha}\, dt +$$

$$+\int\limits_{S_2}\mu_0(z)ds_z\int\limits_0^{b(z)}\frac{\partial_x}{\partial\alpha}\frac{\partial_\xi P(x-\xi)}{\partial n_z}\,|_{\xi=z-t\alpha}\,dt,\quad x\in G.$$

Consequently,

$$\frac{\partial u_0(x)}{\partial\alpha}=\int\limits_{S_1}\mu_0(z)ds_z\int\limits_0^{a(z)}\left(\frac{\partial_\xi}{\partial\alpha}\frac{\partial_\xi P(x-\xi)}{\partial n_z}\right)|_{\xi=z+t\alpha}\,dt-$$

$$-\int\limits_{S_2}\mu_0(z)ds_z\int\limits_0^{b(z)}\left(\frac{\partial_\xi}{\partial\alpha}\frac{\partial_\xi P(x-\xi)}{\partial n_z}\right)|_{\xi=z-t\alpha}\,dt,\quad x\in G.$$

Hence

$$\frac{\partial u_0(x)}{\partial\alpha}=\int\limits_{S_1}\mu_0(z)ds_z\int\limits_0^{a(z)}\frac{\partial}{\partial t}\left\{\frac{\partial_\xi P(x-\xi)}{\partial n_z}\,|_{\xi=z+t\alpha}\right\}dt+$$

$$+\int\limits_{S_2}\mu_0(z)ds_z\int\limits_0^{b(z)}\frac{\partial}{\partial t}\left\{\frac{\partial_\xi P(x-\xi)}{\partial n_z}\,|_{\xi=z-t\alpha}\right\}dt,\quad x\in G.$$

Consequently,

$$\begin{aligned}\frac{\partial u_0(x)}{\partial\alpha}=-\int\limits_S\mu_0(z)\frac{\partial_z P(x-z)}{\partial n_z}ds_z+\int\limits_{S_1}\mu_0(z)\frac{\partial_\xi P(x-\xi)}{\partial n_z}\,\Big|_{\xi=z+a(z)\alpha}\,ds_z+\\+\int\limits_{S_2}\mu_0(z)\frac{\partial_\xi P(x-\xi)}{\partial n_z}\,\Big|_{\xi=z-b(z)\alpha}\,ds_z,\quad x\in G.\end{aligned}\tag{6}$$

Conducting similar calculations, from (5) we have

$$\begin{aligned}\frac{\partial u_1(x)}{\partial\alpha}=-\int\limits_S\mu_1(z)P(x-\xi)ds_z+\int\limits_{S_1}\mu_1(z)P(x-\xi)\,\Big|_{\xi=z+a(z)\alpha}\,ds_z+\\+\int\limits_{S_2}\mu_1(z)P(x-\xi)\,\Big|_{\xi=z-b(z)\alpha}\,ds_z,\quad x\in G.\end{aligned}\tag{7}$$

Further, we note that if $\mu(y)$ is integrable on S and continuous at the point $z\in S$, then for the potentials

$$\nu_0(x)=\int\limits_S P(x-y)\mu(y)ds_y,\quad \nu_1(x)=\int\limits_S\frac{\partial_y P(x-y)}{\partial n_y}\mu(y)ds_y.$$

the following jump formulae [2] hold:

$$\begin{aligned}\left(\frac{\partial\nu_0}{\partial_n}\right)_-(z)=-\tfrac{1}{2}\mu(z)+\int\limits_S\frac{\partial_z P(z-y)}{\partial n_z}\mu(y)ds_y,\\(\nu_1)_-(z)=\frac{1}{2}\mu(z)+\int\limits_S\frac{\partial_y P(z-y)}{\partial n_y}\mu(y)ds_y,\ z\in S.\end{aligned}\tag{8}$$

Using (6), (7) and (8), for defining the density $\mu_i(z)$, $(i = 0, 1)$ from boundary conditions (2) and (3) we get

$$\begin{gathered}\mu_i(z) = (-1)^{i+1} 2\varphi_i(z) + \int\limits_S (H_{il}(z,y) + H_{i2}(z,y)) \times \\ \times \mu_i(y) ds_y,\ z \in S,\ i = 0, 1,\end{gathered} \tag{9}$$

where

$$H_{01}(z,y) = -2\frac{\partial_y P(z-y)}{\partial n_y},\quad H_{02}(z,y) = 2\frac{\partial_\xi P(z-\xi)}{\partial n_y}\Big|_{\xi=\psi(y)},$$

$$H_{11}(z,y) = 2\frac{\partial_z P(z-y)}{\partial n_z},\quad H_{12}(z,y) = -2\frac{\partial_z P(z-\xi)}{\partial n_z}\Big|_{\xi=\psi(y)},$$

$$\psi(y) = y + a(y)\alpha \ \text{ for } \ y \in S_1 \ \text{ and } \ \psi(y) = y - b(y)\alpha \ \text{ for } \ y \in S_2. \tag{10}$$

For $z, y \in S$, using the inequality

$$|z - \psi(y)| \geq R - |z - y| \geq R - d_m,$$

from (10) we have

$$|H_{i2}(z,y)| \leq \frac{2}{\sigma_m (R - d_m)^{m-1}},\quad m \geq 2,\ \sigma_2 = 2\pi,\ i = 0, 1. \tag{11}$$

Remark 1. Inequality (11) shows that $|H_{i2}(z,y)|$ (for $z, y \in S$) may be arbitrarily small at the expense of choice of the number R sufficiently large.

As is known [2], the integral equations

$$\rho_i(z) = g_i(z) + \int\limits_S H_{i1}(z,y)\rho_i(y) ds_y,\quad z \in S,\quad g_i(z) \in C(S), i = 0, 1,$$

are second kind Fredholm equations with weak singularities and they have unique continuous solutions $\rho_i(z)$, respectively. Consequently, according to remark 1, under conditions (5_1) and (5_2) integral equations (9) have unique continuous solutions $\mu_i(z)$, respectively.

Thus, we established

Theorem 1. *Under restrictions* $1^0 - 3^0$*, problem (1), (2) has a solution represented by formula (4), where the density* $\mu_0(z)$ *is determined from (9).*

Theorem 2. *Under restrictions* $1^0 - 3^0$*, problem (1), (3) has a solution represented by formula (5), where the density* $\mu_1(z)$ *is determined from (9).*

Remark 2. As it is seen from (7), the principal part for $\dfrac{\partial u_1(x)}{\partial \alpha}$ is a simple layer potential $\nu_0(x)$.

Consequently, if by means of simple layer potential $\nu_0(x)$ one can find the solution of equation (1) satisfying the boundary condition

$$\left(\frac{\partial u}{\partial \theta}\right)_{-}(z) = \varphi_1(z),\quad z \in S,$$

where $\theta \equiv \theta_z$ is some direction at the point $z \in S$, then the solution of equation (1) satisfying the boundary condition

$$\left(\frac{\partial^2 u}{\partial \alpha \partial \theta} \right)_- (z) = \varphi_1(z), \quad z \in S,$$

may be find in the form of potential (5).

Remark 3. As it is seen from the reasonings mentioned above, if we'll add to boundary conditions (2) and (3) a linear combination of small order derivatives of $u(x)$, (with respect to order contained in (2) and (3), respectively) then we can solve such problems by the suggested potentials (4) and (5).

In conclusion we note that the solution of the considered problems (1), (2) and (1), (3) as Neumann's classic problems, is not unique. For example, on the plane, the solution of equation (1) satisfying the boundary condition

$$\left(\frac{\partial^2 u}{\partial x_1 \partial n} \right)_- (z) = 0, \quad z \in S,$$

is defined to within linear summand

$$ax_1 + bx_2 + c,$$

where a, b, c are arbitrary constants.

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