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ON INEQUALITIES BETWEEN "WEIGHT" NORMS OF PARTIAL DERIVATIVES OF DIFFERENTIABLE FUNCTIONS

Abstract

One form of "weight" integral representation of differentiable functions $f = f(x)$ at the points $x = (x_1, ..., x_n) \in G \subset E_n$ is cited, and by means of this integral representation, validity of imbedding theorem type "weight" integral inequalities

$$\|b(\cdot) D^{\nu} f(\cdot)\|_{L_q(G)} \leq c \left\{ \sum_{k=0}^{n} h^{æ_k} \left\| b_k(\cdot) D^{m^k} f(\cdot) \right\|_{L_p(G)} \right\}, \tag{0.1}$$

$$D^{\nu} : \bigcap_{k=0}^{n} L_P^{<m^k>}(G; b_k) \subset L_q(G; b). \tag{0.2}$$

are proved.

§ 1. One form of "weight" integral representation of differentiable functions. Let the vectors

$$\left. \begin{array}{l} m^0 = \left(m_1^0, ..., m_n^0\right), \\ m^k = \left(m_1^k, ..., m_n^k\right), \\ (k = 1, ..., n) \end{array} \right\} \tag{1.1}$$

be "integer and non-negative", i.e.

$$m_j^k \geq 0 \quad (j = 1, 2, ..., n) \text{-are entire} \tag{1.1*}$$

for all $k = 0, 1, 2, ..., n$.

Further we assume that system of vectors

$$m^k = \left(m_1^k, ..., m_n^k\right) \quad (k = 1, 2, ..., n), \tag{1.2}$$

from (1.1) define linearly independent system of vectors, i.e. a determinant

$$\nabla = \begin{vmatrix} m_1^1 & m_2^1 & ... & m_n^1 \\ m_1^2 & m_2^2 & ... & m_n^2 \\ ... & ... & ... & ... \\ m_1^n & m_2^n & ... & m_n^n \end{vmatrix} \neq 0. \tag{1.3}$$

We assume, that system of vectors (1.2) satisfy the condition of "$*$- arrangements", i.e.

$$\left. \begin{array}{l} \sup p m^k \supset \{k\} \\ (k = 1, 2, ..., n) \end{array} \right\}, \tag{1.4}$$

where through $\sup pm^k$ we will designate a support of the corresponding vector $m^k = \left(m_1^k, ..., m_n^k\right)$, more precisely, $\sup pm^k$ is a set of indices of non zero coordinates of this vector, $m^k = \left(m_1^k, ..., m_n^k\right)$, hence (1.4) means, that

$$m_k^k > 0 \ \ (k = 1, 2, ..., n) . \tag{1.5}$$

Let now "an integer, non-negative vector"

$$\nu = (\nu_1, ..., \nu_n) \tag{1.6}$$

satisfy the condition "$*$- communications" with vectors (1.1), i.e. this means, that

$$\left\{ \begin{array}{l} \nu_j \geq m_j^0 \quad (j = 1, 2, ..., n) , \\ \left. \begin{array}{l} \nu_j \geq m_j^k \quad (j \neq k) \\ \nu_k < m_k^k \quad (j = k) \end{array} \right\} \ (k = 1, 2, ..., n) . \end{array} \right. \tag{1.7}$$

In the monograph [1], different integral representations of differentiable functions $f = f(x)$ are given at the points $x \in G \subset E_n$, from which in special case "weight" integral representation of differentiable functions $f = f(x)$ follows in the form of integral identity

$$b(x) \ D^\nu f(x) = \mathcal{B}_{0,\delta}\left(D^{m^0} f(x)\right) + \sum_{k=1}^n \mathcal{B}_{k,\delta}\left(D^{m^k} f(x)\right) . \tag{1.8}$$

The integral operators standing in the right part of equalities (1.8), are defined by formulas:

$$\mathcal{B}_{0,\delta}\left(D^{m^0} f(x)\right) = (-1)^{\left|m^0 - \nu\right|} c_0 \prod_{j=1}^n \left(a_j(h)\right)^{m_j^0 - \nu_j} \int_{E_n} b_0(x+y) D^{m^0} f(x+y) \times$$

$$\times \left(\frac{b_0(x)}{b_0(x+y)}\right) \Phi_{0,\delta}\left(\frac{y}{\psi(x,h)}\right) \frac{dy}{mes\left(R_\delta \cdot E_n\right)_h}, \tag{1.9}$$

and for each $k \in e_n = \{1, 2, ..., n\}$

$$\mathcal{B}_{k,\delta}\left(D^{m^k} f(x)\right) =$$

$$= (-1)^{\left|m^k - \nu\right|} c_k \int_0^h \prod_{j=1}^n \left(a_j(v)\right)^{m_j^k - \nu_j} \frac{da_k(v)}{a_k(v)} \int_{E_n} b_k(x+y) D^{m^k} f(x+y) \times$$

$$\times \left(\frac{b_k(x)}{b_k(x+y)}\right) \Phi_{k,\delta}\left(\frac{y}{\psi(x,v)}\right) \frac{dy}{mes\left(R_\delta \cdot E_n\right)_v}. \tag{1.10}$$

In equalities (1.9) and (1.10)

$$mes\left(R_\delta \cdot E_n\right)_v = \prod_{j=1}^n \psi_j(x,v) , \tag{1.11}$$

for each $v \in (0, h]$, and the vector $\delta = (\delta_1, ..., \delta_n)$ has the coordinates

$$\delta_j = +1 \quad \text{or} \quad \delta_j = -1 \quad (j = 1, 2, ..., n)\,. \tag{1.12}$$

The vector function-function

$$\psi(x, v) = (\psi_1(x, v), ..., \psi_n(x, v)) \tag{1.13}$$

with coordinates functions

$$\psi_j(x, v) = a_j(v) \left\{ (b_0(x))^{-w_0 \frac{\nabla_j}{\nabla}} \prod_{k=1}^{n} (b_k(x))^{\left(\frac{A_{k,j}}{\nabla} - w_0 \frac{\Delta_j}{\nabla} \right)} \right\} \tag{1.14}$$

$$(j = 1, 2, ..., n)\,, \qquad (0 < v \leq h)\,,$$

here

$$w_0 = \frac{1}{1 - \frac{1}{\nabla} \sum\limits_{j=1}^{n} m_j^0 \nabla_j}. \tag{1.15}$$

Let's notice that in equalities (1.9) and (1.10)

$$\frac{y}{\psi(x, v)} = \left(\frac{y_1}{\psi_1(x, v)}, ..., \frac{y_n}{\psi_n(x, v)} \right) \tag{1.16}$$

for $v \in (0, h]$, and the functions

$$b_0 = b_0(x)\,,\; b_k = b_k(x) \quad (k = 1, 2, ..., n) \tag{1.17}$$

are measurable and non-negative functions in domain $G \subset E_n$, i.e. these functions (1.17) in equalities (1.9) and (1.10) are "weight", functions in domain G.

The kernels

$$\Phi_{0,\delta}\left(\frac{y}{\psi(x, h)} \right), \quad \Phi_{k,\delta}\left(\frac{y}{\psi(x, v)} \right) \qquad (k = 1, 2, ..., n)$$

in "weight" integral representations (1.8) - (1.10) are sufficiently smooth and finite functions in E_n, the supports

$$\sup p\Phi_{k,\delta}(y) \quad (k = 0, 1, ..., n)$$

belong to corresponding sets

$$\{y \in E_n; 0 < y_j \delta_j \leq 1 \quad (j = 1, 2, .., n)\}$$

for each $k = 0, 1, 2, ..., n$.

The support of integral representation (1.8) - (1.10) is "a varying $\psi(x, h)$-horn"

$$x + R_\delta(\psi(x, h))$$

with a vertex at the point $x \in G \subset E_n$.

In "weight" integral representations of functions $f = f(x)$ in the form of (1.8), in the domain $G \subset E_n$, "weight", function $b = b(x)$ standing before the function $D^{\nu} f(x)$, in the form $b(x) D^{\nu} f(x)$, is defined by the equality

$$b(x) = (b_0(x))^{\beta_0} \prod_{k=1}^{n} (b_k(x))^{\beta_k}, \tag{1.18}$$

where

$$\beta_0 = \frac{1 - \frac{1}{\nabla} \sum\limits_{j=1}^{n} \nu_j \nabla_j}{1 - \frac{1}{\nabla} \sum\limits_{j=1}^{n} m_j^0 \nabla_j}, \tag{1.19}$$

$$\beta_k = \frac{1}{\nabla} \sum_{j=1}^{n} \left(\nu_j - \beta_0 m_j^0\right) A_{k,j}, \quad (k = 1, 2, .., n). \tag{1.20}$$

Let's notice, that in equalities (1.14), (1.15), (1.19), (1.20), by ∇ the determinant from (1.3) is designated, and the determinant ∇_j $(j = 1, 2, .., n)$ differs from a determinant ∇ (see (1.3)) only by the elements of corresponding j-th column where there are units, i.e.

$$\nabla_j = \begin{vmatrix} m_1^1 & m_2^1 & ... & m_{j-1}^1 & 1 & m_{j+1}^1 & ... & m_n^1 \\ m_1^2 & m_2^2 & ... & m_{j-1}^2 & 1 & m_{j+1}^2 & ... & m_j^2 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ m_1^n & m_2^n & ... & m_{j-1}^n & 1 & m_{j+1}^n & ... & m_n^n \end{vmatrix} \tag{1.21}$$

for each $j \in e_n = \{1, 2, ..., n\}$.

In equalities (1.20), by $A_{k,j}$ a complement of determinant ∇ (see (1.3)), corresponding to the element standing on intersection of k-th line and j-th column is designated at corresponding $k, j \in e_n = \{1, 2, ..., n\}$.

The domain $G \subset E_n$ satisfies the condition of "a varying $\psi(x, h)$-horn", thus coordinates of a vector-function

$$\psi(x, h) = (\psi_1(x, v), ..., \psi_n(x, v))$$

are defined by equalities (1.14), where the coordinates of a vector-function

$$a(v) = (a_1(v), ..., a_n(v)) \tag{1.22}$$

in $(0, h]$ are subjected to conditions

$$\left.\begin{array}{l} a_j = a_j(v) > 0 \ \ (v \neq 0), \\ \lim\limits_{v \to 0+} a_j(v) = 0 \quad (j = 1, 2, .., n) \end{array}\right\}, \tag{1.23}$$

and they are smooth in $(0, h]$ and

$$\frac{d}{d\nu} a_j(v) > 0 \ \ (j = 1, 2, ..., n) \tag{1.24}$$

in $(0,h]$.

§ 2. "Weight" integral inequalities

The "weight" integrated identities (1.8) given in § 1, are devices of research of functions from corresponding "weight" spaces of differentiable functions with whose help validity of "weight" integral inequalities of imbedding type theorems are proved.

2.1. Definitions and necessary designations

Definition. *"Weight" space*

$$\bigcap_{k=0}^{n} L_P^{<m^k>}(G,b_k) \tag{2.1}$$

is a closure of a set of enough smooth and finite in E_n functions $f=f(x)$ with respect to

$$\sum_{k=0}^{n} \left\| b_k(\cdot)\, D^{m^k} f(\cdot)\right\|_{L_p(G)} < \infty. \tag{2.2}$$

Let's notice that $1 \le P \le \infty$, and the functions

$$b_0 = b_0(x)\,,\ b_k = b_k(x)\ \ (k=1,2,...,n) \tag{2.3}$$

are arbitrary "weight" functions, in domain $G \subset E_n$.

Let's recall, that the subdomain $\Omega \subset G$ is called as a subdomain, satisfying the condition of "a varying $\psi(x,h)$-horn", if there is a vector $\delta=(\delta_1,...,\delta_n)$ with coordinates

$$\delta_j = +1 \ \text{ or } \ \delta_j = -1 \ \ (j=1,2,...,n)\,, \tag{2.4}$$

for which

$$x + R_\delta(\psi(x,h)) \subset G \tag{2.5}$$

for all $x \in \Omega$.

Here

$$R_\delta(\psi(x,h)) = \bigcup_{0<v\le h} \left\{ \begin{array}{c} y \in E_n; c_j^* \le \dfrac{y_j \delta_j}{\psi_j(x,v)} \le c_j^{**} \\ (j=1,2,...,n) \end{array} \right\} \tag{2.6}$$

is combination of sets of all possible points $y \in E_n$ whose coordinates are subjected to corresponding inequalities

$$c_j^* \le \frac{y_j\delta_j}{\psi_j(x,v)} \le c_j^{**} \qquad \left(c_j^*, c_j^{**} = const\right) \tag{2.7}$$

$$(j=1,2,...,n)$$

for each $v \in (0,h]$.

The set

$$x + R_\delta(\psi(x,h)) \tag{2.8}$$

is called "a varying $\psi(x,h)$-horn", with a vertex at the point $x \in G \subset E_n$.

Definition. *The domain*

$$G \subset E_n \tag{2.9}$$

is called a domain satisfying the condition of "a changing $\psi(x,h)$-horn" if there is a set of subdomains

$$\Omega_1, \Omega_2, ..., \Omega_K \subset G, \tag{2.10}$$

satisfying the condition of "a varying $\psi(x,h)$-horn", covering the domain G, i.e. such, that

$$\bigcup_{i=1}^{K} \Omega_i = G. \tag{2.11}$$

2.2 Formulation of the basic results. We cite the basic results from the series of "weight" integral inequalities, in the case when differentiable functions $f = f(x)$ are from "weight" spaces (2.1), thus they are defined in the domain $G \subset E_n$, satisfying the condition of "a varying $\psi(x,h)$-horn".

We assume, that a vector function

$$\psi(x,v) = (\psi_1(x,v), ..., \psi_n(x,v)) \tag{2.12}$$

in $(0,h]$ has coordinates of the functions defined by equality (1.14) for

$$a_j(v) = v^{\sigma_j} \ (\sigma_j > 0) \quad (j = 1,2,...,n). \tag{2.13}$$

Theorem 1. *Let*

$$f \in \bigcap_{k=0}^{n} L_P^{<m^k>}(G; b_k). \tag{2.14}$$

Here we suppose the followings:

1) $1 < p \le q < \infty$, *and vectors*

$$m^0 = \left(m_1^0, ..., m_n^0\right), \ m^k = \left(m_1^k, ..., m_n^k\right) \ (k = 1,2,...,n) \tag{2.15}$$

are "integer and non-negative" i.e. $m_j^k \ge 0 \ (j = 1,2,...,n)$ *are entire for all* $k = 0,1,...,n$, *thus vectors (2.15) "$*$- located" i.e.*

$$\sup pm^k \supset \{k\} \ (k = 1,2,...,n), \tag{2.16}$$

from which it follows, that

$$m_k^k > 0 \ (k = 1,2,...,n). \tag{2.17}$$

2) Further, it is supposed that a determinant $\nabla \ne 0$ *from (1.3), i.e. a system of vectors*

$$m^k = \left(m_1^k, ..., m_n^k\right) \ (k = 1,2,...,n) \tag{2.18}$$

is a linearly-independent system.

3) It is supposed that domain

$$G \subset E_n \tag{2.19}$$

satisfies the condition of "a varying $\psi(x,h)$-horn", thus coordinates of a vector-function

$$\psi(x,v) = (\psi_1(x,v), ..., \psi_n(x,v)) \tag{2.20}$$

in $(0,h]$ are defined by equalities (1.14) for

$$a_j = a_j(v) = v^{\sigma_j} (\sigma_j > 0) \quad (j = 1,2,...,n), \tag{2.21}$$

i.e. in $(0,h]$ (see (1.14)

$$\psi_j(x,v) = v^{\sigma_j} \left\{ (b_0(x))^{-w_0 \frac{\nabla_j}{\nabla}} \prod_{k=1}^{n} (b_k(x))^{\left(\frac{A_{kj}}{\nabla} - w_0 \frac{\nabla_j}{\nabla}\right)} \right\} \tag{2.22}$$

$$(j = 1,2,...,n).$$

Let now "an integer non-negative vector"

$$\nu = (\nu_1, ..., \nu_n),$$

i.e. the vector, with coordinates $\nu_j \geq 0$ $(j = 1,2,...,n)$ entire, satisfy the condition of "$*$-coordination" with vectors (2.15), in the form

$$\left.\begin{array}{l} \nu_j \geq m_j^0 \,(j = 1,2,...,n) \\ \nu_j \geq m_j^k \,(\textit{for } j \neq k), \ \nu_k < m_k^k \,(\textit{for } j = k) \ (k = 1,2,...,n) \end{array}\right\}, \tag{2.23}$$

thus let the following conditions be satisfied

$$æ_k = \left(m^k - \nu - \frac{1}{p} + \frac{1}{q}, \sigma\right) = \sum_{j=1}^{n} \left(m_j^k - \nu_j - \frac{1}{p} + \frac{1}{q}\right) \sigma_j \geq 0 \tag{2.24}$$

$$(k = 1,2,...,n) \ (1 < p \leq q < \infty),$$

and the equality

$$æ_k = 0 \ (\textit{for some } k \in e_n = \{1,2,...,n\}) \tag{2.25}$$

is supposed, only in the case $1 < p < q < \infty$.

Then, in the specified conditions, there are generalized derivatives $D^\nu f(x)$ in domain $G \subset E_n$, such that

$$b(x) D^\nu f(x) \in L_q(G) \tag{2.26}$$

and the "weight" integral inequalities are valid

$$\|b(\cdot) D^\nu f(\cdot)\|_{L_q(G)} \leq c \sum_{k=0}^{n} h^{æ_k} \left\| b_k(\cdot) D^{m^k} f(\cdot) \right\|_{L_p(G)}, \tag{2.27}$$

where $æ_k$ $(k = 1,2,...,n)$ are defined by equalities (2.24), (2.25), and

$$æ_0 = \left(m^0 - \nu - \frac{1}{p} + \frac{1}{q}, \sigma\right) = \sum_{j=1}^{n} \sigma_j \left(m_j^0 - \nu_j - \frac{1}{p} + \frac{1}{q}\right) < 0, \tag{2.28}$$

c is a constant independent of the function $f = f(x)$ *and of* $h > 0$.

Let's notice, that in inequalities (2.27) "weight" function (in domain $G \subset E_n$) $b = b(x)$ is defined by the equality

$$b(x) = (b_0(x))^{\gamma_0} \prod_{k=1}^{n} (b_k(x))^{\gamma_k}, \tag{2.29}$$

thus the numbers γ_0, γ_k $(k = 1, 2, ..., n)$ are defined by formulas

$$\gamma_0 = \frac{1}{\beta_0} \left(1 - \frac{1}{\nabla} \sum_{k=1}^{n} \nabla_{,k}(1) \right) \tag{2.30}$$

$$\left(\beta_0 = 1 - \frac{1}{\nabla} \sum_{j=1}^{n} \nabla_{j,}\left(m^0\right) \right),$$

$$\gamma_k = \frac{\nabla_{k,}\left(\nu + \frac{1}{p} - \frac{1}{q}\right)}{\nabla} + \frac{\nabla_{k,}\left(m^0\right)}{\nabla} \gamma_0, \quad k = 1, 2, ..., n, \tag{2.31}$$

where (recall that)

$$\nabla_{k,}\left(\nu + \frac{1}{p} - \frac{1}{q}\right) \quad (\text{for each } k \in e_n = \{1, 2, ..., n\})$$

is a determinant which differs from a determinant ∇ (see (3.1)) only by the elements of k-th line instead of which there are corresponding coordinates of a vector

$$v + \frac{1}{p} - \frac{1}{q} = \left(\nu_1 + \frac{1}{p} - \frac{1}{q}, ..., \nu_n + \frac{1}{p} - \frac{1}{q} \right), \tag{2.32}$$

the determinant $\nabla_{,k}(1)$ differs from a determinant ∇ (see (3.1)) only by the elements of k-th columns instead of which there are corresponding coordinates of a vector $1 = (1, ..., 1)$, the determinant $\nabla_{j,}\left(m^0\right)$ differs from a determinant ∇ (see (3.1)) only by the elements of j-th line, instead of which there are corresponding coordinates of a vector $m^0 = \left(m_1^0, ..., m_1^0\right)$.

On "weight" functions

$$b_0 = b_0(x), b_k = b_k(x) \ (k = 1, 2, ..., n) \tag{2.33}$$

the condition of "slow growth" is imposed additionally in the form

$$\frac{b_k(x)}{b_k(x+y)} \leq c_k = const \ \ (k = 0, 1, 2, ..., n) \tag{2.34}$$

for all

$$y \in x + R_{\delta^i}(\psi(x, h)) \tag{2.35}$$

for each $i = 1, 2, ..., K$, and$\delta^i = (\delta_1, ..., \delta_n)$ with coordinates

$$\delta_j = +1 \ \text{ or } \delta_j = -1 \ (j = 1, ..., n),$$

namely the vector, for which

$$x + R_{\delta^i}\left(\psi\left(x,h\right)\right) \subset G \tag{2.36}$$

for all $x \in \Omega_i$.

2.3 Special variants of Theorem 1. Let's consider a variant when vectors (2.15) are given in the form

$$\left.\begin{array}{l} m^0 = (0,...,0)\,, m^k = (0,...,0,m_k,0,...,0) \\ (k = 1,2,...,n) \end{array}\right\}. \tag{2.37}$$

In this case (2.37), the determinant ∇ (see (3.1)) is given by the equality

$$\nabla = \begin{vmatrix} m_1 & 0 & ... & 0 \\ 0 & m_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & m_n \end{vmatrix} = \prod_{j=1}^{n} m_j \neq 0, \tag{2.38}$$

hence

$$\nabla_j = \nabla_{,j}\left(1\right) = \prod_{k \neq j} m_k = m_1 \cdot m_2 ... m_{j-1} m_{j+i} ... m_n. \tag{2.39}$$

Then it is obvious that

$$\frac{\nabla_j}{\nabla} = \frac{1}{m_j} \;\; (j = 1,2,...,n). \tag{2.40}$$

In case of (2.37) it follows, from equalities (1.20) - (1.22), that

$$A_{kj} = \begin{cases} 0 \;\text{ for } j \neq k, \\ \prod\limits_{i \neq k} m_i \;\text{ for } j = k \end{cases} \quad (k,j \in e_n = \{1,2,...,n\}), \tag{2.41}$$

$$\beta_0 = 1 - \sum_{j=1}^{n} \frac{\nu_j}{m_j}, \tag{2.42}$$

$$\beta_k = \frac{\nu_k}{m_k} \;\; (k = 1,2,...,n)\,. \tag{2.43}$$

It follows from (2.39) - (2.43), that the "weight" function $b = b\left(x\right)$ from integral representation (1.8) (see (1.18)) is given by the equality

$$b = b\left(x\right) = \left(b_0\left(x\right)\right)^{\left(1 - \sum\limits_{j=1}^{n} \frac{\nu_j}{m_j}\right)} \prod_{k=1}^{n} \left(b_k\left(x\right)\right)^{\frac{\nu_k}{m_k}}. \tag{2.44}$$

Let's notice, that in case of (2.37), coordinates of vectors of the function

$$\psi\left(x,v\right) = \left(\psi_1\left(x,v\right),...,\psi_n\left(x,v\right)\right), \tag{2.45}$$

from equalities (1.14), for all $\nu \in (0,h]$ it is given by the formula

$$\psi_j(x,v) = a_j(v)\left\{(b_0(x))^{-\frac{1}{m_j}}\prod_{\substack{k=1\\k\neq j}}^{n}(b_k(x))^{-\frac{1}{m_k}}\right\}, \tag{2.46}$$

for all $j = 1,2,...,n$, but in Theorem 1, it is considered the case

$$a_j(v) = v^{\sigma_j}\,(\sigma_j > 0) \quad (j = 1,2,...,n),$$

i.e. in Theorem 1 (in case (2.37)) the domain $G \subset E$ satisfies the condition of "a varying $\psi(x,h)$-horn", when coordinates

$$\psi_j(x,v) = v^{\sigma_j}\left\{(b_0(x))^{-\frac{1}{m_j}}\prod_{k\neq j}(b_k(x))^{-\frac{1}{m_k}}\right\}, \tag{2.47}$$

for $v \in (0,h]$ and $\sigma_j > 0$ $(j = 1,2,...,n)$.

In inequalities (2.27) of Theorem 1, in a case of vectors (2.37), "weight" function $b = b(x)$ is defined by the equality (see (2.29)):

$$b(x) = (b_0(x))^{\left(1-\sum\limits_{j=1}^{n}\frac{1}{m_j}\right)}\prod_{k=1}^{n}(b_k(x))^{\frac{\nu_k+\frac{1}{p}-\frac{1}{q}}{m_k}}. \tag{2.48}$$

Let's notice, that in case (2.37) conditions of "$*$-agreement" of a vector $\nu = (\nu_1,...,\nu_n)$ with vectors (2.37) is written in the form

$$\nu_j \geq 0\ (j = 1,2,...,n),\ \nu_k < m_k \quad (k = 1,2,...,n), \tag{2.49}$$

thus it is supposed, that

$$æ_k = m_k\sigma_k - \sum_{j=1}^{n}\left(\nu_j + \frac{1}{p} - \frac{1}{q}\right)\sigma_j \geq 0 \quad (k = 1,2,...,n), \tag{2.50}$$

$$æ_0 = -\sum_{j=1}^{n}\left(\nu_j + \frac{1}{p} - \frac{1}{q}\right)\sigma_j < 0, \tag{2.51}$$

where $1 < p \leq q < \infty$.

Definition. *"Weight" space*

$$W_P^{m_1,...,m_n}(G;b_0,b_1,...,b_n) = L_p(G;b_0)\bigcap_{k=1}^{n}L_p^{m_k}(G;b_k) \tag{2.52}$$

is a closure of a set of sufficiently smooth and finite in E_n functions $f = f(x)$ with respect to

$$\|b_0(\cdot)f(\cdot)\|_{L_p(G)} + \sum_{k=1}^{n}\left\|b_k(\cdot)\frac{\partial^{m_k}}{\partial x_k}f(\cdot)\right\|_{L_p(G)} < \infty. \tag{2.53}$$

Space (2.52) in a case of "weight" functions

$$b_0 = (\rho(x))^{\alpha_0}, \quad b_k = (\rho(x))^{\alpha_k} \ \ (k = 1, 2, ..., n), \tag{2.54}$$

where $\rho = \rho(x)$ is distance from the point $x \in G \subset E_n$ to boundary ∂G of domain G is "weight" space of S.L. Sobolev – L.D. Kudryavtsev, investigated in the works [4] of L.D. Kudryavtsev and his followers.

It is obvious, that spaces (2.52) are contained in definitions of the general "weight" spaces (2.1), as a special case.

Theorem 1, in special case, for spaces (2.52) is given in the following form.

Theorem 2. *Let*

$$f \in W_p^{m_1,...,m_n}(G, b_0, b_1, ..., b_n), \tag{2.55}$$

where $1 < p < \infty$, *and*

$$m_k > 0 \ \ (k = 1, 2, ..., n)\text{-are entire.} \tag{2.56}$$

1) Assume that "weight" functions

$$b_0 = b_0(x), \ b_k = b_k(x) \quad (k = 1, 2, ..., n) \tag{2.57}$$

possess property of "slow growth" of type (2.34).

2) Let the domain $G \subset E_n$ *satisfy the condition of "a varying* $\psi(x, h)$*-horn", thus coordinates a vector-function*

$$\psi(x, v) = (\psi_1(x, v), ..., \psi_n(x, v)),$$

for $v \in (0, h]$, *are defined by equalities*

$$\psi_j(x, v) = v^{\sigma_j} \left\{ (b_0(x))^{-\frac{1}{m_j}} \prod_{k \neq j} (b_k(x))^{-\frac{1}{m_k}} \right\} \ \ (j = 1, 2, ..., n). \tag{2.58}$$

3) Let "an integer vector"

$$\nu = (\nu_1, ..., \nu_n), \tag{2.59}$$

with coordinates $\nu_j \geq 0 \ \ (j = 1, 2, ..., n)$*-are entire, are subjected to conditions of "$*$-distributions" of kind*

$$\nu_j \geq 0 \ \ (j \neq k), \nu_k < m_k \ \ (k = 1, 2, ..., n), \tag{2.60}$$

thus let

$$æ_k = m_k\sigma_k - \sum_{j=1}^{n} \left(\nu_j + \frac{1}{p} - \frac{1}{q}\right)\sigma_j \geq 0 \ \ (k = 1, 2, ..., n), \tag{2.61}$$

for $1 < p \leq q < \infty$, *and equality* $æ_k = 0$, *for some* $k \in e_n = \{1, 2, ..., n\}$ *is supposed only in the case* $1 < p < q < \infty$.

Then, in the specified conditions, there are generalized derivatives $D^{\nu}f(x)$ (in domain $G\subset E_n$) which with "weight" (2.48) belong to $L_q(G)$ i.e.

$$b(x)\,D^{\nu}f(x)=(b_0(x))^{\left(1-\sum\limits_{j=1}^{n}\frac{1}{m_j}\right)}\prod_{k=1}^{n}(b_k(x))^{\frac{\nu_k+\frac{1}{p}-\frac{1}{q}}{m_k}}D^{\nu}f\in L_q(G), \tag{2.62}$$

and the following integral inequalities are valid

$$\|b(\cdot)\,D^{\nu}f(\cdot)\|_{L_q(G)}\le c\left\{\sum_{k=0}^{n}h^{æ_k}\left\|b_k(\cdot)\,D_{x_k}^{m_k}f(\cdot)\right\|_{L_q(G)}\right\}<\infty, \tag{2.63}$$

where c is a constant independent of the function $f=f(x)$ and of $h>0$.

The numbers $æ_k$ $(k=1,2,...,n)$ will be defined by equalities (2.61), $æ_0$ is defined by the equality

$$æ_0=-\sum_{j=1}^{n}\left(\nu_j+\frac{1}{p}-\frac{1}{q}\right)\sigma_j.$$

Corollary from theorem 2. *We assume that "weight" functions (in Theorem 2) are defined as*

$$b_0=(\rho(x))^{\alpha_0},\ b_k=(\rho(x))^{\alpha_k}\ \ (k=1,2,...,n), \tag{2.64}$$

where by $\rho=\rho(x)$ are designated distance from the point $x\in G$ to boundary ∂G of domain $G\subset E_n$ and $\alpha_0>0,\alpha_k>0$ $(k=1,...,n)$ are the fixed numbers.

Then "weight" function $b=b(x)$ in inequalities (2.63), is defined by equality

$$b=(\rho(x))^{\gamma}, \tag{2.65}$$

where

$$\gamma=\alpha_0\left(1-\sum_{j=1}^{n}\frac{1}{m_j}\right)+\sum_{k=1}^{n}\frac{\alpha_k}{m_k}\left(\nu_k+\frac{1}{p}-\frac{1}{q}\right), \tag{2.66}$$

and the following integral inequalities hold

$$\|\rho^{\gamma}(\cdot)\,D^{\nu}f(\cdot)\|_{L_p(G)}\le$$

$$\le c\left\{h^{æ_0}\|\rho^{\alpha_0}(\cdot)\,f(\cdot)\|_{L_p(G)}+\sum_{k=0}^{n}h^{æ_k}\left\|\rho^{\alpha_k}(\cdot)\,D_{x_k}^{m_k}f(\cdot)\right\|_{L_p(G)}\right\}, \tag{2.67}$$

where c is a constant independent of the function $f=f(x)$ and of $h>0$.

§3. The scheme of the proof of the basic theorem

3.1. A set of auxiliary functions. By the condition of basic Theorem 1, the domain $G\subset E_n$ satisfies the condition of "a varying $\psi(x,h)$-horn", hence there is a set of subdomains

$$G_{\mu}\subset G\ \ (\mu=1,2,....,\mathcal{K}), \tag{3.1}$$

satisfying the condition of "a varying $\psi(x,h)$-horn" and covering the domain $G \subset E$, i.e.

$$G = \bigcup_{\mu=1}^{\mathcal{K}} G_\mu, \tag{3.2}$$

thus there is a corresponding set of vectors

$$\delta^\mu = (\delta_1^\mu, ..., \delta_n^\mu) \quad (\mu = 1, 2, ..., \mathcal{K}) \tag{3.3}$$

with coordinates

$$\delta_j^\mu = +1 \quad \text{or} \quad \delta_j^\mu = -1\,(j = 1, ..., n), \quad (\mu = 1, 2, ..., \mathcal{K}) \tag{3.4}$$

such that

$$G_\mu + R_{\delta^\mu}(\psi(x,h)) \subset G \tag{3.5}$$

at corresponding $(\mu = 1, 2, ..., \mathcal{K})$.

We define, a set of auxiliary functions

$$f_{\nu;G_\mu + R_{\delta^\mu}}(x) \quad (\mu = 1, 2, ..., \mathcal{K})$$

by means of "weight" integral representation (1.8)-(1.10) of functions $f = f(x)$, by the equalities

$$b(x)\, f_{\nu;G_\mu + R_{\delta^\mu}}(x) = \mathcal{B}_{0,\delta^\mu}\left(\chi(G_\mu + R_{\delta^\mu})\, D^{m^0} f(x)\right) +$$

$$+ \sum_{k=1}^{n} \mathcal{B}_{k,\delta^\mu}\left(\chi(G_\mu + R_\delta)\, D^{m^k} f(x)\right) \tag{3.6}$$

at corresponding $\mu = 1, 2, ..., \mathcal{K}$, where $\chi = \chi(G_\mu + R_{\delta^\mu})$ are characteristic functions of the set $G_\mu + R_{\delta^\mu}$.

Let's notice, that the integral operators standing in the right part of equalities (3.6), are defined by formulas:

$$\mathcal{B}_{0,\delta^\mu}\left(\chi(G_\mu + R_{\delta^\mu})\, D^{m^0} f(x)\right) =$$

$$= (-1)^{\left|m^0 - \nu\right|} c_0 h^{(m^0 - \nu, \sigma)} \int\limits_{E_n} \chi(G_\mu + R_{\delta^\mu}) \left\{ b_0(x+y)\, D^{m^0} f(x+y) \right\} \times$$

$$\times \left(\frac{b_0(x)}{b_0(x+y)} \right) \Phi_{0,\delta} \left(\frac{y}{\psi(x,h)} \right) \frac{dy}{mes\,(R_{\delta^\mu} \cdot E_n)_h}, \tag{3.7}$$

for $k = 0$;

$$\mathcal{B}_{k,\delta^\mu}\left(\chi(G_\mu + R_{\delta^\mu})\, D^{m^k} f(x)\right) =$$

$$= (-1)^{\left|m^k - \nu\right|} \sigma_k c_k \int\limits_0^h \frac{dv}{v^{1-(\sigma, m^k - \nu)}} \int\limits_{E_n} \chi(G_\mu + R_{\delta^\mu}) \left\{ b_k(x+y)\, D^{m^k} f(x+y) \right\} \times$$

$$\times \left(\frac{b_k(x)}{b_k(x+y)} \right) \Phi_{k,\delta^{\mu}} \left(\frac{y}{\psi(x,v)} \right) \frac{dy}{mes\,(R_{\delta^{\mu}} \cdot E_n)_v} \qquad (3.8)$$

in the case $k \in e_n = \{1, 2, ..., n\}$ for corresponding $\mu = 1, 2, ..., \mathcal{K}$.

In equalities (3.7) and (3.8)

$$\left| m^k - \nu \right| = \sum_{j=1}^{n} \left(m_j^k - \nu_j \right) \quad (k = 0, 1, ..., n), \qquad (3.9)$$

$$\left(\sigma, m^k - \nu \right) = \sum_{j=1}^{n} \sigma_j \left(m_j^k - \nu_j \right) \quad (k = 0, 1, ..., n), \qquad (3.10)$$

$$\sigma_j > 0 \ \ (j = 1, 2, ..., n),$$

and c_k $(k = 0, 1, ..., n)$ are the defined fixed constants in dependent of the function $f = f(x)$ and of $h > 0$, thus

$$mes\,(R_{\delta^{\mu}} \cdot E_n)_v = \prod_{j=1}^{n} \psi_j(x,v) \qquad (3.11)$$

for $v \in (0, h]$ and the coordinates-functions

$$\psi_j = \psi_j(x,v) \ \ (j = 1, 2, ..., n)$$

are defined from equalities (1.14) at

$$a_j(v) = v^{\sigma_j} \ \ (j = 1, 2, ..., n). \qquad (3.12)$$

We can easily see that differentiable functions $f = f(x)$ from spaces (2.14) are represented in the form of integral identities (1.8) - (1.10) in the conditions (3.12).

From construction of a set of auxiliary functions (3.6) it is seen, that each auxiliary function

$$b(x)\, f_{\nu;G_{\mu}+R_{\delta}}(x), \qquad (3.13)$$

coincides with the function

$$b(x)\, D^{\nu} f(x), \qquad (3.14)$$

in the subdomain $G_{\mu} + R_{\delta^{\mu}} \subset G$ for each $\mu = 1, 2, ..., \mathcal{K}$.

This affirms, that

$$\|b(\cdot)\, D^{\nu} f(\cdot)\|_{L_q(G)} \leq c \sum_{\mu=1}^{\mathcal{K}} \left\| b(\cdot)\, f_{\nu;G_{\mu}+R_{\delta^{\mu}}}(\cdot) \right\|_{L_q(E_n)}, \qquad (3.15)$$

whence it follows, that

$$\|b(\cdot)\, D^{\nu} f(\cdot)\|_{L_q(G)} \leq c \sum_{\mu=1}^{\mathcal{K}} \sum_{k=0}^{n} \left\| \mathcal{B}_{k,\delta^{\mu}} \left(\chi(G_{\mu} + R_{\delta^{\mu}})\, D^{m^k} f(\cdot) \right) \right\|_{L_q(E_n)}. \qquad (3.16)$$

This means, that the proof of the basic theorem is reduced to corresponding estimations of integral operators

$$\left\| \mathcal{B}_{k,\delta^\mu} \left(\chi \left(G_\mu + R_{\delta^\mu} \right) D^{m^k} f(\cdot) \right) \right\|_{L_q(E_n)} \tag{3.17}$$

for each $\mu = 1, 2, ..., \mathcal{K}$ and for $k = 0, 1, ..., n$.

3.2. Estimations of integral operators (3.17) is given in the form of the following lemma.

Lemma 1. *In conditions of Theorem 1, for the functions $f = f(x)$, belonging to "weight" space (2.14), the following "weight" integral inequalities hold*

$$\left\| \mathcal{B}_{k,\delta^\mu} \left(\chi \left(G_\mu + R_{\delta^\mu} \right) D^{m^k} f(\cdot) \right) \right\|_{L_q(E_n)} \leq$$

$$\leq c h^{æ_k} \left\| b_k(\cdot) D^{m^k} f(\cdot) \right\|_{L_p(G_\mu + R_{\delta^\mu})}, \quad (k = 0, 1, ..., n;\ \mu = 1, ..., \mathcal{K}), \tag{3.18}$$

where c is a constant independent of the function $f = f(x)$ and of $h > 0$ and the numbers $æ_k$ $(k = 1, 2, ..., n)$ are defined by equalities (2.24), (2.25), $æ_0$ is defined by equality (2.28).

From two inequalities (3.16) and (3.18), it follows, that

$$\| b(\cdot) D^\nu f(\cdot) \|_{L_q(G)} \leq c \sum_{k=0}^{n} h^{æ_k} \left\{ \sum_{\mu=1}^{\mathcal{K}} \left\| b_k(\cdot) D^{m^k} f(\cdot) \right\|_{L_p(G_\mu + R_{\delta^\mu})} \right\} \leq$$

$$\leq c \sum_{k=0}^{n} h^{æ_k} \left\{ \sum_{\mu=1}^{\mathcal{K}} \left\| b_k(\cdot) D^{m^k} f(\cdot) \right\|_{L_p(G)} \right\}, \tag{3.19}$$

that proves inequality (2.27) of Theorem 1. In a "weightless " case, the results of Theorem 2 coincides with the results from [5] (see [2] - [6]).

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Received March 30, 2009; Revised June 18, 2009.