MATHEMATICS

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ON BOUNDARY PROPERTIES OF ANALYTICAL FUNCTIONS

Abstract

Let ν be an arbitrary finite complex Borel measure on the interval $T_0 = [0, 2\pi)$, $u\left(re^{i\varphi}\right)$ its Poisson integral, and $\vartheta\left(re^{i\varphi}\right)$ a function harmonically conjugated with $u\left(re^{i\varphi}\right)$, $F(z) = u(z) + i\vartheta(z)$, $z = re^{i\varphi}$, $F(t)$ non-tangential boundary value of the function $F(z)$ as $z \to t = e^{i\theta}$. In the paper, the analogy of the Cauchy formula is proved for the analytic function $F(z)$ and the conditions satisfying boundary values $F(t)$ are found.

Let ν be an arbitrary finite complex Borel measure on the interval $T_0 = [0; 2\pi)$, $u\left(re^{i\varphi}\right)$ its Poisson integral, and $\vartheta\left(re^{i\varphi}\right)$ be a function harmonically conjugated with $u\left(re^{i\varphi}\right)$, $F(z) = u(z) + i\vartheta(z)$, $z = re^{i\varphi}$, $F(t)$ be nontangential limit value of the function $F(z)$ as $z \to t = e^{i\theta}$. P.L. Ulyanov [1] shows that if ν is absolutely continuous finite measure and $g(z)$ is a bounded analytical function in the circle $|z| < 1$, then

$$\lim_{\lambda\to+\infty} \int\limits_{\{t:|F(t)g(t)|\le\lambda\}} F(t)\, g(t)\, dt = 0. \tag{1}$$

But when ν is not absolutely continuous, formula (1) becomes valid even in case $g(z) \equiv 1$. For example, for discret measure

$$\nu(X) = \begin{cases} 2\pi, \text{ at } 0 \in X, \\ \\ 0, \quad \text{at } 0 \notin X, \end{cases},$$

the analytical function $F(z) = u(z) + i\vartheta(z)$ and its boundary values will be $F(z) = \dfrac{1+z}{1-z}$ and $F(t) = \dfrac{1+t}{1-t}$, respectively, but

$$\lim_{\lambda\to+\infty} \int\limits_{\left\{t:\left|\frac{1+t}{1-t}\right|\le\lambda\right\}} \frac{1+t}{1-t} dt = \nu.p. \int\limits_{T} \frac{1+t}{1-t} dt = -2\pi i \ne 0,$$

where T is a unique surface on a complex plane C, with a center at the arigin of a coordinates.

Let ν be a finite complex Borel measure on T_0. The function $\tilde{\nu}$ which is conjugated to the measure ν,is determined in the following way:

$$\tilde{\nu}(\theta) = \frac{1}{2\pi} \lim_{\varepsilon\to0+} \int\limits_{\{t\in T_0:|t-\theta|>\varepsilon\}} ctg\frac{\theta-t}{2} d\nu(t),\ \theta \in T_0;$$

in particular, when ν is absolutely continuous $d\nu(t) = f(t)\,dt$, then the function $\tilde{\nu}$ is also called conjugated to the function $f(t)$ and it is denoted by $\bar{f}$.

For any real function f on T_0 we'll denote:

$$(f > \lambda) \overset{def}{=} \{\theta \in T_0 : f(\theta) > \lambda\},\ (f < \lambda) \overset{def}{=} \{\theta \in T_0 : f(\theta) < \lambda\},$$

$$(f \geq \lambda) \overset{def}{=} \{\theta \in T_0 : f(\theta) \geq \lambda\},\ (f \leq \lambda) \overset{def}{=} \{\theta \in T_0 : f(\theta) \leq \lambda\}.$$

Lemma 1. *Let the functions $f(t)$ and $g(t)$ be determined on T_0, there exists the limit $\lim\limits_{\lambda\to+\infty} \lambda m(|g| > \lambda) = \alpha$, where m is a Lebesgue measure and the function $f(t)$ is integrable. Then from the existence of the limit $\lim\limits_{\lambda\to+\infty} \int\limits_{(|g-f|\leq\lambda)} g(t)\,dt$ it follows the existence of the limit $\lim\limits_{\lambda\to+\infty} \int\limits_{(|g|\leq\lambda)} g(t)\,dt$ and their equality.*

Proof. From the inclusion

$$[(|g-f| \leq \lambda) \cap (|g| > \lambda)] \subset (\lambda < |g| \leq (1+\varepsilon)\lambda) \cup$$

$$\cup [(|g| > (1+\varepsilon)\lambda) \cap (|g-f| \leq \lambda)],$$

where $\varepsilon > 0$ is a positive number, it follows that

$$m[(|g-f| \leq \lambda) \cap (|g| > \lambda)] \leq m(\lambda < |g| \leq (1+\varepsilon)\lambda) +$$

$$+ m[(|g| > (1+\varepsilon)\lambda) \cap (|g-f| \leq \lambda)] \leq$$

$$\leq m(|g| > \lambda) - m(|g| > (1+\varepsilon)\lambda) + m(|f| \leq \varepsilon\lambda),$$

where m is a Lebesque measure. Hence

$$\lim_{\lambda\to+\infty} \lambda m[(|g-f| \leq \lambda) \cap (|g| > \lambda)] \leq \frac{\varepsilon\alpha}{1+\alpha}.$$

From the arbitrariness of ε we'll obtain that

$$\lim_{\lambda\to+\infty} \lambda m[(|g-f| \leq \lambda) \cap (|g| > \lambda)] = 0. \tag{2}$$

Analogously, it is proved that

$$\lim_{\lambda\to+\infty} \lambda m[(|g| \leq \lambda) \cap (|g-f| > \lambda)] = 0. \tag{3}$$

Then

$$\int\limits_{(|g|\leq\lambda)} g(t)\,dt = \int\limits_{(|g-f|\leq\lambda)} g(t)\,dt + \int\limits_{(|g|\leq\lambda)\cap(|g-f|>\lambda)} g(t)\,dt -$$

$$- \int\limits_{(|g-f|\leq\lambda)\cap(|g|>\lambda)} g(t)\,dt = J_1 + J_2 + J_3.$$

Since $|J_2| \leq \lambda \cdot m\left[(|g| \leq \lambda) \cap (|g-f| > \lambda)\right] \to 0$ as $\lambda \to \infty$ by virtue of (3) and

$$|J_3| = \left| \int\limits_{(|g-f|\leq\lambda)\cap(|g|>\lambda)} \left[g(t) - f(t) + f(t)\right] dt \right| \leq$$

$$\leq \lambda \cdot m\left[(|g-f| \leq \lambda) \cap (|g| > \lambda)\right] + \int\limits_{(|g-f|\leq\lambda)\cap(|g|>\lambda)} f(t)\, dt \to 0$$

as $\lambda \to +\infty$ by virtue of (2) then from the existence of the limit

$$\lim_{\lambda\to+\infty} \int\limits_{(|g-f|\leq\lambda)} g(t)\, dt$$

it follows the existence of the limit

$$\lim_{\lambda\to+\infty} \int\limits_{(|g|\leq\lambda)} g(t)\, dt$$

and their equality. Lemma is proved.

Theorem 1. *Let ν be a finite complex Borel measure on T_0, $u\left(re^{i\varphi}\right)$ be its Poisson integral, and $\vartheta\left(re^{i\varphi}\right)$ be the harmonically conjugated function with $u\left(re^{i\varphi}\right)$. Then there exist the limits $\lim\limits_{\lambda\to+\infty} \int_{T_\lambda} F(t)\, t^n dt$, $n = 0, 1, 2, ...,$ and the equality*

$$\lim_{\lambda\to+\infty} \int\limits_{T_\lambda} F(t)\, t^n dt + i \int\limits_0^{2\pi} e^{i(n+1)\theta} d\nu_s(\theta) = 0, \; n = 0, 1, 2, ..., \tag{4}$$

is true, where $F(z) = u(z) + i\vartheta(z)$, $z = re^{i\theta}$, $F(t)$ is nontangential limit value of the function $F(z)$ for $z \to t = e^{i\theta}$, $T_\lambda = \left\{t = e^{i\theta} : |F(t) \leq \lambda|\right\}$, ν_s is a singular part of the measure ν.

We need the following theorem proved by the author [2] (theorem A) and S.A.Vinogradov, S.B.Khrushev [3] (theorem B).

Theorem A[2]. *If ν is a finite complex Borel measure on T_0 and the function $g(t)$ is continuous by Hölder on T_0, then there exist the limit $\lim\limits_{\lambda\to+\infty} \int\limits_{(|\tilde{\nu}\cdot g|\leq\lambda)d\theta} \tilde{\nu}(\theta)\, g(\theta)\, d\theta$ and the equality*

$$\int\limits_0^{2\pi} g(\theta)\, d\nu(\theta) = -\lim_{\lambda\to+\infty} \int\limits_{(|\tilde{\nu}\cdot g|\leq\lambda)} \tilde{\nu}(\theta)\, g(\theta)\, d\theta. \tag{5}$$

is true.

Theorem B [3]. *For any finite complex Borel measure μ on a unit circle T the equality*

$$\lim_{\lambda\to+\infty} \lambda \cdot m\left(\left|\tilde{\mu}\left(e^{i\theta}\right)\right| > \lambda\right) = \frac{2}{\pi} \|\mu_s\|,$$

is true, where $\|\mu_s\| = \int\limits_0^{2\pi} |d\mu_s(\theta)|$.

Proof of theorem 1. Let $d\nu(\theta) = f(\theta)\,d\theta + d\nu_s(\theta)$. Then [4]

$$F(t) = F\left(e^{i\theta}\right) = f(\theta) + i\tilde{\nu}(\theta).$$

Since

$$\overline{e^{i(n+1)\theta}} = -ie^{i(n+1)\theta},$$

then assuming $g(\theta) = e^{i(n+1)\theta}$, from equality (5) of theorem A we'll obtain that

$$\int\limits_0^{2\pi} e^{i(n+1)\theta} d\nu(\theta) = -i \lim_{\lambda\to+\infty} \int\limits_{(|\tilde{\nu}|\le\lambda)} e^{i(n+1)\theta}\tilde{\nu}(\theta)\,d\theta =$$

$$= -\lim_{\lambda\to+\infty} \int\limits_{(|\tilde{\nu}|\le\lambda)} e^{i(n+1)\theta}\left[F\left(e^{i\theta}\right) - f(\theta)\right] d\theta =$$

$$= -\lim_{\lambda\to+\infty} \int\limits_{(|\tilde{\nu}|\le\lambda)} e^{i(n+1)\theta} F\left(e^{i\theta}\right) d\theta + \int\limits_0^{2\pi} f(\theta)\,e^{i(n+1)} d\theta.$$

Hence from the integrability of the function $f(\theta)$ and from lemma 1 and from theorem B it follows that

$$i\int\limits_0^{2\pi} e^{i(n+1)\theta} d\nu_s(\theta) = i\left[\int\limits_0^{2\pi} e^{i(n+1)\theta} d\nu(\theta) - \int\limits_0^{2\pi} f(\theta)\,e^{i(n+1)\theta} d\theta\right] =$$

$$= -\lim_{\lambda\to+\infty} \int\limits_{(|\tilde{\nu}|\le\lambda)} e^{in\theta} F\left(e^{i\theta}\right) ie^{i\theta} d\theta =$$

$$= -\lim_{\lambda\to+\infty} \int\limits_{\left(\left|F\left(e^{i\theta}\right)\right|\le\lambda\right)} e^{in\theta} F\left(e^{i\theta}\right) ie^{i\theta} d\theta = -\lim_{\lambda\to+\infty} \int\limits_{T_\lambda} F(t)\,t^n dt,$$

i.e., equality (4) is true. Theorem is proved.

In future the functions, determined on T_0 will be considered as periodic function determined on R with period 2π.

For any real function f on T_0 we'll denote [see 2]

$$P(f;q;\theta_0;\theta) = \frac{\pi}{2} ctg \frac{\pi q^{n-1}}{2} \cdot f(\theta)$$

for $\theta \in \left(\theta_0 + \pi q^n; \theta_0 + \pi q^{n-1}\right) \cup \left(\theta_0 - \pi q^{n-1}; \theta_0 - \pi q^n\right),\ n \in N,\ 0 < q < 1,\ \theta_0 \in T_0$,

$$P_1(f;\theta_0) = \lim_{q\to1-} \lim_{\alpha\to+\infty} \alpha \cdot m\left\{\theta \in (\theta_0; \theta_0 + \pi) : |P(f;q;\theta_0;\theta)| > \alpha\right\}, \qquad (6)$$

$$P_2(f;\theta_0) = \lim_{q\to1-} \lim_{\alpha\to+\infty} \alpha \cdot m\left\{\theta \in (\theta_0 - \pi; \theta_0) : |P(f;q;\theta_0;\theta)| > \alpha\right\} \qquad (7)$$

$$r_{\lambda,f}\left(\theta_0\right)=\begin{cases} signP_2\left(f;\theta_0\right)-P_1\left(f;\theta_0\right), & \text{at } f\left(\theta_0\right)>\lambda, \\ 0, & \text{at } \left|f\left(\theta_0\right)\right|\leq\lambda, \\ signP_1\left(f;\theta_0\right)-P_2\left(f;\theta_0\right) & \text{at } f\left(\theta_0\right)<-\lambda, \end{cases}$$

if there exists the limits on the right part of equalities (6), (7) almost for all $\theta_0\in[0,2\pi)$.

In the paper [2] the author proved that for any finite complex measure ν and continuous by Holder complex function f the equality

$$\frac{\pi}{2}\int_0^{2\pi} f\left(\theta\right)d\nu_s\left(\theta\right)=\lim_{\lambda\to+\infty}\lambda\cdot\int_0^{2\pi} r_{\lambda,\mathrm{Re}(f\cdot\tilde{\nu})}\left(\theta_0\right)d\theta_0+i\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi} r_{\lambda,\mathrm{Im}(f\cdot\tilde{\nu})}\left(\theta_0\right)d\theta_0. \quad (8)$$

is fulfilled.

Theorem 2. *Let ν be a finite complex Borel measure on T_0, $u\left(re^{i\varphi}\right)$ its Poisson integral, and $\vartheta\left(re^{i\varphi}\right)$ be a harmonically conjugated function with $u\left(re^{i\varphi}\right)$. Then*

$$\lim_{\lambda\to+\infty}\int_{T_\lambda} F\left(t\right)\cdot t^n dt+\frac{\pi}{2}\lim_{\lambda\to+\infty}\lambda\cdot\int_0^{2\pi} r_{\lambda\,\mathrm{Re}\,h_n}\left(\theta_0\right)d\theta_0+$$

$$+\frac{\pi i}{2}\lim_{\lambda\to+\infty}\lambda\cdot\int_0^{2\pi} r_{\lambda,\mathrm{Im}\,h_n}\left(\theta_0\right)d\theta_0=0,\ n=0,1,2,... \quad (9)$$

where $h_n\left(\theta\right)=e^{i(n+1)\theta}F\left(e^{i\theta}\right)$, $F\left(z\right)=u\left(z\right)+i\nu\left(z\right)$, $z=e^{i\varphi}$, $F\left(t\right)$ are nontangential limit values of the function $F\left(z\right)$ for $z\to t=e^{i\theta}$, $T_\lambda=\left\{t=e^{i\theta}:\left|F\left(t\right)\right|\leq\lambda\right\}$.

Proof. Denote by

$$g\left(\theta\right)=ie^{i(n+1)\theta}.$$

From equality (8) it follows that

$$\frac{2}{\pi}\int_0^{2\pi} g\left(\theta\right)d\nu_s\left(\theta\right)=\lim_{\lambda\to+\infty}\lambda\cdot\int_0^{2\pi} r_{\lambda,\mathrm{Re}(g\cdot\tilde{\nu})}\left(\theta_0\right)d\theta_0+$$

$$+i\lim_{\lambda\to+\infty}\lambda\cdot\int_0^{2\pi} r_{\lambda,\mathrm{Im}(f\cdot\tilde{\nu})}\left(\theta_0\right)d\theta_0. \quad (10)$$

Let $d\nu\left(\theta\right)=f\left(\theta\right)d\theta+d\nu_s\left(\theta\right)$ Then [4]

$$F\left(t\right)=F\left(e^{i\theta}\right)=f\left(\theta\right)+i\tilde{\nu}\left(\theta\right)$$

and therefore

$$g\left(\theta\right)\cdot\tilde{\nu}\left(\theta\right)=ie^{i(n+1)}\tilde{\nu}\left(\theta\right)=$$

$$=e^{i(n+1)\theta}\left[F\left(e^{i\theta}\right)-f\left(\theta\right)\right]=h_n\left(\theta\right)+ig\left(\theta\right)f\left(\theta\right).$$

Since the function $g(\theta) f(\theta)$ is integrable, then

$$\lim_{\lambda\to+\infty} \lambda \cdot m\left(|\operatorname{Im}(gf)| > \lambda\right) = 0$$

and consequently (see the proof of lemma 1),

$$\lim_{\lambda\to+\infty} \lambda \cdot m\left[\left(|\operatorname{Re}(gf)| > \lambda\right) \Delta \left(|\operatorname{Re} h_n| > \lambda\right)\right] = 0. \qquad (11)$$

From (11) it follows that for any $0 < q < 1$ the limits

$$\lim_{\lambda\to+\infty} \alpha \cdot m\left\{\theta \in (\theta_0; \theta_0 + \pi) : \left|P\left(\operatorname{Re}(g \cdot \tilde{\nu}); q; \theta_0; \theta\right)\right| > \alpha\right\}$$

and

$$\lim_{\lambda\to+\infty} \alpha \cdot m\left\{\theta \in (\theta_0; \theta_0 + \pi) : \left|P\left(\operatorname{Re} h_n; q; \theta_0; \theta\right)\right| > \alpha\right\}$$

coincide almost everywhere. Then almost everywhere

$$P_1\left(\operatorname{Re}(g \cdot \tilde{\nu}); \theta_0\right) = P_1\left(\operatorname{Re} h_n; \theta_0\right)$$

Analogously it is proved that

$$P_2\left(\operatorname{Re}(g \cdot \tilde{\nu}); \theta_0\right) = P_2\left(\operatorname{Re} h_n; \theta\right).$$

This means that almost for all $\theta_0 \in \left(|\operatorname{Re} h_n| > \lambda\right) \cap \left(|\operatorname{Re}(g \cdot \tilde{\nu})| > \lambda\right)$

$$r_{\lambda,\operatorname{Re} h_n}(\theta_0) = r_{\lambda,\operatorname{Re}(g\cdot\tilde{\nu})}(\theta_0).$$

Then

$$\int_0^{2\pi} r_{\lambda,\operatorname{Re}(g\cdot\tilde{\nu})}(\theta_0)\, d\theta_0 = \int_{(|\operatorname{Re}(g\cdot\tilde{\nu})|>\lambda)} r_{\lambda,\operatorname{Re}(g\cdot\tilde{\nu})}(\theta_0)\, d\theta_0 =$$

$$= \int_{(|\operatorname{Re}(g\cdot\tilde{\nu})|>\lambda)\cap(|\operatorname{Re} h_n|>\lambda)} r_{\lambda,\operatorname{Re} h_n}(\theta_0)\, d\theta_0 +$$

$$+ \int_{(|\operatorname{Re}(g\cdot\tilde{\nu})|>\lambda)\setminus(|\operatorname{Re} h_n|>\lambda)} r_{\lambda,\operatorname{Re}(g\cdot\tilde{\nu})}(\theta_0)\, d\theta_0 =$$

$$= \int_0^{2\pi} r_{\lambda,\operatorname{Re} h_n}(\theta_0)\, d\theta_0 - \int_{(|\operatorname{Re} h_n|>\lambda)\setminus(|\operatorname{Re}(g\cdot\tilde{\nu})|>\lambda)} r_{\lambda,\operatorname{Re} h_n}(\theta_0)\, d\theta_0 +$$

$$+ \int_{(|\operatorname{Re}(g\cdot\tilde{\nu})|>\lambda)\setminus(|\operatorname{Re} h_n|>\lambda)} r_{\lambda,\operatorname{Re}(g\cdot\tilde{\nu})}(\theta_0)\, d\theta_0,$$

on the other hand

$$\left| \int_{(|\operatorname{Re}(g\cdot\tilde{\nu})|>\lambda)\setminus(|\operatorname{Re} h_n|>\lambda)} r_{\lambda,\operatorname{Re}(g\cdot\tilde{\nu})}(\theta_0)\, d\theta_0 - \right.$$

$$-\int\limits_{(|\mathrm{Re}\,h_n|>\lambda)\backslash(|\mathrm{Re}(g\cdot\tilde{\nu})|>\lambda)} r_{\lambda,\mathrm{Re}\,h_n}(\theta_0)\,d\theta_0\Bigg| \leq$$

$$\leq m\left[\left(\left|\mathrm{Re}\left(g\cdot\tilde{\nu}\right)\right|>\lambda\right)\Delta\left(\left|\mathrm{Re}\,h_n\right|>\lambda\right)\right]$$

Therefore from (11) we'll obtain that

$$\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Re}(g\cdot\tilde{\nu})}(\theta_0)\,d\theta_0=\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Re}\,h_n}(\theta_0)\,d\theta_0. \qquad (12)$$

Analogously it is proved that

$$\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Im}(g\cdot\tilde{\nu})}(\theta_0)\,d\theta_0=\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Im}\,h_n}(\theta_0)\,d\theta_0. \qquad (13)$$

From equalities (10), (12), (13) and from theorem 1 we'll obtain that

$$\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}F(t)\cdot t^n dt+i\int\limits_0^{2\pi}e^{i(n+1)\theta}d\nu_s(\theta)=\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}F(t)\cdot t^n dt+$$

$$+\frac{\pi}{2}\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Re}(g\cdot\tilde{\nu})}(\theta_0)\,d\theta_0+\frac{\pi i}{2}\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Im}(g\cdot\tilde{\nu})}(\theta_0)\,d\theta_0=$$

$$=\lim_{\lambda\to+\infty}\int\limits_{T_\lambda}F(t)\cdot t^n dt+\frac{\pi}{2}\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Re}\,h_n}(\theta_0)\,d\theta_0+$$

$$+\frac{\pi i}{2}\lim_{\lambda\to+\infty}\lambda\cdot\int\limits_0^{2\pi} r_{\lambda,\mathrm{Im}\,h_n}(\theta_0)\,d\theta_0.$$

The theorem is proved.

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