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AN OPTIMAL CONTROL PROBLEM FOR GOURSAT-DARBOUX SYSTEMS WITH BOUNDARY AND DISTRIBUTED CONTROLS UNDER NONLOCAL CONDITIONS

Abstract

In the paper we obtain necessary optimality condition with boundary and distributed controls for Goursat-Darboux system under non-local conditions. The boundary conditions are given on the characteristics in the form of solution of a differential equation and the solution of the differential equation satisfies the integral initial-boundary condition.

1. Introduction

In the paper we consider an optimization problem with distributed and boundary controls for Goursat-Darboux systems with non-local conditions. We get necessary optimality conditions for the considered optimality control problem. Note that while investigating the wear, sorption, drying and other processes we often meet the Goursat-Darboux type boundary value problems [1,2]. The optimal control problems with non-local boundary conditions for the Goursat-Darboux systems are considered also in the papers [3-6] where all the boundary conditions are given in the integral or multi-point form.

2. Problem statement

Let's consider a controlled initial boundary value problem for the Goursat-Darboux system:

$$y_{ts} = f\left(t, s, y(t,s), y_t(t,s), y_s(t,s), u\right), (t,s) \in Q = (0,T) \times (0,l) \tag{2.1}$$

$$y_t(t,0) = \varphi\left(t, y(t,0), v\right), t \in [0,T], \tag{2.2}$$

$$y_s(0,s) = \psi(s), \ \ s \in [0,\lambda], \tag{2.3}$$

$$y(0,0) + Ny\left(T,0\right) = c \tag{2.4}$$

We'll assume that the class of admissible controls consists of the functions $w = (v,u)$, where

$$w = (v(\cdot), u(\cdot)) \in V \times U \subset Lq([0,T]) \times L_2^r(Q). \tag{2.5}$$

Under the solution of the problem (2.1)-(2.4), corresponding to the chosen admissible control $w = (v,u) \in V \times U$ we understand the function $y(t,s) \in L_2^n(Q)$, having general Sobolev derivatives $y_t(t,s), y_s(t,s)$ and $y_{ts}(t,s)$, belonging to $L_2^n(Q)$ and satisfying the equation (2.1) almost everywhere in Q, and the conditions (2.2), (2.3) -in the sense of equality of appropriate traces [7].

On the solutions of initial boundary value problem (2.1)-(2.4) and on the set of admissible controls (2.5) we define the functional

$$J(w) = \sum_{i=1}^{k} \Phi\left(y(t_i, s_i)\right) \tag{2.6}$$

where $(t_i, s_i) \in Q, i = \overline{1, k}$ is an arbitrary fixed collection of points.

Let's consider the following optimal control problem: to minimize the functional (2.6) under restrictions (2.1)-(2.5).

It is assumed that $y \in R^n$ is a state; $u \in R^r$ are controls; t, s are scalar independent variables; T, λ are the given positive numbers; $f, F, \Phi, \varphi, \psi$ are the given functions, and $c \in R^n$ is a fixed point; $n(t) = (n_{ij}(t)), i, j = \overline{1, n}$ is the known matrix function; U and V are the given sets.

The following conditions are imposed on the functions entering into description of problem (2.1)-(2.6).

1. The function $f(t, s, y, p, q, u)$ from the right hand side of (2.1) for almost all $(t, s) \in Q$ is continuous with respect to $(y, p, q, u) \in R^{3n} \times R^r$ and for the fixed $(x, p, q, u) \in R^{3n} \times R^r$ are measurable with respect to $(t, s) \in Q$.

2. The function $\varphi(t, y, v)$ from (2.2) for almost all $t \in [0, T]$ be continuous with respect to $y \in R^n$, be measurable with respect to $t \in [0, T]$ for each fixed $y \in R^n, v \in R^n$

3. $\psi(s) \in L_2^n[0, \lambda]$.

4. $\|N\| < 1$, moreover, $\det[E + N] \neq 0$.

5. $V \times U \subset L_2^q([0, T]) \times L_2^r(Q)$ be a convex closed set.

6. The function $\Phi(y), y \in R^n$ possess continuous partial derivatives $\Phi_y(y)$ for all $y \in R^n$.

7. Let there exist non-negative constants K_1 and K_2 such that $|\varphi(t, 0, 0)| \le K_1$for almost all $t \in [0, T]$, $\left|\varphi_y(t, y, v)\right| \le K_2$ for almost all $t \in [0, T]$, $y \in R^n$, $v \in R^r$.

8. The functions $f(t, s, y, p, q, u)$ and partial derivatives f_y, f_p, f_q, f_u, satisfy the Lipschits condition with respect to $(x, p, q, u) \in R^{3n} \times R^r$.

9. Let the functions $\varphi(t, y, v)$ and partial derivatives $\varphi_y(t, y, v)$ satisfy the Lipschits condition with respect to $y \in R^n$.

We can show that the vector-function $y(t, s)$ is a solution of the initial boundary value problem (2.1)-(2.5) if and only if it satisfies the integral equation

$$y(t, s) = [E + N]^{-1} c - [E + N]^{-1} N \int_0^T \varphi(\tau, y(\tau, 0), v(t))dt+$$

$$+ \int_0^t \varphi(\tau, z(\tau, 0), v(\tau))d\tau + \int_0^s \psi(r)dr+$$

$$+ \int_0^t \int_0^s F(\tau, r, y(\tau, r), y_\tau(\tau, r), y_r(\tau, r), u(\tau, r))d\tau dr.$$

By the successive approximations method we can prove that for

$$K_2 T\left[1+\|N\|\right]\left\|[E+N]^{-1}\right\|<1 \tag{2.7}$$

the initial boundary value problem (2.1)-(2.5) for each fixed admissible control $u \in U$ has a unique solution, where $\tilde{n}_1(T)=\max_{[0,T]}\left|\int_0^t n(\tau)d\tau\right|$.

3. Main results

Theorem 1. *Let the conditions 1-9 and (2.7) be fulfilled. Then the functional (2.6) under restrictions (2.1)-(2.4) is continuous and differentiable with respect to $w \in V\times U$ in the norm $L_2^q(\{[0,T])\times L_2^r(Q)$, and its gradient $J'(w)\in L_2^q(\{[0,T])\times L_2^r(Q)$ at the point $w=(v(t),u(t,x))$ is representable in the form*

$$J'(w)=\left\{H_u(t,s,y,y_z,y_s,u,\psi);H_{1v}(t,y(t,0),\psi_1(t),v)\right\}, \tag{3.1}$$

where $y(t,s)=y(t,s;w),(t,s)\in Q$ is a solution of the initial boundary value problem (2.1)-(2.4), and $(\psi(t,s;w),\ \psi_1(t,w))$ is a solution of the adjoint system:

$$\psi(t,s)=\sum_{i=1}^{k}\Phi_y(y(t_i,s_i))\chi(t_i-t)\chi(s_i-s)+\int_t^T\tilde{H}_q(\tau,s)d\tau+$$

$$+\int_s^\lambda\tilde{H}_p(t,r)dr+\int_t^T\int_s^\lambda H_y(\tau,r)d\tau dr$$

$$\psi_1(t)=\sum_{i=1}^{k}\Phi_y(y(t_i,s_i))\chi(t_i-t)+\int_0^\lambda\int_t^T\tilde{H}_y(\tau,s)d\tau ds+\int_0^\lambda\tilde{H}_p(t,s)ds+$$

$$+\int_t^T\tilde{H}_{1y}(\tau)d\tau-[E+N]^{-1}N'\left[\sum_{i=1}^{k}\Phi_y(y(t_i,s_i))+\int_0^\lambda\int_0^T\tilde{H}_y(t,s)dtds+\int_0^T\tilde{H}_{1y}(t)dt\right],$$

$$\chi(t_i-t)=\begin{cases}0, if\ t_i,t\\ 1,\ if\ t_i\ge t,\end{cases}\qquad \chi(s_i-s)=\begin{cases}0, if\ s_i,s\\ 1,\ if\ s_i\ge s,\end{cases}$$

$$H_1(t,y(t,0),\psi_1(t))=\langle\psi_1(t),\varphi(t,y(t,0))\rangle$$

Proof. Let $y, y+\bar{y}$ be a solution of the problem (2.1)-(2.4) corresponding to the controls $w,\ w+\bar{w}\in U$.

Introduce the system of equations in variations:

$$z_{ts}=\tilde{f}_y(t,s)z(t,s)+\tilde{f}_p(t,s)z_t(t,s)+\tilde{f}_q(t,s)z_s(t,s)+\tilde{f}_u(t,s)\bar{u}(t,s), \tag{3.2}$$

$$z_t(t,0)=\varphi_y(t,y(t,0))z(t,0)+\varphi_v(t,y(t,0),v(t))\bar{v}(t), \tag{3.3}$$

$$z_s(0,s)=0, \tag{3.4}$$

$$z(0,0)+Nz\left(T,0\right)=0 \tag{3.5}$$

Calculating the increments of the functional (2.6) we get

$$J(w+w)-J(w)=\sum_{i=1}^{k}\langle\Phi_y(y(t_i,s_i)),z(t_i,s_i)\rangle+\eta \tag{3.6}$$

where

$$\eta=\sum_{i=1}^{k}\langle\Phi_y(y(t_i,s_i)),\bar{y}(t_i,s_i)-z(t_i,s_s)\rangle+\sum_{i=1}^{k}[\Phi(y(t_i,s_i)+\bar{y}(t_i,s_i))-$$

$$-\Phi(y(t_i,s_i))-\langle\Phi_y(y(t_i,s_i)),\bar{y}(t_i,s_i)\rangle]$$

Multiply (3.2) by some function $\psi(t,s)$, and (3.3) by some function $\psi_1(t)$, integrate with respect to the domain Q and add to (3.6):

$$\Delta J(w)=\sum_{i=1}^{k}\langle\Phi_y(y(t_i,s_i),z(t_i,s_i))\rangle+\int_0^T\int_0^\lambda\left\langle\tilde{H}_y(t,s),z(t,s)\right\rangle+$$

$$+\left\langle\tilde{H}_p(t,s),z_t(t,s)\right\rangle+\left\langle\tilde{H}_q(t,s),z_s(t,s)\right\rangle+\langle H_u(t,s),\bar{u}(t,s)\rangle+$$

$$+\langle-\psi(t,s),z_{ts}\rangle]\,dtds+\left[\int_0^T\langle-\psi_1(t),z_t(t,0)\rangle+\left\langle\tilde{H}_{1y}(t),z(t,0)\right\rangle\right]dt+\eta \tag{3.7}$$

Using the integration by parts formula and Foubini's theorem, we have:

$$\int_0^T\int_0^\lambda\left\langle\tilde{H}_y(t,s),z(t,s)\right\rangle dtds=\left\langle\int_0^T\int_s^\lambda\tilde{H}_y(t,s)dtds,z(0,0)\right\rangle+$$

$$+\int_0^\lambda\left\langle\int_0^T\int_s^\lambda\tilde{H}_y(t,r)drdt,z_s(0,s)\right\rangle ds+\int_0^T\left\langle\int_0^\lambda\int_t^T\tilde{H}_y(\tau,s)d\tau ds,z_t(t,0)\right\rangle dt+$$

$$+\int_0^T\int_0^\lambda\left\langle\int_s^\lambda\int_t^T\tilde{H}_y(\tau,r)d\tau dr,z_{ts}(t,s)\right\rangle dtds, \tag{3.8}$$

$$\int_0^T\int_0^\lambda\left\langle\tilde{H}_p(t,s),z_t(t,s)\right\rangle dtds=\int_0^T\int_0^\lambda\left\langle\tilde{H}_p(t,s),z_t(t,0)\right\rangle dtds+$$

$$+\int_0^T\int_0^\lambda\left\langle\int_s^\lambda\tilde{H}_p(t,r)dr,z_{ts}(t,s)\right\rangle dtds, \tag{3.9}$$

$$\int_0^T\int_0^\lambda\left\langle\tilde{H}_q(t,s),z_s(t,s)\right\rangle dtds=\int_0^T\int_0^\lambda\left\langle\tilde{H}_q(t,s),z_s(0,s)\right\rangle dtds+$$

$$+\int_0^T\int_0^\lambda\left\langle\int_t^T \tilde{H}_q(\tau,s)d\tau, z_{ts}(t,s)\right\rangle dtds. \tag{3.10}$$

Using the equalities

$$z(t_i,s_i)=z(0,0)+\int_0^\lambda z_s(0,s)\chi(s_i-s)ds+$$

$$+\int_0^t z_t(t,0)\chi(t_i-t)dt+\int_0^T\int_0^\lambda z_{ts}(t,s)\chi(t_i-t)\chi(s_i-s)dtds \tag{3.11}$$

taking into account (3.8)-(3.11) in (3.7) we get:

$$\Delta J(w)=J(w+\bar{w})-J(w)=\sum_{i=1}^k\langle\Phi_y(y(t_i,s_i),z(0,0))\rangle+\int_0^\lambda z_s(0,s)\chi(s_i-s)ds+$$

$$+\int_0^T z_t(t,0)\chi(t_i-t)dt+\int_0^T\int_0^\lambda z_{ts}(t,s)\chi(t_i-t)\chi(s_i-s)dtds+$$

$$+\left\langle\int_0^T\int_0^\lambda \tilde{H}_y(t,s)dtds, z(0,0)\right\rangle+\int_0^\lambda\left\langle\int_0^T\int_s^\lambda \tilde{H}_y(t,r)dtdr, z_s(0,s)\right\rangle ds+$$

$$+\int_0^T\left\langle\int_0^\lambda\int_t^T \tilde{H}_y(\tau,s)d\tau ds, z_t(t,0)\right\rangle dt+\int_0^T\int_0^\lambda\left\langle\int_t^T\int_s^\lambda \tilde{H}_y(\tau,r)d\tau dr, z_{ts}(t,s)\right\rangle dtds+$$

$$+\int_0^T\left\langle\int_0^\lambda \tilde{H}_p(t,s)ds, z_t(t,0)\right\rangle dt+\int_0^T\int_0^\lambda\left\langle\int_s^\lambda \tilde{H}_p(t,r)dr, z_{ts}(t,s)\right\rangle dtds+$$

$$+\int_0^\lambda\left\langle\int_0^T \tilde{H}_q(t,s)dt, z_s(0,s)\right\rangle ds+\int_0^T\int_0^\lambda\left\langle\int_t^T \tilde{H}_q(\tau,s)d\tau, z_{ts}(t,s)\right\rangle dtds+$$

$$+\int_0^T\int_0^\lambda\left\langle \tilde{H}_u(t,s),\bar{u}(t,s)\right\rangle dtds+\int_0^T\int_0^\lambda\langle-\psi(t,s),z_{ts}(t,s)\rangle dtds+$$

$$+\int_0^T\langle-\psi_1(t),z_t(t,0)\rangle dt+\left\langle\int_0^T \tilde{H}_{1z}(t)dt, z(0,0)\right\rangle+$$

$$+\int_0^T\left\langle\int_t^T \tilde{H}_{1z}(\tau)d\tau, z_t(t,0)\right\rangle dt+\eta \tag{3.12}$$

In (3.12) we take into account (3.4) and group similar addends then

$$\Delta J(w)=\left\langle \sum_{i=1}^{k}\Phi_y(y(t_i,s_i)+\int_0^T\int_0^\lambda \tilde H_y(t,s)dtds+\int_0^T \tilde H_{1z}(t)dt, z(0,0)\right\rangle +$$

$$+\int_0^T\left\langle \sum_{i=1}^{k}\Phi_y(y(t_i,x_i))\chi(t_i-t)+\int_0^\lambda\int_t^T \tilde H_y(\tau,s)d\tau ds+\int_0^\lambda \tilde H_p(t,s)ds-\psi_1(t)+\right.$$

$$\left.+\int_t^T \tilde H_{1z}(\tau)d\tau, z_t(t,0)\right\rangle dt+\int_0^T\int_0^\lambda\left\langle \sum_{i=1}^{k}\Phi_y(y(t_i,s_i))\chi(t_i-t)\chi(s_i-s)+\right.$$

$$\left.+\int_t^T\int_s^\lambda \tilde H_y(\tau,r)d\tau ds+\int_s^\lambda \tilde H_p(t,r)ds+\int_t^T \tilde H_q(\tau,s)d\tau-\psi(t,s), z_{ts}(t,s)\right\rangle dtds$$

$$+\int_0^T\int_0^\lambda\left\langle \tilde H_u(t,s),\bar u(t,s)\right\rangle dtds+\eta.$$

From (3.3)-(3.5) we have:

$$z(0,0)=-\left[E+N\right]^{-1}N\int_0^T z_t\left(t,0\right)dt \tag{3.13}$$

After some transformations and grouping of similar addends, we get

$$\Delta J(w)=\left\langle \sum_{i=1}^{k}\Phi_y(y(t_i,s_i)+\int_0^T\int_0^\lambda \tilde H_y(t,s)dtds+\int_0^T \tilde H_{1z}(t)dt, z(0,0)\right\rangle +$$

$$+\int_0^T\left\langle \sum_{i=1}^{k}\Phi_y(y(t_i,x_i))\chi(t_i-t)+\int_0^\lambda\int_t^T \tilde H_y(\tau,s)d\tau ds+\int_0^\lambda \tilde H_p(t,s)ds-\psi_1(t)+\right.$$

$$\left.+\int_t^T \tilde H_{1z}(\tau)d\tau, z_t(t,0)\right\rangle dt+\int_0^T\int_0^\lambda\left\langle \sum_{i=1}^{k}\Phi_y(y(t_i,s_i))\chi(t_i-t)\chi(s_i-s)+\right.$$

$$+\int_t^T\int_s^\lambda \tilde H_y(\tau,r)d\tau ds+\int_s^\lambda \tilde H_p(t,r)ds+$$

$$\left.\int_t^T \tilde H_q(\tau,s)d\tau-\psi(t,s), z_{ts}(t,s)\right\rangle dtds+\eta. \tag{3.14}$$

We require that the functions $\psi(t,s)$ and $\psi_1(t)$ are the solutions of the system of adjoint equations. Then

$$J(w+\bar{w})-J(w)=\int_0^T\int_0^\lambda \left\langle \tilde{H}_u(t,s),\bar{u}(t,s)\right\rangle dtds+\int_0^T \left\langle \tilde{H}_{1v}(t),\bar{v}(t)\right\rangle dt+\eta$$

We can show that when the above enumerated conditions are fulfilled $|\eta|\leq C\left\|\bar{u}\right\|^2_{L_2(Q)}$, where $C>0$ are some constants.

Theorem 1 is proved.

Theorem 2. *Let all the conditions of theorem 1 be fulfilled. Let* $w_*(t,s)=(v_*(t),u_*(t,s))\in V\times U$ *be an optimal control in the problem (2.1)-(2.5),* $y^*(t,s)=y(t,s;w_*)$ *be an appropriate solution of the boundary value problem (2.1)-(2.4), and* $(\psi(t,s;w_*),\psi_1(t,w_*))$ *be a solution of the conjugated system corresponding to the control* $(v_*(t),u_*(t,s))$*. Then the following inequalities*

$$\iint_Q \langle H_u(t,s,y^*(t,s),y_t^*(t,s),y_s^*(t,s),\psi(t,x,w_*),u_*(t,s)),u(t,s)-u_*(t,s)\rangle\, dtds+$$

$$+\int_0^T \langle H_{1v}\left(t,z^*(t,0),\psi_1(t,w_*),v_*(t)\right),v(t)-v_*(t)\rangle\, dt\geq 0$$

are fulfilled for all $w=(v(t),u(t,s))\in V\times U$.

The proof of the theorem follows from [7, p.561]

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