Turan Sh. MAMMADOV

SYMMETRIC MODEL OF MOTION OF PARTICLES ON A CIRCLE

Abstract

In the paper symmetric motion model of particles on a circle without overtaking that has a great importance in transport systems is considered. Existence of stationary mode for the considered problem is shown. It is proved that in stationary mode all the particles move by the same parameter for one step. A counter example that proves that in stationary mode the particles don't walk by a binomial law is given.

1. Introduction. The problems on motion of particles on an infinite straight line without overtaking are considered in the papers [1, 2]. The law of motion between particles is determined. In other words, the motion law of a particle or of a group of particles depends on distance to the next particle. It is shown that in stationary mode each separately considered particle randomly walks and distribution of distance between neighbouring particles is found.

For passenger departure transport systems the motion of particles on a closed contour is taken as a mathematical model. While considering motion model of particles on a closed contour without overtaking there arise additional difficulties and they impede the motion laws of particles. It is shown that any two particles affect on each other in motion. In the papers [3, 4] for models consisting of two particles motion laws between particles in stationary mode is determined. It is proved that each observable particle randomly walks with the same parameters $(r, l = 1 - r)$ for models consisting of two particles that involves the advantage of one particle with respect to another one.

Similar results are obtained in [6] for a model of motion of three particles with a leader on equidistant points of a circle.

In [5], the models related with motion of particles on a circle at continuous time are considered. In these models, motion diagram determining the stationary motion mode is found.

When the number of particles is three and more, the motion gets complicated character and as there is a great number of equations that express stationary mode, there arises many difficulties and many problems in this direction remain open. In the present paper, at first a symmetric model of motion with three particles, then with s particles on a circle without overtaking are considered. Hypothesis from [1] an invariant character of a random walk is studied.

2. Description of the model. The motion of particles on the fixed points of a circle is considered. It is assumed that motion happens with jumps at the instants $0, h, 2h, ...(h > 0)$. Assume that there are N number equidistant points on the circle and s is the number of particles $(s < N)$. The particles move counter clockwise (fig. 1). i numbered particle stands before $i + 1$ numbered particle and next to $i - 1$ numbered particle.

Fig 1.

Introduce the following denotation: $\xi_{i,t}$ is a coordinate of the i–th particle at the instant t; Then

$$\rho_{i,t} = \xi_{i+1,t} - \xi_{i,t} \ (i = 1, ..., s-1); \ \ \rho_{s,t} = \xi_{1,t} - \xi_{s,t};$$

is a distance between the successive particles at the instant t; $\varepsilon_{i,t} = \xi_{i,t+h} - \xi_{i,t}$ is motion of the i–th particle at the instant t; $T = \{0, h, 2h, ...\} \ (h > 0), t \in T$;.

There happens jumpiest motion, i.e. either a particle jumps for a given distance $d > 0$ in motion direction or stands motionless; d is a distance between neighbouring points. Note that

$$\varepsilon_{i,t} = \begin{cases} 0, & \text{if the particle is motionless at the instant } t \\ d, & \text{if the particle moves at the instant } t \end{cases} \qquad i = 1, ..., s$$

The motion happens in the following way: probability of jump of each particle depends on a distance to the next particle:

1) if there is no particle for one step distance

$$\begin{matrix} P\left\{\varepsilon_{i,t} = d \mid \rho_{i,t} = kd\right\} = r_k^i, \\ P\left\{\varepsilon_{i,t} = 0 \mid \rho_{i,t} = kd\right\} = l_k^i, \end{matrix} \quad k = 2, ..., N-s+1; r_k^i + l_k^i = 1$$

2) if there is a particle for a step distance and if this particle jumps

$$\begin{matrix} P\left\{\varepsilon_{i,t} = d \mid \rho_{i,t} = d, \varepsilon_{i+1,t} = d\right\} = r_1^i, \\ P\left\{\varepsilon_{i,t} = 0 \mid \rho_{i,t} = d, \varepsilon_{i+1,t} = d\right\} = l_1^i, \end{matrix} \quad r_1^i + l_1^i = 1$$

3) if there is a particle at a step distance and this particle is motionless

$$P\left\{\varepsilon_{i,t} = d \mid \rho_{i,t} = d, \varepsilon_{i+1,t} = 0\right\} = 0,$$

$$P\left\{\varepsilon_{i,t} = 0 \mid \rho_{i,t} = d, \varepsilon_{i+1,t} = 0\right\} = 1,$$

Here for $i = s$, we understand $\varepsilon_{s+1,t} = \varepsilon_{1,t}$.

3. Three particle model. Let's consider the case when the number of particles is three ($s = 3$).

$$E_t = \left(\rho_{1,t}, \rho_{2,t}\right)$$

is the state of the system at the instant t. It is known from the description of the model that E_t makes ergodic Markov chain with finite number states and there is a stationary distribution. So, there is a stationary mode i.e. there is a t independent distribution of E_t, that

$$a_{k_1,k_2} = P\left\{E_t = (k_1,k_2)\right\}, \ \sum_{k_1,k_2} a_{k_1,k_2} = 1$$

Then we can write the following Kolmogorov-Chapman equation

$$a_{1,1} = (r^3_{N-2}r^2_1r^1_1 + l^3_{N-2})a_{1,1} + l^3_{N-3}r^2_2r^1_1a_{1,2} + l^3_{N-3}r^2_3a_{1,2}$$

$$a_{1,2} = r^3_{N-3}l^2_2a_{1,1} + (r^3_{N-3}r^2_2r^1_1 + l^3_{N-3}l^2_2)a_{1,2}+$$
$$+r^3_{N-3}l^2_1r^1_2a_{2,1} + l^3_{N-4}l^2_2r^1_2a_{2,2} + l^3_{N-4}r^3_2r^1_2a_{1,3}$$

$$a_{2,1} = r^3_{N-2}r^2_1l^1_1a_{1,1} + (r^3_{N-3}r^2_1r^1_2 + l^3_{N-3}l^2_1)a_{2,1}+$$
$$+l^3_{N-3}r^2_2l^1_1a_{1,2} + l^3_{N-4}r^2_2r^1_2a_{2,2} + l^3_{N-4}r^1_3a_{3,1}$$

$$a_{k_1,1} = l^3_{N-k_1-1}r^2_2l^1_{k_1-1}a_{k_1-1,2} + r^3_{N-k_1}r^2_1l^1_{k_1-1}a_{k_1-1,1}+$$
$$+(l^3_{N-k_1-1}l^1_{k_1} + r^3_{N-k_1-1}r^2_1r^1_{k_1})a_{k_1,1}+$$
$$+l^3_{N-k_1-2}r^2_2r^1_{k_1}a_{k_1,2} + l^3_{N-k_1-2}r^1_{k_1+1}a_{k_1+1,1}$$
$$k_1 = 3,...,N-1 \tag{1}$$

$$a_{1,k_2} = r^3_{N-k_2}l^2_{k_2-1}a_{1,k_2-1} + r^3_{N-k_2-1}l^2_{k_2-1}r^1_2a_{2,k_2-1}+$$
$$+(r^3_{N-k_2-1}r^2_{k_2}r^1_2 + l^3_{N-k_2-1}l^2_{k_2})a_{1,k_2}+$$
$$+l^3_{N-k_2-2}l^1_{k_2}r^1_2a_{2,k_2} + l^3_{N-k_2-2}r^2_{k_2+1}r^1_1a_{1,k_2+1}$$
$$k_2 = 3,...,N-1$$

$$a_{k_1,k_2} = r^3_{N-k_1-k_2+1}r^2_{k_2}l^1_{k_1-1}a_{k_1-1,k_2} + r^3_{N-k_1-k_2+1}l^2_{k_2-1}l^1_{k_1}a_{k_1,k_2-1}+$$
$$+(r^3_{N-k_1-k_2}r^2_{k_2}r^1_{k_1} + l^3_{N-k_1-k_2}l^2_{k_2}l^1_{k_1})a_{k_1,k_2}+$$
$$+l^3_{N-k_1-k_2}r^2_{k_2+1}l^1_{k_1-1}a_{k_1-1,k_2+1} + r^3_{N-k_1-k_2}l^2_{k_2-1}r^1_{k_1+1}a_{k_1+1,k_2-1}+$$
$$+l^3_{N-k_1-k_2-1}l^2_{k_2}r^1_{k_1+1}a_{k_1+1,k_2} + l^3_{N-k_1-k_2-1}r^2_{k_2+1}r^1_{k_1}a_{k_1,k_2+1}$$
$$k_1 > 1, \ k_2 > 1, \ k_1 + k_2 < N$$

Theorem 1. *The following relations are true for this system:*

$$P\left(\bigcup_{k_1+k_2=j-1}\left\{E_t = (k_1,k_2)\right\}, \bigcup_{k_1+k_2=j}\left\{E_{t+1} = (k_1,k_2)\right\}\right) =$$
$$= P\left(\bigcup_{k_1+k_2=j}\left\{E_t = (k_1,k_2)\right\}, \bigcup_{k_1+k_2=j-1}\left\{E_{t+1} = (k_1,k_2)\right\}\right)$$
$$j = 3,4,...,N-1 \tag{2}$$

Relation (2) will be expressed by the parameters of the system in the following form:

$$\sum_{\substack{k_1+k_2=j-1\\ k_1\neq 1}} r^3_{N-j+1} l^1_{k_1} a_{k_1,k_2} + r^3_{N-j+1}(1-r^2_{j-2}r^1_1)a_{1,j-2} =$$

$$= \sum_{\substack{k_1+k_2=j\\ k_1\neq 1}} l^3_{N-j} r^1_{k_1} a_{k_1,k_2} + l^3_{N-j} r^2_{j-1} r^1_1 a_{1,j-1} \qquad (3)$$

Proof. The proof of the statement is obtained by composition of appropriate equations of system (1). Thus, the case $j=3$ is obtained from the first equation of the system; the case $j=4$ is obtained from side by side composition of the first three equations of the system and so on. If we sum side by side the system's equations by the index $k_1+k_2\leq j$ and take into account that for the index (k_1', k_2') satisfying the condition $k_1'+k_2'=m(m<j)$ the relation

$$\sum_{k_1+k_2\leq j} P(E_{t+1}=(k_1,k_2)\mid E_t=(k_1',k_2'))=1$$

is true, we get proof of the theorem.

Theorem 2. *For the considered system the motion probabilities of each particle for a step in stationary mode coincides:*

$$P(\varepsilon_{1,t}=d)=P(\varepsilon_{2,t}=d)=P(\varepsilon_{3,t}=d)$$

$$P(\varepsilon_{1,t}=0)=P(\varepsilon_{2,t}=0)=P(\varepsilon_{3,t}=0)$$

Proof. The motion probabilities of the first and third particles are described in the following way

$$P\{\varepsilon_{1,t}=d\} = \sum_{\substack{k_1,k_2\\ k_1\neq 1}} r^1_{k_1} a_{k_1,k_2} + \sum_{\substack{k_2\\ k_2\neq 1}} r^2_{k_2} r^1_1 a_{1,k_2} + r^3_{N-2} r^2_1 r^1_1 a_{1,1} \qquad (4)$$

$$P\{\varepsilon_{3,t}=d\} = \sum_{\substack{k_1,k_2\\ k_1+k_2<N-1}} r^3_{N-k_1-k_2} a_{k_1,k_2} +$$

$$+ \sum_{\substack{k_1\\ k_1\neq 1}} r^1_{k_1} r^3_1 a_{k_1,N-k_2-1} + r^2_{N-2} r^1_1 r^3_1 a_{1,N-2} \qquad (5)$$

We can write the first term of expression (4) in the following way

$$\sum_{\substack{k_1,k_2\\ k_1\neq 1}} r^1_{k_1} a_{k_1,k_2} = \sum_{\substack{k_1,k_2\\ k_1\neq 1}} r^1_{k_1}(r^3_{N-(k_1+k_2)} + l^3_{N-(k_1+k_2)})a_{k_1,k_2} =$$

$$= \sum_{\substack{k_1+k_2=j\\ k_1\neq 1, j=3,N-1\\ k_1,k_2}} r^1_{k_1} r^3_{N-j} a_{k_1,k_2} + \sum_{\substack{k_1+k_2=j\\ k_1\neq 1, j=3,N-1\\ k_1,k_2}} r^1_{k_1} l^3_{N-j} a_{k_1,k_2}$$

We can use equality (3) and write

$$\sum_{\substack{k_1+k_2=j\\ k_1\neq 1, j=3,N-1\\ k_1,k_2}} r^1_{k_1} r^3_{N-j} a_{k_1,k_2} + \sum_{\substack{k_1+k_2=j\\ k_1\neq 1, j=3,N-1\\ k_1,k_2}} r^1_{k_1} l^3_{N-j} a_{k_1,k_2} =$$

$$= \sum_{j=3,N-1} \left(\sum_{\substack{k_1+k_2=j-1\\ k_1\neq 1,\\ k_1,k_2}} l^1_{k_1} r^3_{N-j+1} a_{k_1,k_2} + r^3_{N-j+1}(1-r^2_{j-2}r^1_1)a_{1,j-2} + \right.$$

$$\left. - l^3_{N-j} r^2_{j-1} r^1_1 a_{1,j-1} \right) + \sum_{\substack{k_1+k_2=j\\ k_1\neq 1, j=3,N-1\\ k_1,k_2}} r^1_{k_1} r^3_{N-j} a_{k_1,k_2} =$$

$$= \sum_{j=3,N-1} \sum_{\substack{k_1+k_2=j\\ k_1\neq 1,\\ k_1,k_2}} r^3_{N-j} a_{k_1,k_2} + \sum_{\substack{k_1+k_2=N-1\\ k_1\neq 1,\\ k_1,k_2}} r^1_{k_1} r^3_1 a_{k_1,k_2} +$$

$$+ \sum_{j=3,N-1} \left(r^3_{N-j+1}(1-r^2_{j-2}r^1_1)a_{1,j-2} - l^3_{N-j} r^2_{j-1} r^1_1 a_{1,j-1} \right)$$

Here we can write the last term as follows

$$\sum_{j=3,N-1} \left(r^3_{N-j+1}(1-r^2_{j-2}r^1_1)a_{1,j-2} - l^3_{N-j} r^2_{j-1} r^1_1 a_{1,j-1} \right) =$$

$$= \sum_{j=2,N-2} r^3_{N-j}(1-r^2_{j-1}r^1_1)a_{1,j-1} + \sum_{j=2,N-2} l^3_{N-j} r^2_{j-1} r^1_1 a_{1,j-1} =$$

$$= \sum_{j=2,N-2} r^3_{N-j} a_{1,j-1} - \sum_{j=3,N-1} r^2_{j-1} r^1_1 a_{1,j-1} - r^3_{N-2} r^2_1 r^1_1 a_{1,1} + r^3_1 r^2_{N-2} r^1_1 a_{1,N-2}$$

If we take into account the obtained expressions in (4), we get the equality

$$P\{\varepsilon_{1,t} = d\} = P\{\varepsilon_{3,t} = d\}$$

is true. Having renumbered we can prove the other sides of the equality. So, if we number second particle again by 1, then third particle will be first and by the proved fact

$$P\{\varepsilon_{2,t} = d\} = P\{\varepsilon_{1,t} = d\}$$

Thus, the theorem was proved.

Result. In the special case, if motion parameters of the particles are constant,

$$r^i_k = r,\ l^i_k = l,\ i = 1,2;\ k = 1,...,N-1$$

then

$$a_{k_1,k_2} = \frac{2}{N(N-1)}\ ,\ k_1 + k_2 < N-1$$

$$P\left\{\varepsilon_{i,t}=d\right\}=\frac{(N-1)(N-2)r+2(N-2)r^2+2r^3}{N(N-1)},\ i=1,2.$$

4. s **particle model.** We can consider the model in a more general case. Assume that s is the number of particles ($s<N$). The analogies of above stated statements are true for this case as well.

Theorem 3. *The following relations are true for the considered system*

$$P\left(\bigcup_{k_1+k_2+...+k_{s-1}=j-1}\left\{E_t=(k_1,k_2,...,k_{s-1})\right\},\right.$$
$$\left.\bigcup_{k_1+k_2+...+k_{s-1}=j}\left\{E_{t+1}=(k_1,k_2,...,k_{s-1})\right\}\right)=$$
$$=P\left(\bigcup_{k_1+k_2+...+k_{s-1}=j}\left\{E_t=(k_1,k_2,...,k_{s-1})\right\},\right.$$
$$\left.\bigcup_{k_1+k_2+...+k_{s-1}=j-1}\left\{E_{t+1}=(k_1,k_2,...,k_{s-1})\right\}\right)$$
$$j=s,s+1,...,N-s+1 \tag{6}$$

The expression of this relation by the motion parameters of the particles will be in the following form:

$$\sum_{i=1}^{s-1}\sum_{\substack{k_i\neq 1\\ k_1+k_2+...+k_{s-1}=j}} r_1^1 r_1^2 ... r_{k_i}^i l_{N-j}^s a_{1,1,...,k_i,k_{i+1},...,k_{s-1}}=$$
$$=\sum_{i=1}^{s-1}\sum_{\substack{k_i\neq 1\\ k_1+k_2+...+k_{s-1}=j}} (1-r_1^1 r_1^2 ... r_{k_i}^i) r_{N-j}^s a_{1,1,...,k_i,k_{i+1},...,k_{s-1}}$$
$$j=s,s+1,...,N-s+1$$

Theorem 4. *The motion probability of each particle for a step coincides for the considered system in stationary mode:*

$$P(\varepsilon_{1,t}=d)=P(\varepsilon_{2,t}=d)=...=P(\varepsilon_{s,t}=d)$$
$$P(\varepsilon_{1,t}=0)=P(\varepsilon_{2,t}=0)=...=P(\varepsilon_{s,t}=0)$$

These theorems are proved in a similar way noted above. In the statement's proof, the following formulas are used for motion probabilities of particles:

$$P\left\{\varepsilon_{1,t}=d\right\}=\sum_{i=1}^{s-1}\sum_{\substack{k_i\neq 1\\ k_1,k_2,...,k_{s-1}}} r_1^1 r_1^2 ... r_{k_i}^i a_{1,1,...,k_i,k_{i+1},...,k_{s-1}}$$

$$P\{\varepsilon_{s,t}=d\}=\sum_{\substack{k_s\neq 1\\ k_1,k_2,...,k_{s-1}\\ k_1+k_2+...+k_s=N}} r^i_{k_i}a_{k_1,k_2,...,k_{s-1}}+$$

$$+\sum_{i=1}^{s-2}\sum_{\substack{k_i\neq 1\\ k_1,k_2,...,k_{s-1}\\ k_1+k_2+...+k_{s-1}=N-1}} r^s_1 r^1_1 r^2_1 ... r^i_{k_i} a_{1,1,...,k_i,k_{i+1},...,k_{s-1}}$$

5. A counter example. In such type problems a binomial law walk of particles is of great importance. The obtained facts give reason to think that this statement is true for this model as well. So, the motion probabilities of the particles for a step coincide. But the constructed example shows the binomial law walk is not true for the considered model.

Example. We consider motion of two particles on a circle. Each particle moves by the same parameters r, l. For this case we obtain

$$a_k = a_{k+1}$$

As $\sum\limits_{k=1}^{N-1} a_k = 1$, we get $a_k = \dfrac{1}{N-1}$. Then

$$P\{\varepsilon_{1,t}=d\}=\sum_{k=1}^{N-1}P\left\{\varepsilon_{1,t}=d,\rho_{1,t}=k\right\}=r^2a_1+\sum_{k=2}^{N-1}ra_k=\frac{r^2+(N-2)r}{N-1}$$

On other hand

$$P\{\varepsilon_{1,t}=d,\varepsilon_{1,t+1}=d\}=\sum_{k=1}^{N-1}P\left\{\varepsilon_{1,t}=d,\varepsilon_{1,t+1}=d,\rho_{1,t}=k\right\}=$$

$$=\sum_{k=1}^{N-1}P\left\{\varepsilon_{1,t}=d,\varepsilon_{2,t}=d,\varepsilon_{1,t+1}=d,\rho_{1,t}=k\right\}+$$

$$+\sum_{k=1}^{N-1}P\left\{\varepsilon_{1,t}=d,\varepsilon_{2,t}=0,\varepsilon_{1,t+1}=d,\rho_{1,t}=k\right\}=$$

$$=r^4a_1+\sum_{k=2}^{N-1}r^3a_k+r^3la_2+\sum_{k=3}^{N-1}r^2la_k=\frac{2r^3+(N-3)r^2}{N-1}$$

As it is seen

$$P\{\varepsilon_{1,t}=d,\varepsilon_{1,t+1}=d\}\neq P\{\varepsilon_{1,t}=d\}*P\{\varepsilon_{1,t+1}=d\}$$

$$\frac{2r^3+(N-3)r^2}{N-1}\neq\left(\frac{r^2+(N-2)r}{N-1}\right)^2$$

As the assumption is not true for the simplest case, there is no need to consider the complicated case.

References

[1]. Belyayev Yu.K. *On simplified model of motion without overtaking* // Izv. AN SSSR. Tekhnicheskaya Kibernetika, No3, 1969. pp.17-21 (Russian).

[2]. Tsele U. *Generalization of model of motion without overtaking* // Izv. AN SSSR. Tekhnicheskaya Kibernetika, No5, 1972, pp. 100-103 (Russian).

[3]. Hajiev A.H. *A model of motion of particles on a closed contour without overtaking* // Izv. AN SSSR. Tekhnicheskaya Kibernetika, No5,1976. pp.79-84 (Russian).

[4]. Hajiev A.H. *On a random walk of particles on a ring* // Mat. zametki, No6, v. 47,1990, pp.140-143 (Russian).

[5]. Hajiyev A.H., Gafarov I. A. *Mathematical models of moving particles without overtaking* // Proceedings of IMM of NAS of Azerbaijan, v. 16(24), 2002, pp. 50-56.

[6]. Ibadova I.A. *Random walk of particles by the closed trajectory* // Trans. of NAS of Azerb., 2005, v.XXV, No4, pp.153-162.

Turan Sh. Mammadov
Institute of Cybernetics of NAS of Azerbaijan.
9, F. Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 38 69 (off.).

Received March 25, 2009; Revised June 09, 2009.