Rabil A. AMANOV

ON SOME NONUNIFORM WEIGHTED EMBEDDING THEOREMS

Abstract

Sobolev and Poincare type weight spaces containing different weight functions each derivative $\partial u/\partial x_i$ are proved

The paper is devoted to stydying weight variants of Sobolev Poincare classic inequalities containing different weight functions in front of each derivative $\dfrac{\partial u}{\partial x_i}$ $(i = 1, 2, ..., n)$. Such inequalities may be useful while investigating regularity of weak solutions of degeranating elliptic equations of the form

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0,$$

where $A = \|a_{ij}(x)\|$ is a symmetric matrix such that $\exists\mu \in (0, 1]$ for $\forall\xi \in E_n$;

$$\mu\sum_{i=1}^{n}\omega_i(x)\,\xi_i^2 \le A\xi\cdot\xi \le \mu^{-1}\sum_{i=1}^{n}\omega_i(x)\,\xi_i^2$$

when they are studied by a general scheme (see. f.i. [1, 2, 3]). This case has been studied relatively little in comparison with the case of equal weights $\big(\omega_i(x) \equiv \omega(x)$; $i = 1, 2, ..., n\big)$ that are mentioned for example in [4, theorem 5].

Let E_n be n - dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$, $n \ge 1$. Suppose that the vector $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ has positive components. Introduce quasimetric in E_n by the formula

$$\rho(x, y) = \max_{1\le i\le n}\left\{|x_i - y_i|^{1/\sigma_i}\right\}, \quad x, y \in E_n.$$

Assume $\rho(x) = \rho(x, 0)$, $|x|_\sigma = \sum\limits_{i=1}^{n}|x|^{1/\sigma_i}$. Let $Q_R^x = \{y \in E_n : \rho(x, y) < R\}$ be a quasiphere with a center at the point x of radius R in a quasimetric ρ. By $l_j(Q)$ we denote the length of the j-th rib of a quasihere Q, i.e. $l_j(Q) = \sup\{|x_j - y_j| : x, y \in Q\}$, $j = 1, 2, ..., n$. $|E|$ denotes Lebesgue measure of the set $E \subset E_n$. For an integrable function f and a set E we accept the denotation:

$$f(E) = \int_E f(x)\,dx; \quad \oint_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx.$$

By $L_{p,\nu}(D)$ we denote a space of measurable functions $f : D \to R$ with finite norm

$$\|f\|_{p,\nu}^{D} = \left(\int_D |f(x)|^p\,\nu(x)\,dx\right)^{1/p}; \quad \|f\|_{p,D} = \|f\|_{p,1}^{D}, \quad p \ge 1.$$

It is will not lead to misunderstanding, instead of $\|f\|_{p,\nu}$ we'll write $\|f\|_{p,\nu}^{D}$. Denote

$$\overline{f}_{\nu,D} = (1/\nu(D)) \int_D \nu(x) f(x)\, dx; \ \overline{f}_D = \overline{f}_{1,D}.$$

We get the following main results.

Theorem 1. *(Sobolev type inequality). Let $2 \le q < \infty$, $Q_0 = Q_R^a$ be some quasisphere, the non-negative functions V, $\omega_j^{-1} \in L_{1,loc}$ $(j = 1, 2, ..., n)$. Assume that $V \in A_\infty(Q_0, \rho, dx)$: there will be found such $C, \eta > 0$ that for any quasisphere $Q = Q_t^x$ where $x \in Q_0$, $t \in (0, R)$ and it is subset E, it is valid*

$$V(E)/V(Q) \le C\left(|E|/|Q|\right)^\eta. \tag{1}$$

Further, let the conditions:

$$l_j(Q)\,|Q|^{-1}(V(Q))^{1/q}\left(\int_Q \omega_j^{-1}(x)\, dx\right)^{1/2} \le A_{2q} < \infty, \tag{2}$$

be fulfilled, $j = 1, 2, ..., n$ for any quasisphere $Q = Q_t^x$, where $x \in Q_0, t \in (0, R)$.

Then there exists a positive number C_0 dependent only of n, q and C, η from the condition $V \in A_\infty(Q_0, \rho, dx)$ such that for any function $f \in Lip_0(Q_0)$ vanishing on the boundary Q_0 the inequality holds

$$\left(\int_{Q_0} |f|^q V(x)\, dx\right)^{1/q} \le C_0 A_{2q} \sum_{j=1}^n \left(\int_{Q_0} \omega_j f_{xj}^2 dx\right)^{1/2}. \tag{3}$$

Theorem 2. *(Poincare type inequality). Let $2 \le q < \infty$, $Q_0 = Q_R^a$ be a fixed quasisphere, the non-negative functions V, ω_j^{-1} $(j = 1, 2, ..., n)$ belong to $L_{1,loc}$. Further, let the conditions $A_\infty\left(\rho, \chi_{Q_0}, dx\right)$ be fulfilled: there will be found $C, \eta > 0$ such that for any quasisphere $Q = Q_1^x$ where $x \in Q_0$, $t \in (0, R)$ and measurable subset $E \subset Q$, the estimation*

$$\frac{V(E \cap Q_0)}{V(Q \cap Q_0)} \le C\left(\frac{|E \cap Q|}{|Q \cap Q_0|}\right)^\eta; \tag{4}$$

be valid, for any quasisphere $Q = Q_t^x$ $x \in Q_0$, $t \in (0, R)$

$$l_j(Q)\,|Q|^{-1}(V(Q \cap Q_0))^{1/q}\left(\int_{Q \cap Q_0} \omega_j^{-1}(y)\, dy\right)^{1/2} \le A_{2q} < \infty, \tag{5}$$

be fulfilled, where $j = 1, 2, ..., n$ and the constant A_{2q} is independent of Q and j.

Then there exists a positive number C_0 dependent on n, q and C, η from the condition (4) such that for any function $f \in Lip_0(Q_0)$ the inequality

$$\left(\int_{Q_0} \left|f - \overline{f}_{v,D}\right|^q V(x)\, dx\right)^{1/q} \le C A_{2q} \sum_{j=1}^n \left(\int_{Q_0} \omega_j f_{xj}^2 dx\right)^{1/2} \tag{6}$$

is valid.

As application of general theorems 1,2 we give the following two examples.

Example 1. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a vector with non-negative components, $\delta \geq 0$ such that

$$\max_{1\leq j\leq n} \alpha_j < \left(\sum_{k=1}^{n} \alpha_k + n\delta \right) \Big/ 2,$$

are the components of the vector $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$, where

$$\sigma_j = \frac{\alpha_j + \delta}{2}; \; j = 1, 2, ..., n$$

determine the quasimetric p. Let further q be found from the condition

$$\frac{1}{q} - \frac{1}{2} + \frac{\delta}{\delta n + \sum_{k=1}^{n} \alpha_k} = 0. \tag{7}$$

Then there exists a positive constant $C(n, \delta, \alpha)$ dependent on n, δ and the vector α a such that for any function $f \in Lip_0(Q_R^a)$ equal zero on the boundary of a quasisphere Q_R^a it holds the inequality

$$\left(\oint_{Q_R^a} |f|^q \, dx \right)^{1/q} \leq C(n, \delta, \alpha) R^{\delta/2} \sum_{j=1}^{n} \left(\oint_{Q_R^a} |x|_\delta^{\alpha_j} f_{xj}^2 dx \right)^{1/2}. \tag{8}$$

Example 2. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a vector with non-negative components, $\delta \geq 0$ such that

$$\max_{1\leq j\leq n} \alpha_j < \left(\sum_{k=1}^{n} \alpha_k + n\delta \right) \Big/ 2,$$

are the components of the vector $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$, where

$$\sigma_j = \frac{\alpha_j + \delta}{2}; \; j = 1, 2, ..., n$$

determine the quasimetric $\rho(x, y)$. Let further q be found from the condition

$$\frac{1}{q} - \frac{1}{2} + \frac{\delta}{\delta n + \sum_{k=1}^{n} \alpha_k} = 0.$$

Then there exists a positive constant $C(n, \delta, \alpha)$ dependent on n, δ, α and such that for a function $f \in Lip(Q_R^a)$ the inequality

$$\left(\oint_{Q_R^a} \left| f - \overline{f}_{Q_R^a} \right|^q dx \right)^{1/q} \leq C(n, \delta, \alpha) R^{\delta/2} \sum_{j=1}^{n} \left(\oint_{Q_R^a} |x|_\delta^{\alpha_j} f_{xj}^2 dx \right)^{1/2} \tag{9}$$

is valid.

Proof of theorem 1. Let $f \in Lip_0(Q_0)$, assume $Q_0^+ = \{x \in Q_0 : f(x) > 0\}$, $Q_0^- = Q_0 \backslash \overline{Q}_0^+$. Let $D^i\,(i = 1, 2, ...)$ be some connected component Q_0^+. We denote for it

$$D_\beta = \left\{x \in D^i;\ f(x) > \beta\right\},\ \beta > 0.$$

Let β be such that $D_{2,\beta}$ is non-empty. Then for any fixed $x \in D_{2,\beta}$ we can find $\exists Q^x_{r(x)}$:

$$\left| Q^x_{r(x)} \Big\backslash D_\beta \right| = \gamma \left| Q^x_{r(x)} \right|, \tag{10}$$

where $0 < \gamma < 1$ is a number independent of β, x, $r(x)$ will be defined later. Really, in order to show (10), it suffices to assume

$$r(x) = \sup\left\{t > 0 : |Q^x_t \backslash D_\beta| = \gamma\, |Q^x_t|\right\}.$$

For simplicity of denotation, for the fixed $x \in D_{2,\beta}$ we put $Q = Q^x_{r(x)}$.

If 1)

$$|D_{2\beta} \cap Q| < \gamma\, |Q|, \tag{11}$$

then by 1) we have

$$V(D_{2\beta} \cap Q) \le C\gamma^\delta V\, |Q|. \tag{12}$$

Further,

$$V(Q) = V(Q \cap D_\beta) + V(Q \backslash D_\beta) \le C\gamma^\delta V(Q) + V(Q \cap D_\beta)$$

by 1) and (10). Choosing γ from the condition $C\gamma^\delta < 1$ we'll have

$$V(Q) \le \frac{1}{1 - C\gamma^\delta} V(Q \cap D_\beta),$$

therefore, by (12) we'll get

$$V(Q \cap D_{2\beta}) \le \frac{C\gamma^\delta}{1 - C\gamma^\delta} V(Q \cap D_\beta). \tag{13}$$

If 2)

$$|D_{2\beta} \cap Q| \ge \gamma\, |Q|, \tag{14}$$

then by (10),(14) we have

$$\int_A \left(\int_B dy \right) dx \ge \gamma^2\, |Q|^2,$$

where A and B denote $Q \backslash D_\beta$ and $Q \cap D_{2\beta}$ respectively. Let $x \in A$, $y \in B$ be arbitrarily fixed. It is clear that the straight line connecting the point x with y will remain in Q and necessarily intersect the surface $\{x : f(x) = \beta\}$ and $\{x : f(x) = 2\beta\}$ at the points $x' = x + t_1(y - x)$ and $x'' = x + t_2(y - x)$, where $t_2 > t_1 > 0$ are some numbers dependent on x, y. Then $f(x') = \beta$, $f(x'') = 2\beta$, therefore

$$\gamma^2\, |Q|^2 \le \int_A \left(\int_B \frac{|f(x'') - f(x')|}{\beta} dy \right) dx,$$

then

$$\gamma^2 |Q|^2 \le \frac{1}{\beta} \int\limits_A \left(\int\limits_B \left(\int\limits_{t_1(x,y)}^{t_2(x,y)} \left| \frac{\partial f}{\partial t} (x + t(y-x)) \right| dt \right) dy \right) dx.$$

Hence by the Foubini theorem we get

$$\gamma^2 |Q|^2 \le \sum_{j=1}^{n} \frac{l_j(Q)}{\beta} \int\limits_A \left(\int\limits_0^1 \left(\int\limits_{\{y\in B: x+t(y-x)\in G\}} \left| \frac{\partial f}{\partial x_j} (x + t(y-x)) \right| dy \right) dt \right) dx,$$

where $G = Q \cap (D_\beta \setminus D_{2\beta})$. Make substitution $y \to z$ by the formula $z = x + (y - x)$ in the inner integral. Then

$$\gamma^2 |Q|^2 \le \sum_{j=1}^{n} \frac{l_j(Q)}{\beta} \int\limits_A \left(\int\limits_0^1 \left(\int\limits_{\left\{z\in G: \frac{z-x}{t} + x \in B\right\}} \left| \frac{\partial f}{\partial x_j}(z) \right| dz \right) \frac{dt}{t^n} \right) dx.$$

Applying the Foubini formula, we get

$$\gamma^2 |Q|^2 \le \sum_{j=1}^{n} \frac{l_j(Q)}{\beta} \int\limits_0^1 \left(\int\limits_G \left| \frac{\partial f}{\partial z_j} \right| \left(\int\limits_{\{x: |x_j - z_j| \le t l_j(Q), j=1,2,\dots,n\}} dx \right) dz \right) \frac{dt}{t^n} \le$$

$$\le \sum_{j=1}^{n} \frac{|Q|\, l_j(Q)}{\beta} \int\limits_G \left| \frac{\partial f}{\partial z_j} \right| dz,$$

whence

$$1 \le \sum_{j=1}^{n} \frac{l_j(Q)}{\gamma^2 |Q|\, \beta} \int\limits_G \left| \frac{\partial f}{\partial z_j} \right| dz,$$

then

$$1 \le \sum_{j=1}^{n} \left(\frac{n^{\frac{q-1}{n}} l_j(Q)}{\beta \gamma^2 |Q|} \int\limits_G \left| \frac{\partial f}{\partial z_j} \right| dz \right)^q. \tag{15}$$

From (15), by Holder inequality we get

$$1 \le \sum_{j=1}^{n} \left(\frac{n^{\frac{q-1}{n}} l_j(Q)}{\beta \gamma^2 |Q|} \left(\int\limits_Q \omega_j^{-1}(y)\, dy \right)^{1/2} \right)^q \left(\int\limits_G \left| \frac{\partial f}{\partial z_j} \right| \omega_j dz \right)^{q/2}. \tag{16}$$

From (16) condition (2) we get

$$1 \le \sum_{j=1}^{n} \left(\frac{n^{\frac{q-1}{n}} A_{2q}}{\beta \gamma^2} \right)^q \frac{1}{V(Q)} \left(\int\limits_G \left| \frac{\partial f}{\partial z_j} \right| \omega_j dz \right)^{q/2}. \tag{17}$$

Then

$$V\left(Q\cap D_{2,\beta}\right)\leq\frac{n^{q}A_{2q}^{q}}{\gamma^{2q}\beta^{q}}\sum_{j=1}^{n}\left(\int_{G}\left|\frac{\partial f}{\partial z_{j}}\right|\omega_{j}\left(z\right)dz\right)^{q/2}. \tag{18}$$

From (13) and (18) it follows

$$V\left(Q\cap D_{2\beta}\right)\leq\frac{C\gamma^{\delta}}{1-C\gamma^{\delta}}V\left(Q\cap D_{\beta}\right)+$$

$$+\frac{n^{q}}{\gamma^{2q}\beta^{q}}\sum_{j=1}^{n}\left(\int_{Q\cap\left(D_{\beta}\setminus D_{2\beta}\right)}\left|\frac{\partial f}{\partial z_{j}}\right|\omega_{j}\left(z\right)dz\right)^{q/2}. \tag{19}$$

A system of quasisphere $\left\{Q=Q_{r(x)}^{x}:x\in D_{2\beta}\right\}$ forms a covering for $D_{2\beta}$. On the basis of Bezikovich's lemma for quasimetric spaces [5] from the system $\{Q\}$ we can isolate a sub-covering $\{Q_i\}_{i=1}^{\infty}$ covering $D_{2\beta}$ and having finite multiplicity i.e.

$$\sum_{i=1}^{\infty}\chi_{Q_{i}}\left(x\right)\leq C_{n}, \tag{20}$$

where χ_{Q_i} are quasicharacteristical functions of quasispheres Q_i, C_n is a constant independent on i,x. Summing over i all the inequalities

$$V\left(Q\cap D_{2\beta}\right)\leq\frac{C\gamma^{\delta}}{1-C\gamma^{\delta}}V\left(Q\cap D_{\beta}\right)+\frac{n^{q}}{\gamma^{2q}\beta^{q}}\sum_{j=1}^{n}\left(\int_{G}\left|\frac{\partial f}{\partial z_{j}}\right|^{2}\omega_{j}\left(z\right)dz\right)^{q/2}, \tag{21}$$

obtained from (19) for $Q=Q_i$ and considering (20), we get

$$V\left(D_{2\beta}\right)\leq\frac{C_{n}C\gamma^{\delta}}{1-C\gamma^{\delta}}V\left(D_{\beta}\right)+\sum_{j=1}^{n}\frac{C_{n}n^{q}A_{2q}^{q}}{\gamma^{2q}\beta^{q}}\left(\int_{D_{\beta}\setminus D_{2\beta}}\left|\frac{\partial f}{\partial z_{j}}\right|^{2}\omega_{j}\left(z\right)dz\right)^{q/2}. \tag{22}$$

Integrate (22) in the interval $(0,\infty)$:

$$\int_{0}^{\infty}V\left(D_{2\beta}\right)d\beta^{q}\leq\frac{CC_{n}\gamma^{\delta}}{1-C\gamma^{\delta}}\int_{0}^{\infty}V\left(D_{\beta}\right)d\beta^{q}+$$

$$+\sum_{j=1}^{n}\frac{C_{n}n^{q}}{\gamma^{2q}}\int_{0}^{\infty}\frac{d\beta}{\beta}\left(\int_{D_{\beta}\setminus D_{2\beta}}\left|\frac{\partial f}{\partial z_{j}}\right|^{2}\omega_{j}\left(z\right)dz\right)^{q/2}. \tag{23}$$

Whence allowing for the fact that

$$\int_{0}^{\infty}V\left(D_{2\beta}\right)d\beta^{q}=\frac{1}{2^{q}}\int_{D^{i}}f\left(x\right)^{q}V\left(x\right)dx;$$

$$\int_0^\infty V(D_\beta)\,d\beta^q = \int_{D^i} f(x)^q V(x)\,dx$$

by means of Minkowski inequality we'll have

$$\left(\frac{1}{2^q} - \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta}\right)\int_{D^i} f(x)^q V(x)\,dx \le$$

$$\le \frac{C_n q n^q A_{2q}^q}{\gamma^{2q}} \sum_{j=1}^n \left(\int_{D^i} \left|\frac{\partial f}{\partial z_j}\right|^2 \omega_j(z) \left[\int_{f(z)/2}^{f(z)} \frac{d\beta}{\beta}\right] dz\right)^{q/2}, \tag{24}$$

whence choosing γ so small that

$$\frac{1}{2^q} - \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} = \frac{1}{2^{q+1}}$$

from (24) we get the inequality

$$\int_{D^i} f(x)^q V(x)\,dx \le c_0 A_{2q}^q \sum_{j=1}^n \left(\int_{D^i} \left|\frac{\partial f}{\partial x_j}\right|^2 \omega_j(x)\,dx\right)^{q/2}, \tag{25}$$

where the constant c_0 depends on γ, n, q and C, η from condition (1).

Summing all the inequalities of (25) over all D^i, we get

$$\int_{Q_0^+} f(x)^q V(x)\,dx \le c_0 A_{2q}^q \sum_{j=1}^n \left(\int_{Q_0} \left|\frac{\partial f}{\partial x_j}\right|^2 \omega_j(x)\,dx\right)^{q/2}. \tag{26}$$

Similar inequality holds for the function $f(x)$ in Q_0^-:

$$\int_{Q_0^-} (-f(x))^q V(x)\,dx \le c_0 A_{2q}^q \sum_{j=1}^n \left(\int_{Q_0} \left|\frac{\partial f}{\partial x_j}\right|^2 \omega_j(x)\,dx\right)^{q/2}. \tag{27}$$

Inequality (3) follows from (26), (27):

$$\int_{Q_0} |f(x)|^q V(x) \le \int_{Q_0^+} f(x)^q V(x)\,dx + \int_{Q_0^-} (-f(x))^q V(x)\,dx \le$$

$$\le 2c_0 A_{2q}^q \sum_{j=1}^n \left(\int_{Q_0} \left|\frac{\partial f}{\partial x_j}\right|^2 \omega_j(x)\,dx\right)^{q/2}.$$

Theorem 1 is proved.

Proof of theorem 2: There will be found $A \in R$ such that

$$|x \in Q_0 : f(x) > A| \leq \frac{1}{2} |Q_0| \leq |x \in Q_0 : f(x) \geq A| .$$

Assume $Q_0^+ = \{x \in Q_0 : f(x) > A\}$, $Q_0^- = \{x \in Q_0 : f(x) < A\}$. Let D^i $(i = 1, 2, ...)$ be some connected subset of Q_0^+. Denote , $D_\beta = \{x \in D^i : f(x) < A + \beta\}$, $\beta > 0$. Then $|Q_0 \backslash Q^+| \geq \frac{1}{2} |Q_0|$, $|Q_0 \backslash Q^-| \geq \frac{1}{2} |Q_0|$. Let $\beta > 0$ be such that $D_{2\beta}$ is non-empty. For any fixed $x \in D_{2\beta}$ there will be found a quasisphere $Q^x_{r(x)}$:

$$\left| \left(Q^x_{r(x)} \cap Q_0 \right) \setminus D_\beta \right| = \gamma \left| Q^x_{r(x)} \cap Q_0 \right| , \tag{28}$$

where $0 < \gamma < \frac{1}{2}$ is a sufficiently small number whose value will be defined later. Indeed, It suffices to choose $r(x)$ from the condition

$$r(x) = \sup \{t > 0 : |(Q^x_t \cap Q_0) \setminus D_\beta| \leq \gamma |Q^x_t \cap Q_0|\} .$$

Fix some quasisphere $Q = Q^x_{r(x)}$ from the system $\left\{ Q = Q^x_{r(x)} : x \in D_{2\beta} \right\}$. Then, if a) $|D_{2\beta} \cap Q| < \gamma |Q \cap Q_0|$ then $V(Q \cap D_{2\beta}) \leq C\gamma^\delta V(Q \cap Q_0)$. On the other hand by (28) and (4)

$$V(Q \cap Q_0) = V((Q \cap Q_0) \backslash D_\beta) + V(Q \cap D_\beta) \leq$$

$$\leq C\gamma^\delta V(Q \cap Q_0) + V(Q \cap D_\beta) ,$$

whence, if we choose γ from the condition $C\gamma^\delta < 1$ we get

$$V(Q \cap Q_0) \leq \frac{1}{1 - C\gamma^\delta} V(Q \cap D_\beta) ,$$

then

$$V(Q \cap D_{2\beta}) \leq \frac{C\gamma^\delta}{1 - C\gamma^\delta} V(Q \cap D_\beta) \tag{29}$$

now, if now b)

$$|D_{2\beta} \cap Q| \geq \gamma |Q \cap Q_0| , \tag{30}$$

then all the reasonings of theorem 1 are repeated. In this case (30),(28) yield the estimations

$$|D_{2\beta} \cap Q| > \varepsilon\gamma |Q| , \quad |Q \cap Q_0 \backslash D_\beta| > \varepsilon\gamma |Q| , \tag{31}$$

where $\varepsilon \in (0, 1)$ is a number dependent on n. Reasoning as in theorem 1, as a result we get the estimation

$$\left(\int_{Q_0} |f(x) - A|^q V(x) dx \right)^{1/q} \leq c_0 A_{2q} \sum_{j=1}^n \left(\int_{Q_0} \omega_j f^2_{x_j} dx \right)^{1/2} . \tag{32}$$

It remains to show the estimation

$$\left\| f - \overline{f}_{V,Q_0} \right\|^{Q_0}_{q,V} \leq 2 \left\| f - A \right\|^{Q_0}_{q,V} . \tag{33}$$

By Minkowskii inequality

$$\left\|f-\overline{f}_{V,Q_0}\right\|_{q,V}^{Q_0}\leq\|f-A\|_{q,V}^{Q_0}+\left|\overline{f}_{V,Q_0}-A\right|V\left(Q_0\right)^{1/q} \qquad (34)$$

and the Holder inequality

$$\left|f_{V,Q_0}-A\right|\leq\frac{1}{V\left(Q_0\right)}\int\limits_{Q_0}\left|f\left(x\right)-A\right|V\left(x\right)dx\leq\left(V\left(Q_0\right)\right)^{1/Q}\|f-A\|_{q,V}^{Q_0},$$

whence by (34) we get (33).

Proof of statement of example 1. Apply theorem 1 in the case

$$V\left(x\right)\equiv1,\omega_j\left(x\right)=|x|_\sigma^{\alpha_j},\sigma_j=\frac{\alpha_j+\delta}{2};j=1,2,...,n.$$

It suffices to verify condition (2) (condition (1) is obvious). Let's consider two cases: 1) $\rho\left(a\right)<CR$; 2) $\rho\left(a\right)>CR$, where $C>1$ is sufficiently large number independent of R,a.

In the case 1) for any quasisphere $Q=Q_r^x$, where $x\in Q_R^a$, $r<R$ two cases are possible: a) $\rho\left(x\right)<Cr$; b) $\rho\left(a\right)\geq Cr$; ($C$ is the some as in 1)).

If 1) and a) hold, we verify condition 2) for $j=1,2,...,n$:

$$\left(V\left(Q\right)\right)^{1/q}|Q|^{-1}l_j\left(Q\right)\left(\int\limits_Q\omega_j^{-1}dy\right)^{1/2}\leq|Q_r^x|^{\frac{1}{q}-1}r^{\sigma_j}\left(\int\limits_{Q_r^x}|y|_\sigma^{-\alpha_j}dy\right)^{1/2}. \qquad (35)$$

Notice that $\frac{1}{n}|y|_\sigma\leq\rho\left(y\right)\leq|y|_\sigma$, then

$$\int\limits_{Q_r^x}|y|_\sigma^{-\alpha_j}dy\int\limits_{Q_r^x}\rho\left(y\right)^{-\alpha_j}dy\leq Cr^{\sum\limits_{k=1}^n\sigma_k-\alpha_j},$$

therefore, by the choice $\sigma_j=\frac{\alpha_j+\delta}{2}$ the right hand side of (35) doesn't occur

$$|Q_r^x|^{\frac{1}{q}-\frac{1}{2}+\frac{\left(2\sigma_j-\alpha_j\right)}{2\sum\limits_{k=1}^n\sigma_k}}\Big/\left(2\sum_{k=1}^n\sigma_k\right)=C\,|Q_r^x|^{\frac{1}{q}-\frac{1}{2}+\frac{\delta}{2\sum\limits_{k=1}^n\sigma_k}}\Big/\left(2\sum_{k=1}^n\sigma_k\right)\leq C.$$

If 1) and b) hold, then for the left hand side of (35) we have the estimation

$$r^{\sigma_j}|Q_r^x|^{\frac{1}{q}-\frac{1}{2}}|x|_\sigma^{-\frac{\alpha_j}{2}}\leq C$$

by the fact that

$$|x|_\sigma^{-\frac{\alpha_j}{2}}\leq\rho\left(x\right)^{-\alpha_j/2}\leq C_1r^{-\alpha_j/2}.$$

Case 2) is similar to case b) for any quasisphere Q_r^x where $x \in Q_R^a$, $r < R$ the left hand side of (35) is estimated by the expression

$$C\left|Q_r^x\right|^{\frac{1}{q}-\frac{1}{2}} R^{\sigma_j-\alpha_j/2} \leq C.$$

All the conditions of theorem 1 are fulfilled. It remains to apply this theorem to the function $f \in Lip_\circ\left(Q_R^a\right)$:

$$\left(\int\limits_{Q_R^a} |f|^q \, dx\right)^{1/q} \leq C \sum_{j=1}^{n} \left(\int\limits_{Q_R^a} |x|_\sigma^{\alpha_j} f_{xj}^2 dx\right)^{1/2},$$

where C is independent of n, α, δ whence by condition (7) we get inequality (8).

Proof of the statement of example 2. Apply theorem 2 to the case $V(x) \equiv 1$, $\omega_j(x) = |x|_\sigma^{\alpha_j}$, $\sigma_j = \dfrac{\alpha_j+\delta}{2}$. It suffices to verify conditions (4),(5). Condition (4) is obvious, condition (50 is shown in example 1.

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Rabil A. Amanov
Institute Mathematics and Mechanics of NAS of Azerbaijan.
9, F. Agaev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.).

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