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ON A CONICAL SHELL FLUTTER AT INTERNAL STREAMLINE BY SUPERSONIC GAS FLOW

Abstract

A conical shell flutter problem was considered in the papers [1-4], and as a rule the problem statement was based on a formula of piston theory for positive pressure. The paper [5] is devoted to refined statement of the problem. The mathematical model suggested below is based on corollaries of a linearized equation for perturbed flow potential; the matter was reduced to a new non-classic eigen-value problem for a system of two integro-differential equations.

1. Problem statement. Assume a conical shell that in a spherical system of coordinates r, θ, ψ_1 occupies a part $r_1 \leq r \leq r_2$ of a conical surface

$$\{0 \leq r < \infty, \theta = \alpha, 0 \leq \psi_1 \leq 2\pi\}.$$

Interior to a cone gas flows in positive direction of the axis r; we assume that non-perturbed flow is radially steady, its parameters-velocity u_0, density ρ_0, pressure p_0, local velocity of sound a_0 are the known functions of radius. Flow is supersonic, we accept $M^2 = (u_0/a_0)^2 >> 1$; provided small conicity $\alpha^2 << 1$ we can identify the coordinate r with the coordinate x, counted off from the vertex of a cone along the flow. The shell is considered to be elastic. Its mechanical characteristics are: E is Young modulus, v is a Poisson ratio, ρ is density, h is thickness, $D = Eh^3/\left(12(1-v^2)\right)$ is cylindrical rigidity.

Vibrations of the shell are described by the equations of engineering theory in the mixed form (w, F are the flexure and force functions in the median surface)

$$D\Delta^2 w - \frac{1}{xtg\alpha}\frac{\partial^2 F}{\partial x^2} - L(w,F) = p - \rho h \frac{\partial^2 w}{\partial t^2} \qquad (1.1)$$

$$\Delta^2 F + \frac{Eh}{xtg\alpha}\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}L(w,w) = 0\,. \qquad (1.2)$$

The operator $L(u,v)$ is of the form:

$$L(u,v) = \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{x^2}\frac{\partial^2 v}{\partial \psi^2} + \frac{1}{x}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial x^2}\left(\frac{1}{x^2}\frac{\partial^2 u}{\partial \psi^2} + \frac{1}{x}\frac{\partial u}{\partial x}\right) -$$

$$-2\left(\frac{1}{x}\frac{\partial^2 v}{\partial x \partial \psi} - \frac{1}{x^2}\frac{\partial v}{\partial \psi}\right)\left(\frac{1}{x}\frac{\partial^2 u}{\partial x \partial \psi} - \frac{1}{x^2}\frac{\partial u}{\partial \psi}\right).$$

We denote $\psi = \psi_1 \sin\alpha$, Δ is a Laplace operator.

As it is accepted in panel flutter problems, the solution of a nonlinear system is represented by a sum of base and perturbed states $w = w_0 + w_1$, $F = F_0 + F_1$. After substitution in (1.1) (1.2) and linearization by small perturbations we get two systems of equations-quasistatic and vibrations. The solution of the first of them is stipulated by pressure p_0 and is known, therefore we don't write it out. In the equations of vibrations system we introduce dimensionless coordinate $x/\ell = x_1/\ell + y_1/\ell \equiv x_0 + y, 0 \le y \le 1, \quad \ell = x_2 - x_1$ and dimensionless functions $W_i = w_1/\ell, \Phi_0 = F_0/(Eh^2\ell), \Phi_1 = F_1/(Eh^2\ell)$, then it accepts the form:

$$\frac{Dh}{\ell^4}\Delta^2 W - \frac{Eh}{\ell^2(x_0+y)tg\alpha}\frac{\partial^2\Phi_1}{\partial y^2} - \frac{Eh^3}{\ell^3}L(W_1,\Phi_0) = \Delta p_1 - \rho h^2\frac{\partial W_1}{\partial t^2} \tag{1.3}$$

$$\Delta^2\Phi_1 + \frac{1}{(x_0+y)tg\alpha}\frac{\partial^2 W_1}{\partial y^2} = 0 \tag{1.4}$$

here Δp_1 is pressure of aerodynamic interaction between vibrating shell and flow (positive pressure).

System (1.3), (1.4) should be completed by boundary conditions

$$y = 0, L_1(W_1) = 0, M_1(\Phi_1) = 0; y = 1, L_2(W_1) = 0, M_2(\Phi_1) = 0 \tag{1.5}$$

here L_1, L_2, M_1, M_2 are differential operators known in the shell theory. To complete the problem statement it is necessary to determine positive pressure Δp_1.

2. Definition of Δp_1. *Perturbed flow in a shell with good approach may be regarded as potential; we denote velocity vector $\overline{u}$, perturbation potential φ_1, and get*

$$\overline{u} = \left\{u_0 + \frac{\partial\varphi_1}{\partial r}; \frac{\partial\varphi_1}{r\partial\theta}; \frac{1}{r\sin\theta}\frac{\partial\varphi_1}{\partial\psi_1}\right\} \tag{2.1}$$

Perturbation of local sound velocity is found from the Cauchy-Lagrange integral

$$a' = -\frac{\gamma-1}{2a_0}\left(u_0\frac{\partial\varphi_1}{\partial r} + \frac{\partial\varphi_1}{\partial t}\right) \tag{2.2}$$

here γ is a gas polytrope index.

The potential satisfies the nonlinear equation

$$a^2\overline{\Delta}\cdot\overline{u} = \frac{\partial^2\varphi_1}{\partial t^2} + 2\overline{u}\frac{\partial\overline{u}}{\partial t} + \overline{u}\left[(\overline{u}\cdot\overline{\Delta})\overline{u}\right] \tag{2.3}$$

here $a = a_0 + a'$. Introduce dimensionless variable $r' = r/\ell, \ell = r_2 - r_1$ having left its previous denotation and assume $\varphi_1 = \varphi\exp(\omega t)\cos n\psi_1$. Substitute (2.1), (2.2) into (2.3), linearize by small perturbations and as a result get

$$(M^2-1)\frac{\partial^2\varphi}{\partial r^2} + \left[2M\frac{\ell\omega}{a_0} + 2\frac{M^2}{a_0}\frac{\partial u_0}{\partial r}\left(\frac{\gamma-1}{2}M + \frac{1}{M}\right) - \frac{2}{r}\right]\frac{\partial\varphi}{r} +$$
$$+\left[\frac{\ell^2\omega^2}{a_0^2} + (\gamma-1)\frac{M^2}{a_0^2}\frac{\ell\omega}{a_0}\frac{\partial u_0}{\partial r} + \frac{n^2}{r^2\sin^2\theta}\right]\varphi - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{r\partial\theta}\right) = 0. \tag{2.4}$$

Assume $W_1 = W(r)\exp(\omega t)\cos n\psi_1$; then the boundary conditions for φ will take the form

$$\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\Big|_{\theta=\alpha} = hu_0\left(\frac{\partial W}{\partial r} + \frac{\ell\omega}{u_0}W - \frac{1}{r}W\right) \tag{2.5}$$

$$\frac{\partial\varphi}{\partial\theta}\left|_{\theta=\alpha}\right.=0\,. \tag{2.6}$$

Having accepted $\Delta p_1 = \Delta q \exp(\omega t)\cos n\psi_1$, for Δq we get

$$\Delta q = -\frac{\rho_0 u_0}{\ell}\left(\frac{\partial\varphi}{\partial r}+\frac{\omega\ell}{u_0}\varphi\right)_{\theta=\alpha}\,. \tag{2.7}$$

Construct an approximate solution of equation (2.4) based on the condition of small conicity $\alpha^2 << 1$. Within a shell a dimensionless variable r changes not enough ($r >> 1$), therefore we can introduce a new variable $\zeta = r\sin\theta \cong r\theta$, so that $d\zeta = rd\theta$. Allowing for the conditions $M^2 >> 1, 2/r << 1$ from (2.4) we get

$$\frac{\partial^2\varphi}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial\varphi}{\partial\zeta}-\frac{n^2}{\zeta^2}\varphi - M^2\frac{\partial^2\varphi}{\partial r^2}-A(r)\frac{\partial\varphi}{\partial r}-B(r)=0 \tag{2.8}$$

here we introduce the denotation

$$A(r)=2M\frac{\ell\omega}{a_0}+2\frac{M^2}{a_0}\left(\frac{\gamma-1}{2}M+\frac{1}{M}\right)\frac{\partial u_0}{\partial r} \tag{2.9}$$

$$B(r)=\frac{\ell^2\omega^2}{a_0^2}+(\gamma-1)\,\frac{M^2}{a_0}\frac{\ell\omega}{a_0}\frac{\partial u_0}{\partial r} \tag{2.10}$$

the boundary conditions (2.5), (2.6) are transformed

$$\frac{\partial\varphi}{\partial\zeta}\left|_{\zeta=\zeta_0}\right. = hu_0\left(\frac{\partial W}{\partial r}+\frac{\ell\omega}{u_0}W\right),\quad \zeta_0 = r\sin\alpha \tag{2.11}$$

$$\frac{\partial\varphi}{\partial\zeta}\left|_{\zeta=0}\right.=0\,. \tag{2.12}$$

Instead of (2.7) we get

$$\Delta q = -\frac{\rho_0 u_0}{\ell}\left(\frac{\partial\varphi}{\partial r}+\frac{\omega\ell}{u_0}\varphi\right)_{\zeta=\zeta_0}\,. \tag{2.13}$$

We introduce a new variable z counted off from the left end face of the shell; in the domain $z < 0$, flow is not perturbed, therefore there $\varphi = 0$, $u = \partial\varphi/\partial z = 0$. Provided small conicity, the function M, A, B may be approximately considered almost constant parameters and we can apply to (2.8) a Laplace transformation with respect to z; and a s a result obtain (s is a transformation parameter)

$$\frac{\partial^2\varphi^*}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial\varphi^*}{\partial\zeta}-\left(\beta^2+\frac{n^2}{\zeta^2}\right)\varphi^*=0,\quad \beta^2=M^2s^2+As+B\,. \tag{2.14}$$

It follows from (2.11) and (2.12)

$$\frac{\partial\varphi^*}{\partial\zeta}\left|_{\zeta=\zeta_0}\right. = hu_0\left(s+\frac{\ell\omega}{u_0}\right)W^*;\quad \frac{\partial\varphi^*}{\partial\zeta}\left|_{\zeta=0}\right.=0\,. \tag{2.15}$$

The solution of equation (2.14) under conditions (2.15) is written by the Bessel modified function

$$\varphi^* = hu_0\left(s+\frac{\ell\omega}{u_0}\right)\frac{I_n(\beta\zeta)}{\beta I_n'(\beta\zeta_0)}W^*.$$

Positive pressure is determined from (2.13)

$$\Delta q^* = -\frac{\rho_0 u_0^2 h}{\ell}\left(s+\frac{\ell\omega}{u_0}\right)^2 \frac{I_n(\beta\zeta_0)}{\beta I_n'(\beta\zeta_0)}W^* \qquad (2.16)$$

the prime means an argument derivative.

The inverse transformation is found on the base of estimates and asymptotic expansion [6]

$$\frac{I_n(\beta\zeta_0)}{I_n'(\beta\zeta_0)}\Box 1+\frac{1}{2\beta\zeta_0}.$$

From (2.16) we get

$$\Delta q^* = -\frac{\rho_0 u_0^2 h}{M\ell}\frac{(\Omega+s)^2 W^*(s)}{(s+s_1)^{1/2}(s+s_2)^{1/2}} - \frac{\rho_0 u_0^2 h}{2\zeta_0 M^2\ell}\frac{(\Omega+s)^2 W^*(s)}{(s+s_1)(s+s_2)} \qquad (2.17)$$

$$s_{1,2} = \frac{1}{2M^2}\left(A \pm \left[A^2-4M^2B\right]^{1/2}\right);\quad \Omega=\ell\omega/u_0.$$

On the basis of the inequality

$$\frac{s_2-s_1}{s_2+s_1} = \frac{(A^2-4M^2B)^{1/2}}{A} = \frac{\Omega}{(\gamma-1)M^2}\left(\frac{\partial u_0/\partial r}{2\omega_0 a_0}\right)^{1/2} << 1$$

from (2.17) we'll have a principal part of the expression for the positive pressure Δq;

$$\begin{aligned}\Delta q(z) \cong & -\frac{\gamma\rho_0 h}{\ell}\left[\Omega_0 W + M\frac{\partial W}{\partial z} + \frac{W}{2\zeta_0(z)} - \frac{M}{a_0}\frac{\partial u_0}{\partial z}\times\right.\\ & \times\left(\frac{\gamma-1}{2}M+\frac{1}{M}\right)W - \frac{1}{a_0\zeta_0(z)}\frac{\partial u_0}{\partial z}\times\\ & \left.\times\left(\frac{\gamma-1}{2}M+\frac{1}{M}\right)\int\limits_0^z e^{-\Omega_*(z-\tau)}W(\tau)d\tau\right]\end{aligned} \qquad (2.18)$$

here $\Omega_* = \ell\omega/(a_0(z)M)$; while writing integral summand we accept $(s_1+s_2)/2 = A(z)/M_k^2 \cong \Omega_* =$
$= \Omega_0/M$.

For small conicity the coordinate z may be replaced by x and the following formulae

$$a_0 = \frac{\sqrt{\gamma+1}a_{cr}}{(z+(\gamma-1)M^2)^{1/2}} \equiv a_{cr}/f_1(M) = a_{cr}/f(x) \qquad u_0 = a_0 M \qquad (2.19)$$

are valid.

Here a_{cr} is sound velocity in critical section. For great supersonic velocities, when $(\gamma-1)M^2 >> 2$, $a_0 \equiv a_{cr}\left[(\gamma+1)/(\gamma-1)\right]^{1/2}/M$ therefore

$$u_0 \cong \left[(\gamma+1)/(\gamma-1)\right]^{1/2} a_{cr}.$$

Consequently, we can approximately accept

$$\Omega_* = \Omega_0/M \cong \left[(\gamma-1)/(\gamma+1)\right]^{1/2}\omega_0 \equiv \delta\omega_0,\ \omega_0 = \ell\omega/a_{cr};$$

we can use this approximation in writing the integral in (2.18).

3. Mathematical model of a flutter. Substitute Δq from (2.18) allowing for the last remarks, in (1.3), having beforehand accepted

$$W_1 = W \exp(\omega t) \cos n\Phi_1, \;\; \Phi_1 = \Phi \exp(\omega t) \cos n\psi_1$$

and as a result we get

$$\Delta^2 W - B_0 \frac{\partial^2 \Phi}{\partial y^2} - B_0 \frac{h}{\ell} L(W, \Phi_0) = -B_0 \frac{a_{cr}^2}{c_0^2} \omega_0^2 -$$

$$-\frac{\gamma p_0 B_0 \ell}{Eh} \left[\omega_0 f(x) W + M \frac{\partial W}{\partial x} + \frac{W}{2\zeta_0(x)} - \frac{M}{a_0} \frac{\partial u_0}{\partial x} \left(\frac{\gamma - 1}{2} M + \frac{1}{M} \right) W - \right. \quad (2.20)$$

$$\left. -\frac{1}{a_0 \zeta_0(x)} \frac{\partial u_0}{\partial x} \left(\frac{\gamma - 1}{2} M + \frac{1}{M} \right) \int_0^x e^{-\delta \omega_0 (x - \tau)} W(\tau) d\tau \right]$$

here $B_0 = 12(1 - \upsilon^2)\ell^2 / h^2$.

Equation (1.4) doesn't change (index "1" should be omitted). The system (1.4), (2.19) together with boundary conditions (1.5) composes a new problem on eigen values ω_0. As a problem of aeroelastic vibrations, it is stated as follows: to determine the values of the parametrs that would provide steady vibrations of the shell, i.e. the condition $\mathrm{Re}\,\omega_0 < 0$.

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