MECHANICS

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LONGITUDINAL VIBRATIONS OF POLYMERIZED BAR WITH REGARD TO LATERAL MOTION DYNAMICS

Abstract

In the paper, L.Pohhamer's generalized equation on investigation of influence of lateral motion and also polymerization effect on longitudinal eigen vibration of a polymerized bar is suggested. The physical effect consisting in the fact that controlling the polymerization character of power fibrous structure one can create composite materials possessing the property of shock wave damper in dynamical problems of mechanics, is established.

In [1] Pohhammer has investigated influence of lateral motion in homogeneous elastic bar on longitudinal vibration of a bar. As a result he suggested a correction for wave propagation velocity in dynamical problems and also a correction for free vibrations frequency in steel bars that was widely investigated by Rayleigh.

In the suggested paper, we consider longitudinal vibration of a sufficiently long polymerized annular bar consisting of a set of weaved fibers polymerized in polymeric cylindrical matrix.

The goal of the paper is to investigate influence of lateral motions that correspond to compressive and tensile strains of lateral section in its own plane, and also to study influence of polymerization effect on the character of elastic waves propagation in a bar and on frequency of its eigen vibrations.

Consider eigen longitudinal vibration of a sufficiently long polymerized bar consisting of many elementary fibers polymerized in polymeric medium Models of Mechanical strain of such polymerized fibers $\sigma \sim \varepsilon$ were suggested in [3]. In the case of linear elastic strain a mechanical model of polymerized fiber will be of the form [3]:

$$\sigma_H = E_H(\varepsilon_x + \nu_\perp \varepsilon_\perp) \ . \tag{1}$$

Here E_H and $\nu_\perp$ is elasticity modulus and Poisson type ratio of a polymerized fibre determined by special tests that were suggested in [3]; $\varepsilon_x = \dfrac{\partial u}{\partial x}$, $\varepsilon_\perp = \varepsilon_y + \varepsilon_z = \dfrac{\partial \nu}{\partial y} + \dfrac{\partial w}{\partial z}$ are longitudinal and lateral strains; $u, \ \nu, \ w$ are displacements in coordinates $x, \ y, \ z$.

In the case of volumetric incompressibility, i.e. for $\varepsilon_\perp = -\varepsilon_x$ mechanical strain model of polymerized bar (1) is represented in the following one-dimensional form:

$$\sigma_H = E_H \left(1 - \nu_\perp\right) \varepsilon_x \ . \tag{2}$$

We derive equation of motion of a polymerized bar with regard to lateral motion from energy principle. For that we set up kinetic enrgy of lateral motion that corresponds to compressive and tensile strains of cross section passing in its own plane. Let y and z be the coordinates of any point of the section of the bar referred to the axes passing though its centre of gravity. In this case, lateral displacement of this point in the direction of longitudinal axis x is expressed in the form:

$$u_1 = -\nu_\perp y \frac{\partial u}{\partial x}, \quad u_2 = -\nu_\perp z \frac{\partial u}{\partial x} \ . \tag{3}$$

Then total longitudinal displacement along the axis x will equal:

$$u_{total} = u - \nu_\perp (y+z) \frac{\partial u}{\partial x} \ . \tag{4}$$

In this case, allowing for (4), kinetic energy of unit of length of a polymerized bar will be:

$$K = \frac{1}{2}\rho_n F_n \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \nu_\perp^2 (y^2+z^2) \left(\frac{\partial^2 u}{\partial x \partial t}\right)^2 \right\}, \tag{5}$$

where $k^2 = y^2 + z^2$ is section radius of inertia with respect to gravity center. For the case of annular section radius of inertia of the section of a polymerized bar will equal:

$$k^2 = \int_F \int (y^2+z^2) dF = \frac{\pi r^4}{2} \ . \tag{6}$$

Allowing for (2), potential energy of a unit of length of a polymerized bar under small deformations will equal:

$$U = \frac{1}{2}\sigma_H \varepsilon_x = \frac{1}{2} E_H (1-\nu_\perp) \left(\frac{\partial u}{\partial x}\right)^2 . \tag{7}$$

On this basis, allowing for (5) and (7), variational equation of motion of a bar will be represented in the form:

$$\begin{gathered} \delta \int dt \int \left\{ \frac{1}{2}\rho_n F_H \left[\left(\frac{\partial u}{\partial t}\right)^2 + \nu_\perp^2 k^2 \left(\frac{\partial^2 u}{\partial x \partial t}\right)^2 \right] - \right. \\ \left. -\frac{1}{2} F_H E_H (1-\nu_\perp) \left(\frac{\partial u}{\partial x}\right)^2 \right\} dx = 0 \end{gathered} \tag{8}$$

where integration with respect to x is extended to all the length of the bar.

Use the following identities:

$$\begin{gathered} \frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\delta u = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\delta u\right) \\ \frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\delta u\right) \\ 2\left(\frac{\partial^2 u}{\partial x \partial t}\frac{\partial^2 \delta u}{\partial x \partial t} - \frac{\partial^4 u}{\partial x^2 \partial t^2}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial t}\frac{\partial \delta u}{\partial t} - \frac{\partial^3 u}{\partial x \partial t^2}\delta u\right) + \\ +\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x \partial t}\frac{\partial \delta u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t}\delta u\right) . \end{gathered} \tag{9}$$

Integrating (8) by parts, allowing for (9) and equating the coefficient to zero for variation of δu under the integral sign, we get in terminal form an equation of longitudinal vibration of a polymerized bar with regard to influence of lateral motion and also influence of polymerization effect of a bar in the form:

$$\frac{\partial^2 u}{\partial t^2} - \nu_\perp^2 k^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} = \frac{E_H(1-\nu_\perp)\partial^2 u}{\partial x^2} \; . \tag{10}$$

Thus, retaining the term $\rho_H \nu_\perp^2 k^2 \frac{\partial^4 u}{\partial x^2 \partial t^2}$ and also taking into account the coefficient $\nu_\perp$in the right hand side, we get an appropriate correction for longitudinal wave velocity in dynamical problems and correction for free vibrations frequency in a polarized bar.

Write boundary and initial conditions for a problem on longitudinal vibrations of a polymerized bar of length ℓ whose one end $x = 0$ is fixed, the other end $x = \ell$ is free.

In this case, the boundary conditions will be in the form:

$$u(x,t)\,|_{x=0} = 0, \qquad \frac{\partial u}{\partial x}\,|_{x=\ell} = 0 \; . \tag{11}$$

The end $x = \ell$ of the bar is stretched to the length ℓ_1. Then $x = \ell$ is released and there appears longitudinal vibration in the bar. Accept that at zero time, displacement of section with abscissa x is proportional to this abscissa, i.e. in the form:

$$u(x,t)\,|_{t=0} = rx, \qquad (0 < x < \ell) \; . \tag{12}$$

Here r is a proportionality factor and is determined as follows. At zero time, displacement at the end $x = \ell$ of the bar equals $\ell_1 - \ell = \Delta\ell$, i.e.

$$\ell_1 - \ell = r\ell \quad \textit{or} \quad r == \frac{\ell_1 - \ell}{\ell} \; . \tag{13}$$

As the velocities of all intermediate sections of the bar at zero time equal zero, then the following condition will hold:

$$\frac{\partial u(x,t)}{\partial t}\,|_{t=0} = 0, \quad (0 < x < \ell) \; . \tag{14}$$

Thus, initial conditions of the stated problem are of the form (12) and (14).

Introduce the denotation:

$$a^2 = \frac{E_H(1-\nu_\perp)}{\rho_H}, \quad b^2 = \nu_\perp^2 k^2 \; . \tag{15}$$

Then, allowing for (15), the systems (10), (11), (12) and (14) will take the following compact form:
an equation of vibration of a polymerized bar:

$$\frac{\partial^2 u}{\partial t^2} - b^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{16}$$

the boundary conditions:

$$u(x,t)\,|_{x=0} = 0, \qquad \frac{\partial u}{\partial x}\,|_{x=\ell} = 0 \tag{17}$$

the initial conditions:

$$u\left|_{t=0}\right. = rx, \qquad \frac{\partial u}{\partial t}\left|_{t=0}\right. = 0\ . \tag{18}$$

According to Fourier method, we look for the solution of equation (16) in the form of separation of variables [2]:

$$u(x,t) = X(x)T(t)\ . \tag{19}$$

Substituting (19) into (16) we reduce the problem to the system of two ordinary differential equations:

$$X(x) + \lambda^2 X(x) = 0 \tag{20}$$

$$T''(t) + \frac{a^2\lambda^2}{1+\lambda^2 b^2} T(t) = 0\ . \tag{21}$$

Here the parameter λ is an eigen value of equation (20) under boundary conditions:

$$X(0) = 0, \qquad X'(\ell) = 0\ . \tag{22}$$

Non-trivial solutions of the problem on eigen values of the parameter λ for equation (20) under boundary conditions (22) are possible only for the values:

$$\lambda_n = \frac{(2n+1)\pi}{2\ell}. \quad (n \text{ is an integer})\ . \tag{23}$$

The eigen functions of the form:

$$X_n(x) = \mathrm{Si}\,n\frac{(2n+1)\pi x}{2\ell}, \qquad (n = 0,1,2,3...) \tag{24}$$

will correspond to the values λ_n^2.

Determine general solution of equation (21) under initial conditions (18).

Presubstitute the value λ^2 to the coefficient of equation (21) and get:

$$A_n^2 = \frac{a^2\lambda^2}{1+\lambda^2 b^2} = \frac{a^2}{1+\left[\frac{(2n+1)\pi}{2\ell}\right]^2 b^2}\left[\frac{(2n+1)\pi}{2\ell}\right]^2 . \tag{25}$$

Considering denotation (25), equation (21) will be of the form:

$$T''(t) + A^2 T(t) = 0\ . \tag{26}$$

For $\lambda = \lambda_n$, general solution of equation (26) has the form:

$$T_n(t) = a_n Cos A_n t + b_n \mathrm{Si}\,n A_n t, \tag{27}$$

where a_n and b_n are arbitrary constants, and A_n is of the form:

$$A_n = \frac{\frac{(2n+1)\pi}{2\ell}}{\sqrt{1+\left[\nu_\perp k\frac{(2n+1)\pi}{2\ell}\right]^2}}\sqrt{\frac{E_n(1-\nu_\perp)}{\rho_n}}\ . \tag{28}$$

By (19) we get that the function:

$$u(x,t) = T_n(t)X_n(t) = [a_n Cos A_n t + b_n \operatorname{Si} n A_n t\ ] \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell} \tag{29}$$

satisfies equation (10) and boundary conditions (11) for any a_n and b_n.

Make up a series:

$$u(x,t) = \sum_{n=0}^{\infty} [a_n Cos A_n t + b_n \operatorname{Si} n A_n t\ ] \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell}\ . \tag{30}$$

In order to satisfy initial conditions (18), it is necessary that

$$f(x) = rx = \sum_{n=0}^{\infty} a_n \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell} \tag{31}$$

$$F(x) = 0 = \sum_{n=0}^{\infty} b_n A_n \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell}\ . \tag{32}$$

Assuming that series (31) and (32) converge uniformly, we determine the coefficients a_n and b_n. For that, it suffices to multiply the both hand sides of (31) and (32) by $\operatorname{Si} n \frac{(2n+1)\pi x}{2\ell}$ and integrate with respect to x within $x = 0$ and $x = \ell$.

Having taking into attention

$$\int_0^{\ell} \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell} \operatorname{Si} n \frac{(2k+1)\pi x}{2\ell} dx \begin{Bmatrix} 0 & for \quad k \neq n \\ \frac{\ell}{2} & for \quad k = n \end{Bmatrix}, \tag{33}$$

the coefficients a_n and b_n will be of the form:

$$a_n = \frac{2r}{\ell} \int_0^{\ell} x \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell} dx = \frac{(-1)^n 8\ell r}{(2n+1)^2 \pi^2}$$

$$b_n = \frac{4}{(2n+1)\pi a} \int_0^{\ell} F(x) \operatorname{Si} n \frac{(2n+1)\pi x}{2\ell} dx \Bigg|_{F(x)=0} = 0\ . \tag{34}$$

Substituting (34) into (29), we establish that relative displacement of bar's section with abscissa x is expressed in the form:

$$u(x,t) = \frac{8\ell r}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} Cos \left\{ \frac{\frac{(2n+1)\pi}{2\ell}}{\sqrt{1 + \left[\nu_{\perp} k \frac{(2n+1)\pi}{2\ell} \right]^2}} \times \right. \tag{35}$$

$$\left. \times \sqrt{\frac{E_n(1-\nu_{\perp})}{\rho_H}} \right\} t \operatorname{Si} n \frac{(2n+1)\pi}{2\ell}.$$

It is seen from (35) that longitudinal vibrational motion of the polymerized bar is a result of summation of simple harmonic vibrations of the form:

$$\frac{8\ell R}{\pi^2}\frac{(-1)^n}{(2n+1)^2}\,\mathrm{Si}\,n\frac{(2n+1)\pi x}{2\ell}\times$$

$$Cos\left\{\frac{\frac{(2n+1)\pi}{2\ell}}{\sqrt{1+\left[\nu_\perp k\frac{(2n+1)\pi}{2\ell}\right]^2}}\sqrt{\frac{E_H(1-\nu_\perp)}{\rho_H}}\right\}t \qquad 36)$$

executed with amplitude of the form:

$$\frac{8\ell R}{\pi^2}\frac{(-1)^n}{(2n+1)^2}\,\mathrm{Si}\,n\frac{(2n+1)\pi x}{2\ell} \tag{37}$$

and frequencies of the form:

$$\omega_n=\frac{\frac{(2n+1)\pi}{2\ell}}{\sqrt{1+\left[\nu_\perp k\frac{(2n+1)\pi}{2\ell}\right]^2}}\sqrt{\frac{E_H(1-\nu_\perp)}{\rho_H}}\,. \tag{38}$$

Frequency and period of vibration of fundamental tone of a polymerized bar by means of (39) for $n=0$ will be of the form:

$$\omega_0=\frac{\pi}{2\ell}\frac{1}{\sqrt{1+\left(\frac{\nu_\perp k\pi}{2\ell}\right)^2}}\sqrt{\frac{E_H(1-\nu_\perp)}{\rho_H}} \tag{39}$$

$$T_0=\frac{2\pi}{\omega_0}=4\ell\sqrt{1+\left(\frac{\nu_\perp k\pi}{2\ell}\right)^2}\sqrt{\frac{\rho_H}{E_H(1-\nu_\perp)}}\,. \tag{40}$$

For the case of not polymerized bar, i.e. for the case $\nu_\perp=0$, we define eigen frequency ω_{00} and oscillations period T_{00} from (35) and (40) in the form:

$$\omega_{00}=\frac{\pi}{2\ell}\sqrt{\frac{E_{0H}}{\rho_{0H}}}\,, \tag{41}$$

$$T_{00}=\frac{2\ell}{\omega_{00}}=2\ell\sqrt{\frac{\rho_{0H}}{E_{0H}}}\,. \tag{42}$$

Here E_{0H} and ρ_{0H} is modulus of elasticity and density of a not polymerized bar. Considering (35)-(42), determine the relative quantities $\frac{\omega_0}{\omega_{00}}$ and $\frac{T_0}{T_{00}}$ in the form:

$$\frac{\omega_0}{\omega_{00}}=\frac{1}{\sqrt{1+\left(\frac{\nu_\perp k\pi}{2\ell}\right)^2}}\sqrt{\frac{E_H(1-\nu_\perp)}{E_{OH}}\frac{\rho_{OH}}{\rho_H}}\,, \tag{43}$$

$$\frac{T_0}{T_{00}} = \sqrt{1 + \left(\frac{\nu_\perp k\pi}{2\ell}\right)^2} \sqrt{\frac{E_{OH}}{E_H(1-\nu_\perp)} \frac{\rho_H}{\rho_{OH}}} \,. \tag{44}$$

Thus, dependence of eigen frequency ω_0 and oscillation period T_0 of a polymerized bar on eigen frequency ω_{00} and oscillation period T_{00} of a not polymerized bar is established by formulae (42) and (44)

Eigen vibrations frequency at infinity $\omega_\infty = \omega_n \left|_{n\to\infty}\right.$ and also appropriate vibrations period will equal:

$$\omega_\infty = \omega_n \left|_{n\to\infty}\right. = \lim_{n\to\infty} \frac{\frac{(2n+1)\pi}{2\ell}}{\sqrt{1+\left[\nu_\perp k \frac{(2n+1)\pi}{2\ell}\right]^2}} \sqrt{\frac{E_H(1-\nu_\perp)}{\rho_H}} =$$

$$= \frac{1}{\nu_\perp k} \sqrt{\frac{E_H(1-\nu_\perp)}{\rho_H}}, \qquad T_\infty = 2\pi \nu_\perp k \sqrt{\frac{\rho_H}{E_H(1-\nu_\perp)}}.$$

It follows from these relations that eigen frequency at infinity ω_∞ is inversely proportional to polymerized effect of the bar $\nu_\perp$. From the physical point of view this means that the higher is the value of Poisson type coefficient $\nu_\perp$, the lower is longitudinal vibration frequency of a polymerized bar at infinity, and the longer is the vibration period.

Example. Investigate numerical influence of above-stated effects on eigen frequency and oscillations period of a polymerized bar. Appropriate number calculation where-from the following laws of vibrations of polymerized bars follow, is given in table 1.

Table 1.

Polymerization effect of fibrous structures in reinforced structural elements reduces to essential changes of dynamical characteristics of composite materials.

So, it is numerically shown that a longitudinal elastic wave $a = \sqrt{\dfrac{E_H(1-\nu_\perp)}{\rho_H}}$ and also frequency of free vibrations $\omega_0 = \dfrac{\pi}{2\ell}\dfrac{1}{\sqrt{1+\left(\dfrac{\nu_\perp k\pi}{2\ell}\right)^2}}\sqrt{\dfrac{E_H(1-\nu_\perp)}{\rho_H}}$ in a polymerized bar is lower for 40% than in not a polymerized bar. And vibration period $T_0 = \dfrac{2\pi}{\omega_0}$ is higher for 25-30% than in not polymerized case.

- practical value of the established fact is that managing the character of polymerization power of fibrous structure, we can create composite materials having the properties of shock wave dampers in dynamical problems of mechanics.

References

[1]. Liav A. *Mathematical theory of elasticity.* ONII NKT SSSR, Leningrad, 1935, 674 p. (Russian).

[2]. Tikhonov A.N., Samarskii A.A. *Equations of mathematical physics.* Moscow, 1953, 679 p. (Russian).

[3]. Aliyev G.G. *Bases of mechanics of reinforced flexible pipes.* "Elm", Baku, 1987, 263 p. (Russian).

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