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NECESSARY CONDITION OF BASICITY OF A SYSTEM OF POWERS IN LEBESGUE SPACES

Abstract

In the paper, some necessary conditions are obtained for the power system of the form

$$\{A(t)\,\varphi^n(t)\,;B(t)\,\overline{\varphi}^n(t)\}_{n\geq 0}$$

to be a basis in Lebesgue spaces.

Investigation of spectral properties of differential operator pencils, especially in concrete cases, requires studying the basis properties (completeness, minimality, basicity) of systems of the form

$$a(t)\,\varphi^n(t)+b(t)\,\overline{\varphi}^n(t)\,,\quad n=0,1,... \tag{1}$$

in the Lebesgue spaces $L_p\equiv L_p(a,b)\,,\quad 1\leq p<\infty$, where $a(t)\,,b(t)\,,\varphi(t)$ are complex-valued functions on $[a,b]$. In this connection, the basis properties of (1) have been studied by many authors [1-10]. Under some conditions on the functions $a(t)\,,b(t)$ and $\varphi(t)$ completeness and minimality of the system (1) have been completely studied in [7-9]. Thus, depending on approaches to the considered problems various criterions of completeness and minimality have been obtained. Mainly, these questions are reduced to the study of special boundary value problems of analytic functions with Karleman translation. B.T. Bilalov [9,11] suggested another approach to the study of the basis properties of (1) i.e. introduced "double" power system

$$\left\{A(t)\,W^n(t)\,;\quad B(t)\,\overline{W}^k(t)\right\}_{n\geq 0,k\geq 0}\,, \tag{2}$$

and investigation of basis properties of (1) have been reduced to the investigation of analogous properties of this system, where $A(t)\,,B(t)$ are complex-valued functions on $[a,b]\,,\ W\{[a,b]\}\equiv\Gamma$ is a Jordan curve ($W(a)=W(b)$). Under some conditions on the functions $A(t)\,,B(t)$ he completely studied completeness, minimality and basicity (when $W(t)\equiv e^{it},[a,b]\equiv[-\pi,\pi]$) of this system in L_p. It is proved that system (2) can be a basis for L_2 only when $W(t)\equiv const$. The proof of this fact essentially relies on the connection between basis properties and solvability of the corresponding conjugation boundary value problem in the Hardy classes $H_p^{\pm}$.

This method is not applicable for $p\neq 2$. In this paper we develop the idea of [11] to the case $p\neq 2$.

Let B be a Banach space over the field of complex numbers with the norm $\|\cdot\|_B$ in which a "double" system of elements $\{x_n^+;x_n^-\}_{n\geq 0}$ is given.

Definition [11]. *A system* $\{x_n^+; x_n^-\}_{n\geq 0}$ *is called a basis for* B *if for every* $x \in B$ *there is a unique sequence* $\{a_n^+; a_n^-\}_{n\geq 0}$ *of complex numbers such that*

$$\left\| \sum_{n=0}^{N^+} a_n^+ x_n^+ + \sum_{n=0}^{N^-} a_n^- x_n^- - x \right\| \to 0 \quad as \quad N^-, N^+ \to \infty.$$

Consider the following system of functions

$$\{A(t)\varphi^n(t); B(t)\overline{\varphi}^n(t)\}_{n\geq 0} \ . \tag{3}$$

We suppose that the complex-valued functions $A(t), B(t)$ and $\varphi(t)$ satisfy the following conditions:

1) The functions $|A(t)|, |B(t)|$ are measurable on (a,b); moreover

$$\sup vrai \left\{ |A(t)|^{\pm 1}; |B(t)|^{\pm 1} \right\} < \infty;$$

2) $\varphi(t)$ is a continuous function on $[a,b]$.

1. Degenerate case. In this section we consider the basicity of the systems of the form (3) in $L_p\,(1 \leq p < \infty)$.

Theorem 1. *Suppose that the functions* $A(t), B(t)$ *and* $\varphi(t)$ *satisfy conditions 1) and 2). If the system (3) is a basis for* $L_p\,(1 \leq p < \infty)$*, then* $|\varphi(t)| \equiv const$ *n* $[a,b]$.

Proof. Suppose the contrary. Let

$$R = \max_{[a,b]} |\varphi(t)| > \min_{[a,b]} |\varphi(t)| = r.$$

Since system (3) is a basis for L_p, every $f \in L_p$ has a unique expansion (in L_p)

$$f(t) = A(t) \sum_{n=0}^{\infty} a_n \varphi^n(t) + B(t) \sum_{n=0}^{\infty} b_n \overline{\varphi}^n(t).$$

As in [11] we can prove that $R_1 \geq R, R_2 \geq R$, where R_1, R_2 are radii of convergence of power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$.

Put

$$S_m(t) = A(t) \sum_{n=0}^{m} a_n \varphi^n(t) + B(t) \sum_{n=0}^{m} b_n \overline{\varphi}^n(t), \quad as \quad m \in N.$$

Since

$$S_m(t) \xrightarrow{L_p} f(t), \ as \ \ m \to \infty,$$

there exists a subsequence $\{S_{m_k}(t)\}_{k=1}^{\infty}$ of $\{S_m(t)\}_{m=1}^{\infty}$such that $S_{m_k}(t) \to f(t)$ almost everywhere on $[a,b]$.

Since $r < R$, there exists a δ-neighborhood of some point $\tau \in (a,b)$ such that $|\varphi(t)| = |\overline{\varphi}(t)| < R$ for every $t \in [\tau-\delta, \tau+\delta]$. Then, the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are uniformly convergent on $[\tau-\delta,\tau+\delta]$. Therefore, it is evident that $\{S_{m_k}(t)\}_{k=1}^{\infty}$ converges a.e. on $[\tau-\delta,\tau+\delta]$ to the limit function

$$A(t)\sum_{n=0}^{\infty} a_n\varphi^n(t) + B(t)\sum_{n=0}^{\infty} b_n\overline{\varphi}^n(t),$$

which is an essentially bounded function on $[\tau-\delta,\tau+\delta]$. Thus, restriction of every function $f \in L_p$ (note that choice of τ and δ do not depend on $f \in L_p$) to the segment $[\tau-\delta,\tau+\delta]$ is essentially bounded. This gives a contradiction, since the function

$$f(t) = \frac{1}{(t-\tau)^a}, \quad \left(0 < \alpha < \frac{1}{p}\right)$$

is an element of L_p, but the restriction of this function to any segment that contains τ is not essentially bounded. The theorem is proved.

In the sequel, the symbol I or J will denote empty or finite set. Put

$$A_1(t) = A(t)\prod_{i\in I}(t-\xi_i)^{\alpha_i}, \quad t\in[a,b],$$

$$B_1(t) = B(t)\prod_{j\in J}(t-\theta_i)^{\beta_j}, \quad t\in[a,b],$$

where $A(t), B(t)$ satisfy condition 1, $\xi_i\,(i\in I), \theta_j\,(j\in J)$ are some points in $[a,b]$ and $\alpha_i\,(i\in I), \beta_j\,(j\in J)$ are some scalars.

The following generalization of theorem 1 is valid:

Theorem 2. *If the system*

$$\{A_1(t)\varphi^n(t); B_1(t)\overline{\varphi}^n(t)\}_{n\geq 0} \tag{4}$$

is a basis for $L_p\,(1\leq p<\infty)$, *then* $|\varphi(t)| \equiv const$ *on* $[a,b]$.

Proof. Suppose the contrary, let

$$R = \max_{[a,b]}|\varphi(t)| > \min_{[a,b]}|\varphi(t)| = r.$$

It follows from the condition of the theorem that every $f \in L_p$ has a unique expansion (in L_p)

$$f(t) = \sum_{n=0}^{\infty} a_n A_1(t)\varphi^n(t) + \sum_{n=0}^{\infty} b_n B_1(t)\overline{\varphi}^n(t).$$

Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$. Their radii of convergence will be denoted by R_1 and R_2, respectively.

It can be shown that $R_1, R_2 \geq R$. Suppose the contrary: $R_1 < R$ (the case $R_2 < R$ is analogous). Since $|\varphi(t)| \in C[a,b]$, there are points $\tau_m, \tau_M \in (a,b)$ and some neighborhoods of these points (by neighbarhood $U(t)$ of the point t we understand any interval or semi-interval containing t) $U(\tau_m), U(\tau_M), U(\xi_i), U(\theta_j)$, $i \in I$, $j \in J$ such that

1) $\forall\ t \in U(\tau_M)$, $R_1 < |\varphi(t)| \leq R$; $\forall\ t \in U(\tau_M) : |\varphi(t)| < R$;

2) $r_0 = \min\limits_{U(\tau M)} |\varphi(t)| > R_1$;

3) $U(\tau_m)$, $U(\tau_M) \in E$, where $E \equiv [a,b] / \left\{ \bigcup\limits_{i \in I} U(\xi_i) \cup \bigcup\limits_{j \in J} U(\theta_j) \right\}$.

It is obvious that every function $f \in L_p(E)$ has an expansion (in $L_p(E)$)

$$f(t) = A_1(t) \sum_{n=0}^{\infty} a_n \varphi^n(t) + B_1(t) \sum_{n=0}^{\infty} b_n \overline{\varphi}^n(t).$$

Since the series in the last formula converges in $L_p(E)$, $\|a_n \varphi^n(t)\|_{L_p(E)} \to 0$ as $n \to \infty$. Therefore (since $U(\tau_M) \in E$)

$$\|a_n \varphi^n(t)\|_{L_p(U(\tau_M))} \to 0 \ as \ n \to \infty.$$

It is known that

$$R_1 = \frac{1}{\overline{\lim\limits_{n}} \sqrt[n]{|a_n|}},$$

i.e. there is a subsequence $\{n_k\}$ such that

$$R_1 = \frac{1}{\lim\limits_{k} \sqrt[n_k]{|a_{n_k}|}}.$$

Since $r_0 > R_1$, for sufficiently large k we find

$$r_0 = \min_{U(\tau M)} |\varphi(t)| > \frac{1}{\sqrt[n_k]{|a_{n_k}|}}, \quad i.e. \ |a_{n_k} \varphi^{n_k}(t)| > 1, \quad \forall t \in U(\tau_M).$$

This implies

$$|a_{n_k} \varphi^{n_k}(t)|_{L_p(U(\tau_M))} > \mu(U(\tau_M)) > 0$$

for sufficiently large k and the contradiction proves $R_1 \geq R$ (and $R_2 \geq R$). Arguing as in the proof of theorem 1 we find that under our assumptions, restriction of every function $f \in L_p$ to $U(\tau_M)$ is essentially bounded. But the example in the proof of theorem 1 shows that it is not so. The theorem is proved.

2. The case of weighted spaces. The results obtained in the previous section allow as obtain analogous results for weighted spaces $L_{p,\rho(t)}(a,b)$, where $1 \leq p < \infty$ and $\rho(t)$ is a nonnegative continuous function with finite number of degenerations, exactly

$$\rho(t) \equiv \prod_{i \in I} |t - \mu_i|^{\omega_i},$$

where $\mu_i\,(i\in I)$ are some points in $[a,b]$ and $\omega_i\,(i\in I)$ are some scalars. Put $v(t)\equiv\rho^{1/p}(t)$.

Lemma. *The system*

$$\{A(t)\,\varphi^n(t)\,;B(t)\,\overline{\varphi}^n(t)\}_{n\geq 0}$$

is a basis in $L_{p,\rho(t)}\,(1\leq p<\infty)$ *if and only if the system*

$$\{v(t)\,A(t)\,\varphi^n(t)\,;v(t)\,B(t)\,\overline{\varphi}^n(t)\}_{n\geq 0}$$

is a basis in $L_p\,(1\leq p<\infty)$.

The proof of this lemma is obvious.

Using this lemma and taking into account theorem 2, we find

Theorem 3. *If the system (3) is a basis for* $L_{p,\rho(t)}\,(a,b)$, *then* $|\varphi(t)|\equiv const$ *on* $[a,b]$.

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