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# ON BASICITY OF DOUBLE AND UNITARY SYSTEMS OF FUNCTIONS

**Abstract**

*In the paper the unitary and double systems of functions that are generalizations of classic systems of exponents, cosines and sines are considered. Relation between basicity of these systems in Lebesque spaces is established.*

Under the bases properties of systems in Banach spaces we understand the classical notion as completeness, minimality and Shauder's basicity. It is well known that the basicity properties of the exponent system $\left\{e^{int}\right\}_{n\in Z}$ ($Z$ is a set of integers) in $L_p\left(-\pi,\pi\right),\ 1\le p\le+\infty$, are equivalent to the corresponding basis properties of the system of sines $\{\sin nt\}_{n\in N}$ and cosines $1\cup\{\cos nt\}_{n\in N}$ ($N$ are natural numbers) in $L_p\left(0,\pi\right)$. About these questions we can consider the monographs [1-4]. In the present paper we suggest some generalizations of these results.

Let's consider unitary systems of the form

$$\vartheta_n^{\pm}\left(t\right)\equiv a\left(t\right)\omega_n^{+}\left(t\right)\pm b\left(t\right)\omega_n^{-}\left(t\right),\ n\in N.$$

Without losing generality, we'll assume that $\left\{\vartheta_n^{\pm}\left(t\right)\right\}_{n\in N}$ is determined on the segment $[0,a]$.

We'll assume

$$A\left(t\right)\equiv\begin{cases}a\left(t\right), & t\in\left[0,a\right],\\ b\left(-t\right), & t\in\left[-a,0\right),\end{cases}$$

$$W_n\left(t\right)\equiv\begin{cases}\omega_n^{+}\left(t\right), & t\in\left[0,a\right],\\ \omega_n^{-}\left(-t\right), & t\in\left[-a,0\right).\end{cases}$$

Let

$$V_{n;m}\equiv\left(A\left(t\right)W_n\left(t\right);A\left(-t\right)W_m\left(-t\right)\right),n,m\in N.$$

We'll denote

$$V_n^{+}\equiv\left(A\left(t\right)W_n\left(t\right);0\right);\ V_n^{-}\equiv\left(0;A\left(-t\right)W_n\left(-t\right)\right)$$

It is absolutely evident that the minimality of the system $\left\{V_{n;m}\right\}_{n;m\in N}$ in $L_p\left(-a,a\right)$ is equivalent to the existence of the system $\{h_n^{+};h_n^{-}\}_{n\in N}\subset L_q\left(-a,a\right):\dfrac{1}{p}+\dfrac{1}{q}=1$ (for $p=1$ we'll assume that $q=\infty$)

$$\left.\begin{array}{l}<h_n^{+},V_m^{+}>=\delta_{nm};<h_n^{+},V_m^{-}>=0;\\ <h_n^{-},V_m^{+}>=0;\ <h_n^{-},V_m^{-}>=\delta_{nm},\end{array}\right\}\qquad(1)$$

where $< f,g >= \int\limits_{-a}^{a} g(t)\,\overline{f(t)}dt$, $\delta_{nm}$ -is a Kroneker symbol, $\overline{(\cdot)}$ is a complex conjugation.

Prior to the statement of basic results we'll give definition of basicity of a double system.

Let $X$ be some Banach space with the norm $\|\cdot\|$ and $\{x_n^+;x_n^-\}_{n\in n} \subset X$ be a double system.

**Definition.** *The system $\{x_n^+;x_n^-\}_{n\in n}$ is called a basis in $X$, if for $\forall x\in X$ there exists a unique sequence of the complex numbers $\left\{\lambda_n^+;\lambda_n^-\right\}_{n\in n}$:*

$$x=\lim_{m\to\infty}\sum_{n=1}^{m}\lambda_n^+x_n^+ + \lim_{m\to\infty}\sum_{n=1}^{m}\lambda_n^-x_n^-.$$

This is a usual definition of Shauder's basicity for a double system.

Under the basicity of double system we'll understand the basicity namely in this sense. Unconventional basicity has the analogous definition.

We'll need the following easily proved lemma.

**Lemma 1.** *1) If $\{h_n^+;h_n^-\}_{k\in N}\subset L_q(-a,a)$ is a biorthogonal to $\{V_{n;n}\}_{n\in N}\subset L_p(-a,a)$, $\frac{1}{p}+\frac{1}{q}=1$ system, then the system $\left\{\vartheta_k^+\right\}_{k\in N}\subset L_q(0,a)$ $\left(\left\{\vartheta_k^-\right\}_{k\in N}\subset L_q(0,a)\right)$ is biorthogonal to $\left\{\vartheta_n^+\right\}_{k\in N}\subset L_p(0,a)$ $\left(\left\{\vartheta_n^-\right\}_{n\in N}\subset L_p(0,a)\right)$, where*

$$\nu_k^+(t)\equiv h_k^+(t)+h_k^+(-t)\,.\ \left(\nu_k^-(t)\equiv h_k^-(-t)-h_k^-(-t)\right),\forall k\in N. \qquad (2)$$

*2) If $\left\{\vartheta_n^\pm\right\}_{n\in N}\subset L_p(0,a)$ and $\{\nu_k\}_{k\in N}\subset L_p(0,a)$ are biorthogonal, then $\{V_{n;n}\}_{n\in N}\subset L_p(-a,a)$ and $\left\{h_k^+;h_k^-\right\}_{k\in N}\subset L_q(-a,a)$ are also biorthogonal, where*

$$h_k^\pm(t)=\frac{1}{2}\left[\tilde{\nu}_k(t)\pm\tilde{\nu}_k^-(t)\right],$$

$$\nu_k^\pm(t)\equiv\begin{cases}\nu_k^\pm(t)\,,\ t\in(0,a)\\ \\ \pm\nu_k^\pm(-t)\,,\ t\in(-a,0)\,.\end{cases} \qquad (3)$$

**Theorem 1.** *The double system $\{V_{n;n}\}_{n\in N}$ forms a basis (unconditional basis) in $L_p(-a,a)$ iff each of unitary systems $\left\{\vartheta_n^+\right\}_{n\in N}$ and $\left\{\vartheta_n^-\right\}_{n\in N}$ forms a basis (unconditional basis) in $L_p(0,a)$.*

**Proof.** Let $\{V_{n;n}\}_{n\in N}$ form a basis in $L_p(-a,a)$ and $\{h_n^+;h_n^-\}_{n\in N}\subset L_q(-a,a)$ be the corresponding biorthogonal system, i.e.,

$$\int\limits_{-a}^{a}A(t)\,W_n(t)\,\overline{h_n^\pm(t)}dt=\frac{1}{2}\left[\delta_{nk}\pm\delta_{nk}\right],$$

$$\int\limits_{-a}^{a}A(-t)\,W_n(-t)\,\overline{h_n^\pm(t)}dt=\frac{1}{2}\left[\delta_{nk}\mp\delta_{nk}\right],\forall n,k\in N.$$

We have

$$\int_{-a}^{a} A(-t)\, W_n(-t)\, \overline{h_n^{\pm}(t)}dt = \int_{-a}^{a} A(t)\, W_n(t)\, \overline{h_n^{\pm}(-t)}dt = \frac{1}{2}\left[\delta_{nk} \mp \delta_{nk}\right],$$

From uniqueness of biorthogonal system to the complete system from the previous relation we obtain that $h_n^-(t) = h_n^+(-t)$. Let's take $\forall f \in L_p(-a,a)$ and consider ($\|\cdot\|_p$ is an ordinary norm in $L_p$):

$$I_m^{(-a,a)} = \left\| \sum_{n=1}^{m} \left[ \int_{-a}^{a} f(t)\, \overline{h_n^+(t)}dt A(x)\, W_n(x) + \int_{-a}^{a} f(t)\, \overline{h_n^-(t)}dt A(-x)\, W_n(-x) \right] - \right.$$

$$\left. - f(x) \right\|_p .$$

By the lemma 1, the systems $\{\vartheta_n^{\pm}\}_{n\in N}$ are minimal in $L_p(0,a)$ and the systems biorthgonal to them and to $\{V_{n;n}\}_{n\in N}$ are connected by the relations (2), (3). Taking into account these relations we have:

$$I_m^{(-a,a)} = \left\| \sum_{n=1}^{m} \left[ \int_{-a}^{a} f(t)\, \overline{h_n^+(t)}dt A(x)\, W_n(x) + \right.\right.$$

$$\left.\left. + \int_{-a}^{a} f(-t)\, \overline{h_n^+(t)}dt A(-x)\, W_n(-x) \right] - f(x) \right\|_p$$

$$= \left\| \sum_{n=1}^{m} \left[ \frac{1}{2}\int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^+(t)}dt A(x)\, W_n(x) + \frac{1}{2}\int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^-(t)}dt A(-x)\, W_n(-x) + \right.\right.$$

$$\left.\left. + \frac{1}{2}\int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^-(t)}dt A(-x)\, W_n(-x) \right] - f(x) \right\|_p .$$

We'll denote

$$I_1^n(x) = \int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^+(t)}dt A(x)\, W_n(x) = \int_{0}^{a} \left[f(t) + f(-t)\right] \overline{\nu_n^+(t)}dt A(x)\, W_n(x);$$

$$I_2^n(x) = \int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^-(t)}dt A(x)\, W_n(x) = \int_{0}^{a} \left[f(t) - f(-t)\right] \overline{\nu_n^-(t)}dt A(x)\, W_n(x);$$

$$I_3^n(x) = \int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^+(t)}dt A(-x)\, W_n(-x) =$$

$$= \int_{0}^{a} \left[f(t) + f(-t)\right] \overline{\nu_n^+(t)}dt A(-x)\, W_n(-x);$$

$$I_4^n(x) = \int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^-(t)}dt A(-x)\, W_n(-x) =$$

$$= -\int_{0}^{a} \left[f(t) - f(-t)\right] \overline{\nu_n^-(t)}dt A(-x)\, W_n(-x).$$

Adding these relations we obtain:

1). Let $x \in (0,a)$

$$I_1^n(x) + I_3^n(x) = \int_0^a [f(t) + f(-t)] \overline{\nu_n^+(t)} dt \vartheta_n^+(x);$$

$$I_2^n(x) + I_4^n(x) = \int_0^a [f(t) - f(-t)] \overline{\nu_n^-(t)} dt \vartheta_n^-(x);$$

2). Let $x \in (-a,0)$. We have:

$$I_1^n(x) = \int_0^a [f(t) + f(-t)] \overline{\nu_n^+(t)} dt b(-x) \omega_n^-(x);$$

$$I_2^n(x) = \int_0^a [f(t) - f(-t)] \overline{\nu_n^-(t)} b(-x) \omega_n^-(-x);$$

$$I_3^n(x) = \int_0^a [f(t) + f(-t)] \overline{\nu_n^+(t)} dt a(-x) \omega_n^+(-x);$$

$$I_4^n(x) = -\int_0^a [f(t) - f(-t)] \overline{\nu_n^-(t)} dt a(-x) \omega_n^+(-x).$$

Absolutely analogously, adding these expressions we obtain:

$$I_1^n(x) + I_3^n(x) = \int_0^a [f(t) + f(-t)] \overline{\nu_n^+(t)} dt \vartheta_n^+(-x);$$

$$I_2^n(x) + I_4^n(x) = -\int_0^a [f(t) - f(-t)] \overline{\nu_n^-(t)} dt \vartheta_n^-(-x);$$

Finally we have:

$$\left| I_m^{(-a,a)} \right|^p = \int_{-a}^a \left| \frac{1}{2} \sum_{n=1}^m \sum_{k=1}^4 I_k^n(x) - f(x) \right|^p dx =$$

$$= \int_0^a |S_m(x)|^p dx + \int_{-a}^0 |S_m(x)|^p dx, \text{ where } S_m = S_m^+ + S_m^-,$$

$$S_m^+(x) \equiv \sum_{n=1}^m \frac{I_1^n(x) + I_3^n(x)}{2} - \frac{f(x) + f(-x)}{2}, \tag{4}$$

$$S_m^-(x) \equiv \frac{I_2^n(x) + I_4^n(x)}{2} - \frac{f(x) - f(-x)}{2}. \tag{5}$$

Further, taking into account the inequality $|a+b|^p \leq 2^p (|a|^p + |b|^p)$, we obtain:

$$\int_0^a |S_m(x)|^p dx \leq 2^p \left[ \int_0^a \left| S_m^+(x) \right|^p dx + \int_0^a \left| S_m^-(x) \right|^p dx \right]$$

$$\int_{-a}^0 |S_m(x)|^p dx \leq 2^p \left[ \int_{-a}^0 \left| S_m^+(x) \right|^p dx + \int_{-a}^0 \left| S_m^-(x) \right|^p dx \right] =$$

$$= 2^p \left[ \int_0^a \left| S_m^+(-x) \right|^p dx + \int_0^a \left| S_m^-(-x) \right|^p dx \right] =$$

$$= 2^p \left[ \int_0^a \left| \sum_{n=1}^m - \int_0^a \frac{f(t) - f(-t)}{2} \overline{\nu_n^-(t)} dt \vartheta_n^-(x) - \frac{f(-x) - f(x)}{2} \right|^p dx \right].$$

Now we'll assume that the systems $\{\vartheta_n^\pm\}_{n\in N}$ form the bases in $L_p(0,a)$.Then from the previous relations it immediately follows that $\int_0^a |S_m(x)|^p dx \to 0$, $\int_{-a}^0 |S_m(x)|^p dx \to 0$ as $m \to \infty$, and so $I_m^{(-a,a)} \to 0$ as $m \to \infty$. Thus the biorthogonal series of arbitrary function $f(t)$ from $L_p(-a,a)$ by the system $\{V_{n;n}\}_{n\in N}$ converges to it in $L_p(-a,a)$, and as a result the system $\{V_{n;n}\}_{n\in N}$ forms a basis in $L_p(-a,a)$. The fact that from unconventional basicity of each of the systems $\{\vartheta_n^\pm\}_{n\in N}$ in $L_p(0,a)$ it follows the unconventional basicity of the system $\{V_{n;n}\}_{n\in N}$ in $L_p(-a,a)$ is proved by the analogical scheme.

Now we'll assume that the system $\{V_{n;n}\}_{n\in N}$ forms a basis in $L_p(-a,a)$. Let's take $\forall g \in L_p(0,a)$. We'll continue this function evenly (odd) on the whole segment $[-a,a]$ and denote it by $f^+(x)\,(f^-(x))$. Let's consider the expression $S_m(x) = S_m^+ + S_m^-(x)$, $\forall m \in N$, where $S_m^\pm(x)$ are determined by the relations (4), (5).

At first let's consider the even case. In this case it is easy to see that $S_m^-(x) \equiv 0$, $\forall m \in N$.

We have

$$\left| I_m^{(-a,a)} \right|^p = \int_0^a \left| S_m^+(x) \right|^p dx + \int_{-a}^0 \left| S_m^+(x) \right|^p dx = 2\int_0^a \left| S_m^+(x) \right|^p dx, \ \forall m \in N.$$

From the basicity of the system $\{V_{n;n}\}_{n\in N}$ in $L_p(-a,a)$ it follows that $I_m^{(-a,a)} \to 0$, $m \to \infty$. As a result, $\int_0^a |S_m^+(x)|^p dx \to 0$, $m \to \infty$ i.e., the biorthogonal series of arbitrary function $g(t)$ from $L_p(0,a)$ converges to it in $L_p(0,a)$, since $\frac{f^+(x) + f^+(-x)}{2} = g(x)$. By this, the basicity of the system $\{\vartheta_n^+\}_{n\in N}$ in $L_p(0,a)$ is proved. Analogously, we consider the odd case and with this the basicity of the system $\{\vartheta_n^+\}_{n\in N}$ in $L_p(0,a)$ is proved. By these considerations it is proved that if the system $\{V_{n;n}\}_{n\in N}$ forms on unconventional basis $L_p(-a,a)$, then each of the systems $\{\vartheta_n^+\}_{n\in N}$ is also form the basis in $L_p(0,a)$. Theorem is proved.

By the same way we prove the following theorem.

**Theorem 2.** *The double system $1 \cup \{V_{n;n}\}_{n\in N}$ forms a basis (unconditional basis) in $L_p(-a,a)$ iff each of the system $\{\vartheta_n^-\}_{n\in N}$ and $1 \cup \{\vartheta_n^+\}_{n\in N}$ forms a basis (unconditional basis) in $L_p(0,a)$. In particular, if in the place of $\omega_n^+(t) \equiv e^{i(\lambda_n t + \mu_n)}$; $\omega_n^-(t) \equiv e^{-i(\lambda_n t + \mu_n)}$ on $[0,\pi]$, from these theorems we get the following statement.*

**Corollary 1.** *Let $\{\lambda_n; \mu_n\}_{n\in N}$ be some sequences of complex numbers. The exponent system $\{e^{i(\lambda_n t + \mu_n signt)}\}_{n\in N}$ forms a basis (unconditional basis) in $L_p(-\pi,\pi)$, $1 \leq p > +\infty$, iff each of the systems of cosines $\{\cos(\lambda_n t + \mu_n)\}_{n\in N}$ and sines $\{\sin(\lambda_n t + \mu_n)\}_{n\in N}$ forms a basis (unconditional basicity) in $L_p(0,\pi)$, respectively.*

**Corollary 2.** *The system of exponents* $1\cup\left\{e^{\pm i(\lambda_n t+\mu_n sign t)}\right\}_{n\in N}$ *forms a basis (unconditional basis) in* $L_p(-\pi,\pi)$ *iff each of the systems of cosines* $1\cup\{\cos(\lambda_n t+\mu_n)\}_{n\in N}$ *and sines* $\{\sin(\lambda_n t+\mu_n)\}_{n\in N}$ *forms a basis (unconditional basis) in* $L_p(0,\pi)$*, respectively.*

It should be noted that some variants and particular cases of these corollaries were considered earlier in [5;6].

We pass to the question on Riesz basicity in $L_2$. Let's remind that the system $\{x\}_{n\in N}\subset X$ in the Banach space $X$ with the norm $\|\cdot\|$ is called almost normed if $0<\inf_n\|x_n\|\le\sup_n\|x_n\|<+\infty$. We need the following Lorch theorem [7].

**Lorch theorem.** *In order a basis in Hilbert space be the Riesz basis, it is necessary and sufficient it be unconditional and almost normed.*

Relative to the Riesz basicity the following theorem is true.

**Theorem 3.** *Let the condition* $\lim_{n\to\infty}\int_0^a a(t)\,b(t)\,\omega_n^+(t)\cdot\omega_n^+(t)\cdot\overline{\omega_n^-(t)}dt=0$ *hold. Then the system* $\{V_{n;n}\}_{n\in N}$ *forms the Riesz basis in* $L_2(-a,a)$ *iff the systems* $\{\vartheta_n^\pm\}_{n\in N}$ *form the Riesz basis in* $L_2(0,a)$.

**Proof.** Let $\{V_{n;n}\}_{n\in N}$ form the Riesz basis in $L_2(-a,a)$. Then by theorem 1 the systems $\{\vartheta_n^\pm\}_{n\in N}$ form the unconditional bases in $L_2(0,a)$. We have:

$$\int_0^a\left|\vartheta_n^\pm(t)\right|^2dt=\int_0^a|A(t)\,W_n(t)\pm A(-t)\,W_n(-t)|^2\,dt\le$$

$$\le 2\left[\int_0^a|A(t)\,W_n(t)|^2\,dt+|A(-t)\,W_n(-t)|^2\,dt\right]=2\int_{-a}^a|A(t)\,W_n(t)|^2\,dt. \qquad (6)$$

It is evident that $0<\inf_n\|V_{n;n}\|_2\le\sup_n\|V_{n;n}\|_2<+\infty$. Then from the previous inequality it follows that $\sup_n\left\|\vartheta_n^\pm\right\|_2<+\infty$, where

$$\left\|\vartheta_n^\pm\right\|_2^2=\int_0^a\left|\vartheta_n^\pm\right|^2dt;\ \ \|V_{n;n}\|_2^2=\int_{-a}^a|V_{n;n}(t)|^2\,dt.$$

We'll show that $\inf_n\left\|\vartheta_n^\pm\right\|_2>0$. It suffices to consider the case of the system $\{\vartheta_n^\pm\}_{n\in N}$. So:

$$\left\|\vartheta_n^\pm\right\|_2^2=\int_0^a|A(t)\,W_n(t)+A(-t)\,W_n(-t)|^2\,dt=$$

$$=\int_0^a[A(t)\,W_n(t)+A(-t)\,W_n(-t)]\cdot\left[\overline{A(t)W_n(t)}+\overline{A(-t)W_n(-t)}\right]dt=$$

$$=\int_0^a|A(t)\,W_n(t)|^2\,dt+\int_0^a|A(-t)\,W_n(-t)|^2\,dt+\int_0^a A(-t)\,W_n(-t)\,\overline{A(t)W_n(t)}dt+$$

$$+\int_0^a A(t)\,W_n(t)\,\overline{A(-t)W_n(-t)}dt=\int_{-a}^a|A(t)\,W_n(t)|^2\,dt+$$

$$+\int_{-a}^a A(t)\,W_n(t)\,\overline{A(-t)W_n(-t)}dt=\left\|V_n^+\right\|_2^2+J_n,\ \text{where}$$

$$J_n = \int_{-a}^{a} A(t) W_n(t) \overline{A(-t) W_n(-t)} dt$$

From the relation:

$$J_n = \int_0^a a(t) \omega_n^+(t) \overline{b(t) \omega_n^-(t)} dt + \int_0^a b(-t) \omega_n^-(-t) \cdot$$

$$\cdot a(-t) \cdot \overline{\omega_n^+(-t)} dt = 2 \operatorname{Re} \int_0^a a(t) \omega_n^+(t) \cdot \overline{b(t)} \omega_n^-(t) dt,$$

it follows that $\lim_{n\to\infty} J_n = 0$. Then from equality $\left\|\vartheta_n^+\right\|_2 > 0$ we directly obtain $\inf_n \left\|\vartheta_n^-\right\|_2 > 0$. Analogously, it is proved that $\inf_n \left\|\vartheta_n^-\right\|_2 > 0$. As a result, by the Lorch theorem the systems $\left\{\vartheta_n^\pm\right\}_{n\in N}$ form the Riesz basis in $L_2(0,a)$.

Now, let the system $\left\{\vartheta_n^\pm\right\}_{n\in N}$ form the Riesz basis in $L_2(0,a)$. Then by theorem 1 system $\{V_{n;n}\}_{n\in N}$ forms the unconditional basis in $L_2(-a,a)$. From inequality (6) we directly obtain that $\inf_n \|V_{n;n}\|_2 > 0$. From equality $\left\|\vartheta_n^\pm\right\|_2^2 = \|V_{n;n}\|_2^2 + J_n$ and from relation $\lim_{n\to\infty} J_n = 0$ it follows that $\sup_n \|V_n^+\|_2 < +\infty$. Analogously, it is shown that $\sup_n \|V_n^-\|_2 < +\infty$. Further ones follows from the Lorch theorem.

The theorem is proved.

The same assertion holds for the system with a unit.

**Theorem 4.** *Let the* $\lim_{n\to\infty} \int_0^a a(t) \overline{b(t)} \omega_n^+(t) \cdot \overline{\omega_n^-(t)} dt = 0$,*hold. The system* $1 \cup \{V_{n;n}\}_{n\in N}$ *forms the Riesz basis in* $L_2(-a,a)$ *iff the systems* $1 \cup \left\{\vartheta_n^+\right\}_{n\in N}$ *and* $\left\{\vartheta_n^-\right\}_{n\in N}$ *form the Riesz basis in* $L_2(0,a)$.

From these theorems we obtain the analogous assertions for the system of exponents

$\left\{e^{\pm i(\lambda_n t + \mu_n signt)}\right\}_{n\in N}$, cosines $\{\cos(\lambda_n t + \mu_n)\}_{n\in N}$ and sines $\{\sin(\lambda_n t + \mu_n)\}_{n\in N}$.

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