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COMPACT WEIGHTED COMPOSITION OPERATORS ON FUNCTION SPACES ON LOCALLY CONNECTED SETS

Abstract

Let X be a locally connected compact metric space and $C(X)$ denote the space of all continuous complex-valued functions on X equipped with sup-norm. In this paper we investigate compactness of weighted composition operators on a uniformly closed subspaces of $C(X)$ and give as some applications compactness criterion on uniform algebras.

1. Introduction

Let X be a compact metric space and $C(X)$ denote the space of all continuous complex-valued functions defined on X equipped with sup-norm. Let $A = A(X)$ be a uniformly closed subspace of $C(X)$. We will consider the operators $T : A \to C(X)$ of the form $T : f \longmapsto u \cdot f \circ \varphi$ (the symbol "$\circ$" denote the composition of functions), where $u \in C(X)$ is a fixed function and $\varphi : X \to X$ is a selfmapping of X which is continuous on the support of u, i.e., on the open set

$$S(u) = \{x \in X : u(x) \neq 0\}$$

(in particular, we can choose the function u and the selfmapping φ such that the operator may be acting in $A(X)$, i.e., $T : A(X) \to A(X)$). The operators of these forms are called the weighted composition operators induced by the function u (the weighted function) and by selfmapping φ. Since the endomorphisms of any semisimple commutative Banach algebras (also, any bounded linear operator on a Banach space) can be represented as operators of these forms, so the weighted composition operators are very interesting to study. Composition operators (i.e., the operators of the forms as operator T with the weighted function $u \equiv 1$) and weighted composition operators on the concrete uniform algebras are being investigated from different points of view (such as compactness, nuclearity, spectrum, closedness of ranges, etc.) by many authors. The aim of this paper is to clarify the compactness conditions of such operators. Kamowitz [1] in particular, gave the compactness criterion for the weighted composition operators acting in the disc-algebra $A(D)$ (the uniform algebra of functions analytic on the open unit disc $D = \{z \in C : |z| < 1\}$ of complex plane C and continuous on its closure $\bar{D}$), when $u, \varphi \in A(D)$ and $\|\varphi\| \leq 1$: the operator T of the form $f \longmapsto u \cdot f \circ \varphi$ in $A(D)$ is compact if, and only if, φ is constant, or $|\varphi(z)| < 1$ whenever $u(z) \neq 0$. In [2] (see also [3]) was given sufficiently simple necessity condition of compactness of the operator T in general case (i.e., when $A(X)$ has no any additional structures, such as algebraic, analytic, etc.), which in concrete situations for the algebras with good structure turn into compactness criterion. Here we give analogous general compactness criterion as above mentioned necessity condition in the case, when X is a locally connected compact set and its applications for concrete uniform algebras (in particular, including multidimensional analogues of disc-algebra), which reduces to easily verifiable constructive compactness criterion (note that for an arbitrary compact set the converse of above mentioned necessity condition is not true; see the proof of necessity of theorem in the next section).

2. Compactness criterion on closed subspaces

In this section we investigate compactness of operator $T : A \to C(X)$ in general case, when A has no any special structure, such as algebraic, analytic, etc. Except for easy degenerate cases, we will consider the nontrivial weighted composition operators, i.e., we assume that $\varphi \neq const$ and u is no identically zero.

Definition 2.1. *A closed subset E of X is called peak set with respect to $A(X)$, if there exists a sequence $\{f_n\} \subset A(X)$, such that $\|f_n\| = f_n(x) = 1$ for all n and all $x \in E$, moreover, outside any neighbourhood of the set E the sequence $\{f_n\}$ tends to 0 uniformly. A peak set consisting of only one point is called peak point.*

We denote the set of all peak points by P, assume that it is a nonempty set and X is a locally connected (we recall that the compact set X is locally connected if any point of X has a fundamental system of connected compact neighbourhoods). Put $G = X \backslash P$. To each point $x \in X$ corresponds a functional $\delta_x : f \longmapsto f(x)$, which lies in init ball of conjugate space A^*. This induces on X A^*-topology which is, generally speaking, more stronger than the original one. We also assume that the original topology on G coincides with A^*-topology (a typical example is given by the disc-algebra, where $X = \bar{D}$, $P = \partial\bar{D} = \bar{D}\backslash D$, $G = D$). Then we have following easily verifiable compactness criterion.

Theorem 2.2. *The operator $T : A(X) \to C(X)$ is compact if, and only if for any connected compact subset Y of $S(u)$ either $\varphi(Y)$ is one-point, or $\varphi(Y) \subset G$.*

Proof. Necessity. It is directly corollary of the Lemma 2 [2] (it is independent from locally connectedness of X and from that the original topology on G coincides with A^*-topology) (also see Theorem 1.5 [3]):

If the operator $T : A(X) \to C(X)$ of the form $f \longmapsto u \cdot f \circ \varphi$ is compact, then for any connected compact set $Y \subset S(u)$ and for any peak set E with respect to $A(X)$, we have either $\varphi(Y) \subset E$, or $\varphi(Y) \cap E = \emptyset$. (for an arbitrary compact set the converse of this statement is not true; indeed if X is compact with only one limit point, then this statement holds for any weighted composition operator, because there is no connected subset in X other than one-point sets).

Sufficiency. We shall identify the unit ball in conjugate of $C(X)$ with the set of Borel measures (complex) on X of variation less than 1, and identify the points $x \in X$ with corresponding δ-measures. Furher, as the compactness of operator is equivalent to compactness of conjugate one, then, it is easy to show that the compactness of operator T is equivalent to the continuity of mapping $x \longmapsto T^*x$ acting from X with original topology into A^* with a metric topology : if $x \to \varsigma$ in X in original topology, then the sequence $u(x)\varphi(x)$ must converges in A^*-topology to $u(\varsigma)\varphi(\varsigma)$. Indeed, if the operator T is compact, then T^*-image of unit ball of $C(X)^*$ is a compact set in A^*. In particular, image of X under mapping $x \longmapsto T^*x$ is compact with respect to metric topology of conjugate space, so mentioned above mapping is continuous. Conversely, if the mapping $x \longmapsto T^*x$ from X with original topology into A^* with a metric topology is continuous, then $T^*(X)$ is a compact set in A^*. Further, since absolutely convex hull of X is predcompact and contained unit ball, so we obtain that T is compact operator (see also Theorem VI 7.1 of [4]).

Consequently, we must demonstrate the continuity of mapping $x \longmapsto T^*x$, i.e., we must show that, when $x \to \varsigma$ on X (for any $\varsigma \in X$) in original topology, then we have $\|T^*x - T^*\varsigma\|_{A^*} \to 0$. First of all, we note that last condition is equivalent to the condition $|u(\varsigma)| \, \|\varphi(x) - \varphi(\varsigma)\|_{A^*} \to 0$ (because the function u is continuous everywhere on X). Further since

$$\|\lambda_1 x_1 + \lambda_2 x_2\|_{C(X)^*} = \begin{cases} |\lambda_1 + \lambda_2|, & if \quad x_1 = x_2 \\ |\lambda_1| + |\lambda_2|, & if \quad x_1 \neq x_2 \end{cases}$$

(where $x_1, x_2 \in X$ and λ_1, λ_2 are complex numbers), then for the points $\varsigma \in X$ such that $u(\varsigma) = 0$, we have

$$|u(\varsigma)| \, \|\varphi(x) - \varphi(\varsigma)\|_{A^*} = 0$$

for any case when x converges to ς on X in original topology.

If $\varsigma \in S(u)$, then there exists a connected neighbourhood U_ς of the point ς such that its closure $\bar{U}_\varsigma$ belong to $S(u)$ and by the condition of the theorem we have $\varphi\left(\bar{U}_\varsigma\right) \in G$ (if $\varphi\left(\bar{U}_\varsigma\right)$ is not a one –point set). Then, since on G the original topology coincides with A^*-topology we obtain that

$$\|\varphi(x) - \varphi(\varsigma)\|_{A^*} \to 0,$$

i.e.,

$$|u(\varsigma)| \, \|\varphi(x) - \varphi(\varsigma)\|_{A^*} \to 0$$

for any case when x converges to ς in original topology. In other side, if $\varphi\left(\bar{U}_\varsigma\right)$ is singleton, then it is clear that for all x in U_ς, since $\varphi(x) = \varphi(\varsigma)$ we obtain that $\|\varphi(x) - \varphi(\varsigma)\|_{A^*} = 0$ i.e., in this case, again we have that

$$|u(\varsigma)| \, \|\varphi(x) - \varphi(\varsigma)\|_{A^*} \to 0,$$

when x converges to ς in original topology.

So, continuous of above mentioned mapping is proved, i.e., the operator T is compact. The theorem is proved.

Since on the set when a Hausdorff topology weakly than compact topology then they coincides, so we have following corollary:

Corollary 2.3. *If for any compact subset K of X the natural restriction operator $A(X) \to C(K)$ is compact, then the operator T is compact if and only if, when for every connected compact $Y \subset S(u)$ either $\varphi(Y)$ is singleton, or $\varphi(Y) \subset G$.*

Remark 2.4. *It is clear that for subspaces $A(X)$ of $C(X)$ which $P = X$ the theorem is true even without from locally connectedness. For example when $A(X) = C(X)$ we have $P = X$. This condition may be true for non-trivial cases, e.g., A is the disc-algebra viewed as the algebra of continuous functions on the boundary of the disc.*

3. Compactness criterion on function algebras

In this section we give some applications for the function algebras defined on locally connected compact sets. Recall that a function algebra is a sup-norm closed subalgebra of continuous functions on a compact set X which separates points of X and contains the constants. A peak set of a function algebra is a closed subset E of X for which there exists a function f in the algebra with $\|f\| = f(x) = 1$ for $x \in E$

and $|f(x)| < 1$ for $x \in X \backslash E$. A singleton peak set is a peak point (a generalized peak point is a point x in X such that $\{x\}$ is an intersection of peak sets).

First of all we note that since for the algebra $C(X)$ the set of all peak points coincides with X, then without from any locally connected conditions we have following theorem from [2] (see Remark 2.3):

Theorem 3.1. *The operator T of the form $f \longmapsto u \cdot f \circ \varphi$ is compact on $C(X)$ if, and only if, for each $\varepsilon > 0$, the set $\varphi(\{x \in X : |u(x)| \geq \varepsilon\})$ is finite. In particular, if $S(u) = X$ and X is connected compact set, then T is compact on $C(X)$ if, and only if, φ is constant.*

From this theorem it is clear that if T is a compact operator on $C(X)$, then the range φ on the set $S(u)$ is at most countable. The Theorem 2.2 shows that this result is convertible. Unlike to $C(X)$ in analytical situations a nontrivial weighted endomorphism may be a compact operator, as in the case disc-algebra.

Lemma 3.2. *In the disc-algebra for any $x, y \in D$ we have*

$$\|\delta_x - \delta_y\|_{A^*} = d(x, y) = d,$$

where

$$\frac{d}{1 + \frac{1}{4}d^2} = \left|\frac{x - y}{1 - \bar{x}y}\right|,$$

in particular,

$$\left|\frac{x - y}{1 - \bar{x}y}\right| \leq \|x - y\|_{A^*} \leq 2\left|\frac{x - y}{1 - \bar{x}y}\right|.$$

Proof. Let $|x|, |y| < 1$. We perform the linear-fractional transformation, such that $x \to t$, $y \to -t$, $t > 0$. Then,

$$\|\delta_x - \delta_y\|_{A^*} = \|\delta_t - \delta_{-t}\|_{A^*}$$

and the number t may be defined from the equality $\left|\frac{x - y}{1 - \bar{x}y}\right| = \frac{2t}{1 + t^2}$, because the expression $\left|\frac{x - y}{1 - \bar{x}y}\right|$ is invariant under the linear-fractional transformations (the Schwartz lemma). For the points $x, y = \pm t$ and the function $f(z) = z$ we obtain $(\delta_x - \delta_y) f = 2t$, i.e., $\|\delta_x - \delta_y\|_{A^*} \geq 2t$. In other side, let f be a function from the unit ball of $A(D)$ such that $|f(t) - f(-t)| = C$, and put $g(z) = \frac{f(z) - f(-z)}{2}$. Then

$$\|g\| \leq \|f\| \leq 1, \quad |g(t) - g(-t)| = C$$

and g is a odd function. In particular, $g(0) = 0$. By using the Schwartz Lemma we have that $|g(z)| \leq |z|$, consequently $C \leq 2t$. So, we obtain that

$$\|\delta_x - \delta_y\|_{A^*} = \|\delta_t - \delta_{-t}\|_{A^*} = 2t.$$

The lemma is proved.

Since on the open unit disc A^*-topology coincides with the original topology and every point of ∂D is a peak point, then from the Theorem 2.2 immediately we obtain Kamowitz compactness criterion [1]. We may replace the disc in Kamowitz theorem by more general domains. In multidimensional case different domains give rise to additional properties. The case of ball is easier than the case of polydisc.

Let $A(B^n)$ be the algebra of analytic functions in the interior of the ball

$$B^n = \left\{ z = (z_1, \cdots, z_n) \in C^n : \sum_{k=1}^{n} |z_k|^2 < 1 \right\}$$

and continuous on its closure. Since every point of the topological boundary of ball is a peak point, so our Theorem 2.2 enables us to generalize Kamowitz theorem (see also [2]):

Theorem 3.3. *The operator T on $A(B^n)$ (u and φ are analytic) is compact if, and only if, either $\varphi = const$, or $\|\varphi(z)\| < 1$ (Eucledian norm) for all $z \in S(u)$.*

Now we consider the case polydisc. Let

$$D^n = \{z = (z_1, \cdots, z_n) \in C^n : \ |z_k| < 1, \quad 1 \le k \le n\},$$

$A(D^n)$–the algebra of analytic functions in D^n and continuous on its closure. The Shilov's boundary of $A(D^n)$ is the torus

$$T^n = \{z = (z_1, \cdots, z_n) \in C^n : \ |z_k| = 1, \quad 1 \le k \le n\}$$

and it is contained in the topological boundary ∂D^n as a proper subset for $n > 1$ (this is the main difference of B^n with polydisc). So from Theorem 2.2 we have another generalization of Kamowitz theorem:

Theorem 3.4. *The operator T acting on $A(D^n)$ is compact iff, for any k, either $\varphi_k(z) = const$, or $|\varphi_k(z)| < 1$ for all z with $u(z) \neq 0$.*

We finish this section with an example which combines Theorem 3.1 (in case of closed unit interval) and Kamowitz theorem. Similar combination can also be made in more general cases (for those operators which act on complete tensor product spaces), but in the following case the compactness is very easy.

Let X be the subset of $C \times R$ (C and R denote the fields of complex and real numbers, respectively) defined by

$$X = \{(z, 0) : |z| \le 1\} \cup \{(0, t) : 0 \le t \le 1\}$$

and let

$$A(X) = \{f \in C(X) : \ z \longmapsto f(z, 0) \ \ is\ analyticin\, |z| < 1\}.$$

Then $A(X)$ is a function algebra whose Shilov boundary is

$$\{(z, 0) : |z| = 1\} \cup \{(0, t) : 0 \le t \le 1\}.$$

The point $(0, 0)$ is in the Shilov boundary, but is not a peak point. So, $G = D$ and as above we noted A^*-topology coincides with original one. In this algebra the functions $u \in A(X)$ and continuous mappings on $S(u)$, $\varphi : X \to X$, such that $\varphi(z, 0) : \ D \cap S(u) \to D$ is analytic are induced the weighted composition operators T of the form $f \longmapsto u \cdot f \circ \varphi$ which acting in $A(X)$ (in other words these functions and selfmappings are multiplicators and compositors respectively, with respect to $A(X)$; i.e., for any $f \in A(X)$ we have $u \cdot f \in A(X)$ and $f \circ \varphi \in A(X)$); where in generally (since real-valued complex variable analytic functions are constants) φ defined as form:

$$\varphi(z,0) = (\alpha(z), 0), \quad \varphi(0, t) = (0, \beta(t))$$

on the set

$$X = \{(z,0) : |z| \leq 1\} \cup \{(0,t) : 0 \leq t \leq 1\}.$$

Since

$$P = \{(z,0) : |z| = 1\} \cup \{(0,t) : 0 < t \leq 1\},$$

so from Theorem 2.2 we obtain following compactness criterion for the weighted composition operators in $A(X)$:

Theorem 3.5. *The weighted composition operator T of the form $f \longmapsto u \cdot f \circ \varphi$ in algebra $A(X)$ is compact if, and only if, either $\varphi = const(\alpha = const,\ \beta = 0;$ or $\alpha = 0,\ \beta = const)$, or $\beta \equiv 0$, and $|\alpha(z)| < 1$ whenever $z \in S(u)$.*

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