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ON A BOUNDARY VALUE PROBLEM FOR SINGULARLY PERTURBED QUASILINEAR ELLIPTIC EQUATION IN CURVILINEAR TRAPEZOID

Abstract

A boundary value problem for an elliptic type quasilinear equation of second order containing a small parameter for heigher derivatives is considered in a curvilinear trapezoid. Asymptotic expansion of generalized solution of the considered problem is constructed to within any power of small parameter and remainder term is estimated.

Let $x = \varphi_1(y)$, $x = \varphi_2(y)$ be sufficiently smooth functions determined in $[a, b]$ and satisfy the following conditions:

I. $\varphi_1(y) < \varphi_2(y)$ for $y \in [a, b]$;

II. $\varphi_1(y) < y$ for $y \in [a, b]$, $\varphi_2(y) > y$, $y \in [a, b]$;

III. $\varphi_1(a) = a, \varphi_2(b) = b$

IV. $\varphi_1'(y) < 1, \varphi_2'(y) < 1$ for $y \in [a, b]$.

Introduce the denotation:

$$\Gamma_1 = \{(x,y)|\, x = \varphi_1(y),\ a \le y \le b\},\quad \Gamma_2 = \{(x,y)|\, \varphi_1(y) \le x \le \varphi_2(y),\quad y = b\},$$

$$\Gamma_3 = \{(x,y)|\, x = \varphi_2(y),\ a \le y \le b\},\quad \Gamma_4 = \{(x,y)|\, \varphi_1(y) \le x \le \varphi_2(y),\quad y = a\}.$$

In $\Omega = \{(x,y)|\, \varphi_1(y) \le x \le \varphi_2(y),\quad a \le y \le b\}$ we consider the following boundary value problem

$$L_\varepsilon U \equiv -\varepsilon^p \left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^p + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)^p\right] -$$

$$-\varepsilon \Delta U + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + F(x, y.U) = 0, \tag{1}$$

$$U|_\Gamma = 0, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $p = 2k + 1$, k is an arbitrary natural number, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, $F(x, y, U)$ is a given smooth function satisfying the condition

$$\frac{\partial F(x,y,U)}{\partial U} \ge \gamma^2 > 0 \ \text{ for } \ (x, y, U) \in (\Omega \setminus \{(x,y) \in \Omega\,|x = y\}) \times (-\infty, +\infty). \tag{3}$$

It is assumed that $F(x, y, U)$ may depend on U both linearly, i.e. $F(x, y, U) = d(x, y) U - f(x, y), d(x, y) \ge \gamma^2 > 0$ and non-linearly.

It is known that for each fixed ε there exists a unique generalized solution of problem (1),(2) in space $\overset{\circ}{W}{}^{1}_{p+1}(\Omega)$. Obviously, if $F(x,y,0) \equiv 0$, the problem (1),(2) has only a trivial solution. Therefore, assume that

$$F(x,y,0) \not\equiv 0 \ \text{ for } \ (x,y) \in \Omega. \tag{4}$$

Asymptotics of the solution of boundary value problem for a second order quasi-linear elliptic equation in n-dimensional bounded domain with smooth boundary is constructed in the paper [1]. In the paper [2], a boundary value problem is investigated for equation (1) in a rectangular domain.

Our goal is to construct asymptotic expansion of the solution of boundary value problem (1),(2) in small parameter $\varepsilon > 0$.

In the connection, we'll conduct iteration processes.

In the first iteration process, approximate solution of the equation is sought in the form

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n, \tag{5}$$

and the functions $W_i(x,y); i = 0,1,...,n$ will be chosen so that

$$L_\varepsilon W = 0\left(\varepsilon^{n+1}\right). \tag{6}$$

From (1),(5) and (6) we get the following equations:

$$\frac{\partial W_0}{\partial x} + \frac{\partial W_0}{\partial y} + F(x,y,W_0) = 0, \tag{7}$$

$$\frac{\partial W_i}{\partial x} + \frac{\partial W_i}{\partial y} + \frac{\partial F(x,y,W_0)}{\partial W_0} W_i = f_i; \ i = 1,2,...,n, \tag{8}$$

where the functions f_i depend on $W_0, W_1, ..., W_{i-1}$ and their derivatives. Boundary conditions on the lines Γ_1, Γ_4 that are parts of the boundary Γ of domain Ω will be used for equations (7),(8). Then boundary condition (2) may not be fulfilled on the lines Γ_2 and Γ_3. The boundary layer type functions should be constructed near Γ_2 and Γ_3 in order to compensate, the lost boundary conditions.

As it was noted above, we'll solve equations (7),(8) under the following boundary conditions:

$$\begin{gathered} W_i|_{x=\varphi_1(y)} = 0, \ (a \le y \le b); \\ W_i|_{y=a} = 0, \ (\varphi_1(a) \le x \le \varphi_2(a)); \ i = 0,1,...,n. \end{gathered} \tag{9}$$

Problem (7),(9) (for $i = 0$) will be called a degenerated problem corresponding to problem (1),(2).

It holds the following

Theorem 1. *Let $F(x,y,U) \in C^m(\Omega \times (-\infty,+\infty))$, the function $F(x,y,U)$ satisfy conditions (3),(4) and the condition*

$$\left.\frac{\partial^i f(x,y)}{\partial x^{i_1} \partial y^{i_2}}\right|_{x=y} = 0; \ \ y \in [a,b], \ i = i_1 + i_2; \ i = 0,1,...,m, \tag{10}$$

in the case of linear dependence of F from U, the conditions

$$F\left(x,y,U\right)|_{x=y}=0;\ y\in[a,b]\,,\quad U\in(-\infty,+\infty)\,, \tag{11}$$

$$\left.\frac{\partial^{i} f\left(x,y,0\right)}{\partial x^{i_1}\partial y^{i_2}\partial U^{i_3}}\right|_{x=y}=0;\quad y\in[a,b]\,,\ i=i_1+i_2+i_3;\ i=0,1,...,m, \tag{12}$$

in the case of nonlinear dependence of F on U (m is an arbitrary natural number). The degenerate problem has a unique solution, moreover $W_0\left(x,y\right)\in C^m\left(\Omega\right)$ and

$$\left.\frac{\partial^{i} W_0\left(x,y\right)}{\partial x^{i_1}\partial y^{i_2}}\right|_{x=y}=0;\quad y\in[a,b]\,,\ i=i_1+i_2;\ i=0,1,...,m. \tag{13}$$

Proof. The characteristic line of equation (7) passing through the origin of coordinates divides the domain Ω into two parts:

$$\Omega_1=\{(x,y)|\,(x,y)\in\Omega,\ y\geq x\}\quad\text{and}\quad\Omega_2=\{(x,y)|\,(x,y)\in\Omega,\ y\leq x\}\,.$$

We can look for the solution of the degenerate problem in the form

$$W_0=\begin{cases} W_0^{(1)} & \text{for}\quad (x,y)\in\Omega_1,\\ \\ W_0^{(2)} & \text{for}\quad (x,y)\in\Omega_2,\end{cases} \tag{14}$$

moreover $W_0^{(1)}$ and $W_0^{(2)}$ are the solutions of the following Cauchy problems:

$$\begin{gathered}\frac{\partial W_0^{(1)}}{\partial x}+\frac{\partial W_0^{(1)}}{\partial y}+F\left(x,y,W_0^{(1)}\right)=0,\quad (x,y)\in\Omega_1;\\ \left.W_0^{(1)}\right|_{x=\varphi_1(y)}=0,\ y\in[a,b]\,,\end{gathered} \tag{15}$$

$$\begin{gathered}\frac{\partial W_0^{(2)}}{\partial x}+\frac{\partial W_0^{(2)}}{\partial y}+F\left(x,y,W_0^{(2)}\right)=0,\quad (x,y)\in\Omega_2;\\ \left.W_0^{(2)}\right|_{y=a}=0,\ x\in[\varphi_1\left(a\right),\varphi_2\left(a\right)]\,.\end{gathered} \tag{16}$$

When $F\left(x,y,W_0\right)$ depends linearly on W_0, the obvious representation of the solution of degenerate problem is of the form

$$W_0=\begin{cases}\displaystyle\int\limits_{\psi(y_1)}^{x_1} f\left(\xi,\xi+y_1\right)\exp\left[\int\limits_{x_1}^{\xi} d\left(\tau,\tau+y_1\right)d\tau\right]d\xi,\ x_1= \\ \qquad =x,\ y_1=y-x,y>x,\\ \displaystyle\int\limits_{a}^{y_1} f\left(x_1+\xi,\xi\right)\exp\left[\int\limits_{y_1}^{\xi} d\left(x_1+\tau,\tau\right)d\tau\right]d\xi,\ x_1= \\ \qquad =x-y,\ y_1=y,x>y,\end{cases} \tag{17}$$

where $x_1 = \psi(y_1)$ is a solution of the equation $x_1 = \varphi_1(x_1 + y_1)$ with respect to x_1. Using formula (17), it is easily proved that if condition (10) is fulfilled, then $W_0(x, y) \in C^m(\Omega)$ and condition (13) is satisfied.

In the case of nonlinear dependence of $F(x, y, W_0)$ on W_0, the problems (15),(16) are reduced to the following Cauchy problems for ordinary differential equations:

$$\frac{dW_0^{(1)}}{dx_1} = -F\left(x_1, x_1 + y_1, W_0^{(1)}\right), \quad W_0^{(1)}\Big|_{x_1=\psi(y_1)} = 0, \tag{18}$$

$$\frac{dW_0^{(2)}}{dy_1} = -F\left(x_1 + y_1, y_1, W_0^{(2)}\right), \quad W_0^{(2)}\Big|_{y_1=a} = 0. \tag{19}$$

Existence of local solutions of problems $(18), (19)$ is obvius. Using condition (3), we can get a priori estimations for these local solutions. Possibility of continuous continution of local solutions on Ω_1 and Ω_2 follows from the obtained a priori estimations.

In order to study differential properties of the solution of the degenerate problem, in the non-linear case we reduce this problem to the following non-linear integral equations:

$$W_0(x, y) = \begin{cases} -\displaystyle\int_{\psi(y_1)}^{x_1} F(\tau, \tau + y_1, W_0(\tau, \tau + y_1))\, d\tau, \\ \qquad x_1 = x,\ y_1 = y - x, y > x, \\ -\displaystyle\int_{a}^{y_1} F(x_1 + \tau, \tau, W_0(x_1 + \tau, \tau))\, d\tau, \\ \qquad x_1 = x - y,\ y_1 = y,\ x > y. \end{cases} \tag{20}$$

Using formula (20), we can prove that if conditions $(11), (12)$ are satisfied, then $W_0(x, y) \in C^m(D)$ and (13)is satisfied. Theorem 1 is proved.

The problems $(8), (9)$ for $i = 1, 2, ..., n$ wherefrom the functions $W_1, W_2, ..., W_n$ will be successively determined, and linear. We can write the solutions of these problems in the obvious form by formula (17). Notice that the functions $W_i(x, y)\,;\ i = 1, 2, ..., n$ will also vanish for $y = x$ together with their own derivatives.

If in theorem 1 we take $m = 2n + 2$, then from this theorem in the case of linear dependence of $F(x, y, U)$ on U it follows that $W_i \in C^{2(n-i)+2}(\Omega)\,;\ i = 0, 1, ..., n$. Hence and from (5) we get $W \in C^2(\Omega)$ for each fixed value of $\varepsilon \in [0, \varepsilon_0]$. Consequently the operator L_ε may operate on the constructed function W.

Thus, we constructed the function W that is an approximate solution of equation (1) in the sense of (6) and satisfies the boundary conditions:

$$W|_{\Gamma_1} = 0, \quad W|_{\Gamma_4} = 0. \tag{21}$$

In order to construct a boundary layer function near the boundary Γ_3, at first we should write a new decomposition of the operator L_3 near this line. Make change of variables: $\varphi_2(y) - x = \varepsilon\tau,\ y = y_1$. Consider the auxiliary function

$$r = \sum_{j=0}^{n+1} \varepsilon^j r_j(\tau, y_1)$$

where $r_j(\tau, y_1)$ are some functions determined near the line $x = \varphi_2(y)$. Considering this change, substituting the expressions r in $L_\varepsilon r$, expanding the function $F(\varphi_2(y_1) - \varepsilon\tau, y_1, r)$ and other nonlinear members in power of ε, after certain transformations we get a new decomposition of the operator $L_\varepsilon r$ in the coordinates (τ, y_1) in the form

$$L_{\varepsilon,1} \equiv \varepsilon^{-1}\left\{-\left[\delta_1^2(y_1)\frac{\partial}{\partial\tau}\left(\frac{\partial r_0}{\partial\tau}\right)^{2k+1} + \delta_2^2(y_1)\frac{\partial^2 r_0}{\partial\tau^2} + \delta_3^2(y_1)\frac{\partial^2 r_0}{\partial\tau}\right] + \right.$$

$$+\sum_{j=1}^{n+1}\varepsilon^j\left[-(2k+1)\delta_1^2(y_1)\frac{\partial}{\partial\tau}\left(\left(\frac{\partial r_0}{\partial\tau}\right)^{2k}\frac{\partial r_j}{\partial\tau}\right) - \delta_2^2(y_1)\frac{\partial^2 r_j}{\partial\tau^2} - \delta_3^2(y_1)\frac{\partial r_j}{\partial\tau} + \right.$$

$$\left.\left. + h_j(r_0, r_1, ..., r_{j-1})\right] + 0\left(\varepsilon^{n+2}\right)\right\}. \qquad (22)$$

Here h_j, are the known functions dependent on $\tau, y_1, r_0, r_1, ..., r_{j-1}$ and their first and second derivatives. The functions $\delta_1^2(y_1), \delta_2^2(y_1), \delta_3^2(y_1)$ are determined by the following formulae:

$$\delta_1^2(y_1) = 1 + \left[\varphi_2'(y_1)\right]^{2k+2},\quad \delta_2^2(y_1) = \left[1 + \varphi_2'(y_1)\right]^2,\ \delta_3^2(y_1) = 1 - \varphi_2'(y_1).$$

We'll look for a boundary layer type function near the boundary Γ_3 in the form

$$V = V_0(\tau, y_1) + \varepsilon V_1(\tau, y_1) + ... + \varepsilon^{n+1} V_{n+1}(\tau, y_1), \qquad (23)$$

as a solution of the equation

$$L_{\varepsilon,1}(w + V) - L_{\varepsilon,1}W = 0\left(\varepsilon^{n+1}\right). \qquad (24)$$

Before we substitute expressions $(5), (23)$ in (24), we should expand each function $W_i(\varphi_2(y_1) - \varepsilon\tau, y_1)$ by Taylor formula at the point $(\varphi_2(y_1)\, y_1)$ and get a new expansion in powers of ε of the function W in the coordinates (τ, y_1). A new expansion of the function W is of the form

$$W = \sum_{j=0}^{n+1} \varepsilon^j \omega_j(\tau, y_1) + 0\left(\varepsilon^{n+2}\right), \qquad (25)$$

where $\omega_0(\tau, y_1) = W_0(\varphi_2(y_1)\, y_1)$ is independent on τ, the remaining functions ω_k are determined by the formula

$$\omega_k = \sum_{i+j=k} \frac{(-1)^i}{i!}\frac{\partial^i W_j(\varphi_2(y_1), y_1)}{\partial x^i}\tau^i;\ k = 1, 2, ..., n+1.$$

It follows from $(22)-(25)$ that the functions V_j contined in the right hand side of (23) are the solutions of the following equations:

$$\delta_1^2(y_1)\frac{\partial}{\partial\tau}\left(\frac{\partial V_0}{\partial\tau}\right)^{2k+1}+\delta_2^2(y_1)\frac{\partial^2 V_0}{\partial\tau^2}+\delta_3^2(y_1)\frac{\partial V_0}{\partial\tau}=0, \tag{26}$$

$$\frac{\partial}{\partial\tau}\left\{\left[(2k+1)\,\delta_1^2(y_1)\left(\frac{\partial V_0}{\partial\tau}\right)^{2k}+\delta_2^2(y_1)\right]\frac{\partial V_j}{\partial\tau}\right\}+ \\ +\delta_3^2(y_1)\frac{\partial V_j}{\partial\tau}=H_j(\tau,y_1) \tag{27}$$

where $H_j; j=1,2,...,n+1$ are the known functions dependent on $\tau, y_1, V_0, V_1, ..., V_{j-1}$, $\omega_0,\omega_1,...,\omega_j$ and their first and second derivatives.

Boundary conditions for equations $(26),(27)$ are obtained from the requirement that the sum $W+V$ should satisfy the boundary condition

$$(W+V)|_{\Gamma_3}=0 \tag{28}$$

Substituting the expressions for W and V from (5) and (23), respectively to (28) and considering also the fact that we look for $V_j, j=0,1,...,n+1$ as a boundary layer type function, we have

$$V_0|_{\tau=0}=g_0(y_1),\ \lim_{\tau\to+\infty}V_0=0, \tag{29}$$

$$Vj|_{\tau=0}=g_j(y_1),\ \lim_{\tau\to+\infty}V_j=0,\ j=1,2,...,n+1 \tag{30}$$

where $g_i(y_1)=-W_i(\varphi_2(y_1),y_1)$ for $i=0,1,...,n; g_{n+1}(y_1)\equiv 0$.

The following thorem is true.

Theorem 2. *For each fixed $y_1\in[a,b]$, problem $(26),(29)$ has a unique solution which is differentiable with respect to τ and has continuous derivatives up to $(2n+2)-$th order inclusively with respect to y_1, and the function $V_0(\tau,y_1)$ and its derivatives exponentially tend to zero as $\tau\to+\infty$.*

Proof. At first we prove uniqueness of the solution of problem $(26),(29)$.

If $V_0^{(1)}(\tau,y_1), V_0^{(2)}(\tau,y_2)$are two solutions of problem $(26),(29)$, we integrate by parts and get

$$\delta_1^2(y_1)\int_0^{+\infty}\left(\frac{\partial V_0^{(1)}}{\partial\tau}-\frac{\partial V_0^{(2)}}{\partial\tau}\right)^{2k+1}d\tau+2^{2k+2}\delta_1^2(y_1)\int_0^{+\infty}\left(\frac{\partial V_0^{(1)}}{\partial\tau}-\frac{\partial V_0^{(2)}}{\partial\tau}\right)^{2}d\tau\le 0,$$

hence $V_0^{(1)}(\tau,y)\equiv V_0^{(2)}(\tau,y)$ follows.

Passing to the proof of the existence of the solution of problem $(26),(29)$, we note that the variable y_1 plays as a parameter in this problem. Since $W_0(\varphi_2(a),a)=W_0(\varphi_2(b),b)=0$, then for $y_1=a$ and $y_1=b$ the function $V_0\equiv 0$ will satisfy problem $(26),(29)$. It follows from uniquenes of the solution of the problem that in the cases $y_1 = a$ and $y_1 = b$ the solution of problem $(26),(29)$ is predetermined by an identity zero.

We should prove the existence of the solution of problem $(26),(29)$ for $y_1 \in (a,b)$.

In a similar way as it was made in [3], we can prove that for each $y_1 \in (a,b)$ the solution of problem $(26),(29)$ in parametric form has the view:

$$\tau = \frac{2k+1}{2k}\frac{\delta_1^2(y_1)}{\delta_3^2(y_1)}\left(t_0^{2k} - t^{2k}\right) + \frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\ln\left|\frac{t_0}{t}\right|, \tag{31}$$

$$V_0 = -\frac{\delta_1^2(y_1)}{\delta_3^2(y_1)}t^{2k+1} - \frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}t. \tag{32}$$

Here t is a parameter, $t_0(y_1)$ is a unique real root of the algebraic equation

$$t_0^{2k+1} + t_0 + g_0(y_1) = 0. \tag{33}$$

Using the obvious form of the solution of V_0, we can prove that the function V_0 is infinitely differentiable with respect to τ and the estimation

$$\left|\frac{\partial^k V_0}{\partial \tau^k}\right| \le c\exp\left[-\frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\tau\right], (c>0);\ k = 0,1,... \tag{34}$$

is valid for all $y_1 \in [a,b]$.

Investigate the behavior of $V_0(\tau, y_1)$ with respect to y_1. At first, note that as equation (33) has a unique real root $t_0(g_0(y_1))$ for all $y_1 \in [a,b]$ and the function $g_0(y_1) \in C^{2n+2}[a,b]$, then the function $t_0(g_0(y_1))$ also will have continuous derivatives up to $(2n+2)$ order inclusively. Hence and from $(31),(32)$ the smoothness of the function $V_0(\tau, y_1)$ with respect y_1 follows.

Now estimate as $\tau \to +\infty$ the derivatives with respect to y_1 of the function $V_0(\tau, y_1)$.

The function $z = \dfrac{\partial V_0}{\partial y_1}$ satisfies the equation in variations that is obtained from equation (26) by differentiating with respect to y_1:

$$\frac{\partial}{\partial \tau}\left[A(\tau, y_1)\frac{\partial z}{\partial \tau}\right] + \delta_3^2(y_1)\frac{\partial z}{\partial \tau} = \Phi_1 \tag{35}$$

where

$$A(\tau, y_1) = (2k+1)\,\delta_1^2(y_1)\left(\frac{\partial V_0}{\partial \tau}\right)^{2k} + \delta_2^2(y_1), \tag{36}$$

$$\Phi_1(\tau, y_1) = -\left[\delta_1^2(y_1)\right]'\frac{\partial}{\partial \tau}\left(\frac{\partial V_0}{\partial \tau}\right)^{2k+1} - \left[\delta_2^2(y_1)\right]'\frac{\partial^2 V_0}{\partial \tau^2} - \left[\delta_3^2(y_1)\right]'\frac{\partial V_0}{\partial \tau}. \tag{37}$$

Obviously, the function z should satisfy the boundary conditions:

$$z|_{\tau=0} = g_0'(y_1), \quad \lim_{\tau\to+\infty} z = 0. \tag{38}$$

The solution of problem $(35),(38)$ is of the form

$$z=\left\{\int\limits_0^{\tau}\Phi_2(\xi_1,y_1)\exp\left[\delta_3^2(y_1)\int\limits_0^{\xi_1}\frac{d\xi}{A(\xi,y_1)}\right]d\xi_1+\right.$$

$$\left.+g_0'(y_1)\right\}\exp\left[-\delta_3^2(y_1)\int\limits_0^{\tau}\frac{d\xi}{A(\xi,y_1)}\right], \tag{39}$$

where

$$\Phi_2(\tau,y_1)=-\frac{1}{A(\tau,y_1)}\int\limits_{\tau}^{\infty}\Phi_1(\xi,y_1)\,d\xi. \tag{40}$$

Using $(36),(37),(39),(40)$ and estimation (34), from (39) we can get the following estimations:

$$|z|=\left|\frac{\partial V_0}{\partial y_1}\right|\leq(C+C_2\tau)\exp(-\tau)\quad\text{for}\quad\delta_3^2(y_1)=\delta_2^2(y_1), \tag{41}$$

$$|z|=\left|\frac{\partial V_0}{\partial y_1}\right|\leq C_3\exp\left[-\frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\tau\right]+$$

$$+C_4\exp\left[-\frac{\delta_3^2(y_1)}{\delta_2^2(y_1)}\tau\right]\quad\text{for}\quad\delta_3^2(y_1)\neq\delta_2^2(y_1), \tag{42}$$

where C_1,C_2,C_3,C_4 are positive constants.

We can establish estimations for the next derivatives of V_0 with respect to y_1 and for mixed derivatives in a similar way. Theorem 2 is proved.

Now construct the functions $V_1,V_2,...,V_{n+1}$ that are the solutions of equations (27) satisfying boundary conditions (30) for $j=1,2,...,n+1$. Equations (27) and (35) differ only by the right hand sides. Therefore, the functions $V_1; j=1,2,...,n+1$ will also be determined by formula (39), only changing the functions $\Phi_2(\xi_1,y_1)$, $g_0'(y_1)$ by another appropriate functions. Using the obvious forms of the right hand sides of equations (27), we can prove validity of the estimation of the form

$$\left|\frac{\partial^k V_j(\tau,y_1)}{\partial\tau^{k_1}\partial y^{k_2}}\right|\leq\left(\sum_{i=0}^{j+1}C_i\tau^i\right)\exp(-\tau)\quad\text{for}\ \delta_3^2(y_1)=\delta_2^2(y_1), \tag{43}$$

$$\left|\frac{\partial^k V_j(\tau,y_1)}{\partial\tau^{k_1}\partial y^{k_2}}\right|\leq\left(\sum_{i=0}^{j}C_i^{(1)}\tau^i\right)\exp\left[-\frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\right]+$$

$$+\left(\sum_{i=0}^{j}C_i^{(2)}\tau^i\right)\exp\left[-\frac{\delta_3^2(y_1)}{\delta_2^2(y_1)}\right]\quad\text{for}\ \ \delta_3^2(y_1)\neq\delta_2^2(y_1), \tag{44}$$

where $k=k_1+k_2$, $k_2+2(j-1)=2n+2$; $j=1,2,...,n+1$; $C_i,C_i^{(1)},C_i^{(2)}$ are positive constants.

Multiply all the functions by $V_j; j = 1, 2, ..., n+1$ a by smoothing multiplier and leave previous denotation for obtained new functions. Note that at the expense of smoothing functions, V doesn't violate fulfilment of the first condition from (21), i.e. the sum $W + V$, in addition to condition (28), satisfies the condition

$$(W + V)|_{\Gamma_1} = 0. \tag{45}$$

But the function V may violate fulfilment of the second condition from (21) for the sum $W + V$. In order the condition

$$(W + V)|_{\Gamma_4} = 0, \tag{46}$$

be fulfilled, all the functions V_j for $y = a$ should vanish, i.e.

$$V_j|_{y=a} = 0;\ j = 0, 1, ..., n+1. \tag{47}$$

Obviously, condition (47) is fulfilled for $j = 0$. Assume that the function $F(x, y, U)$ satisfies the condition

$$\frac{\partial^k f(\varphi_2(a), a)}{\partial x^{k_1} \partial y^{k_2}} = 0;\ k = k_1 + k_2;\ k = 0, 1, ..., 2n+1, \tag{48}$$

in the case of linear dependence of F on U, the condition

$$\frac{\partial^k F(\varphi_2(a), a, 0)}{\partial x^{k_1} \partial y^{k_2} \partial U^k} = 0;\ k = k_1 + k_2 + k_3;\ k = 0, 1, ..., 2n+1, \tag{49}$$

in the case of nonlinear dependence of F on U. Then condition (47) will be fulfilled for all $j = 1, 2, ..., n+1$.

Thus, the constructed sum $W+V$ will satisfy the boundary conditions $(28), (45), (46)$. But generally speaking, this sum doesn't satisfy the homogeneous boundary condition on Γ_2. Therefore, it is necessary to construct the boundary layer type function

$$\eta = \eta_0 + \varepsilon\eta_1 + ... + \varepsilon^{n+1}\eta_{n+1}\ , \tag{50}$$

near the boundary Γ_2 that should satisfy the fulfilment of the boundary condition

$$(W + V + \eta)|_{\Gamma_2} = 0, \tag{51}$$

Therewith, the equations whence the functions $\eta_j; j = 0, 1, ..., n+1$ are determined, are obtained from the equality

$$L_{\varepsilon,2}(W + V + \eta) - L_{\varepsilon,2}(W + V) = 0\left(\varepsilon^{n+1}\right), \tag{52}$$

where $L_{\varepsilon,2}$ is another decomposition of the operator L_ε near the boundary Γ_2.

Here, change of variables near the boundary Γ_2 is conducted by the formula: $x = x, b - y = \varepsilon\xi$. Expanding each function $W_i(x, b - \varepsilon\xi)$ and $V_i(\tau, b - \varepsilon\xi)$ by

Taylor formula at the points (x, b) and (τ, b), from (52) we obtain the following equations:

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\eta_0}{\partial\xi}\right)^{2k+1}+\frac{\partial^2\eta_0}{\partial\xi^2}+\frac{\partial\eta_0}{\partial\xi}=0,$$

$$\frac{\partial}{\partial\xi}\left\{\left[\left(\frac{\partial\eta_0}{\partial\xi}\right)^{2k+1}+1\right]\frac{\partial\eta_j}{\partial\xi}\right\}+\frac{\partial\eta_j}{\partial\xi}=G_j,$$

where $G_j; j=1,2,...,n+1$ are the known functions.

Comparison of obtained equations with equations $(26), (27)$ shows that the construction of the function η_j in the right hand side of (50) will very little differ from the procedure on finding the functions $V_j; j=0,1,...,n+1$. Therefore, we'll not stop on constructions of η_j:

Multiply all the functions by $\eta_0, \eta_1, ..., \eta_{n+1}$ by the smoothing functions. At the expense of smoothing multipliers the functions η_j vanish on Γ_4. Therefore, in addition to condition (51) the sum $W+V+\eta$ satisfies the condition

$$\left.(W+V+\eta)\right|_{\Gamma_4}=0. \tag{53}$$

Using the conversion to zero of the functions $W_i(x,y);\ i=0,1,...,n$ and their derivatives for $x=\varphi_2(b), y=b$, we can prove

$$\left.\eta_j\right|_{x=\varphi_2(y)}=0;\ j=0,1,...,n+1.$$

Hence and from (28) it follows that the sum $W+V+\eta$ satisfies also the boundary condition

$$\left.(W+V+\eta)\right|_{\Gamma_3}=0, \tag{54}$$

Assume that the function $F(x,y,U)$ satisfies the condition

$$\frac{\partial^k f(\varphi_1(b), b)}{\partial x^{k_1}\partial y^{k_2}}=0;\ k=k_1+k_2;\ k=0,1,...,2n+1, \tag{55}$$

when the function F linearly depends on U, the condition

$$\frac{\partial^k F(\varphi_1(b), b, 0)}{\partial x^{k_1}\partial y^{k_2}\partial U^{k_3}}=0;\ k=k_1+k_2+k_3;\ k=0,1,...,2n+1. \tag{56}$$

when F nonlinearly depends on U. Then along with boundary conditions $(51), (53), (54)$ the sum $W+V+\eta$ will also satisfy the boundary condition

$$\left.(W+V+\eta)\right|_{\Gamma_1}=0. \tag{57}$$

Thus, we constructed the sum $\widetilde{U}=U+V+\eta$ that following $(51), (53), (54), (57)$ satisfies the boundary condition

$$\left.\widetilde{U}\right|_{\Gamma}=0. \tag{58}$$

Summing up $(6), (24), (52)$ we have that $\widetilde{U}$ satisfies the equation

$$L_\varepsilon\widetilde{U}=0\left(\varepsilon^{n+1}\right). \tag{59}$$

Having denoted $U-\widetilde{U}=z$, we get the following asymptotic expansion in small parameter of the solution of problem $(1),(2)$:

$$U=\sum_{i=0}^{n}\varepsilon^{i}W_{i}+\sum_{i=0}^{n+1}\varepsilon^{j}V_{i}+\sum_{i=0}^{n+1}\varepsilon^{j}\eta_{j}+z, \tag{60}$$

where z is a remainder term.

It follows from (2) and (58) that the remainder term z satisfies the boundary condition

$$z|_{\Gamma}=0. \tag{61}$$

Subtracting (59) from (1), multiplying the both hand sides of the obtained equality by $z=U-\widetilde{U}$ and integrating the obtained expressions on domain Ω, allowing for condition (61), after certain transformations we get the estimation

$$\varepsilon^{p}\iint_{\Omega}\left[\left(\frac{\partial z}{\partial x}\right)^{p+1}+\left(\frac{\partial z}{\partial y}\right)^{p+1}\right]dxdy+\varepsilon\iint_{\Omega}\left[\left(\frac{\partial z}{\partial x}\right)^{2}+\left(\frac{\partial z}{\partial y}\right)^{2}\right]dxdy+$$

$$+C_{1}\iint_{\Omega}z^{2}dxdy\leq C_{2}\varepsilon^{2(n+1)}, \tag{62}$$

where $C_1>0, C_2>0$ are the constants independent of ε.

Combining the results obtained above, we arrive at the following statement.

Theorem 3. *Assume* $F(x,y,U)\in C^{2(n+1)}(\Omega\times(-\infty,))$, *the conditions* $(3),(4)$ *and conditions* $(10),(48),(55)$ *are fulfilled in the case of linear dependence of* F *on* U*, the conditions* $(11),(12),(49),(56)$ *in the case of nonlinear dependence of* F *on* U*. Then for generalized solution of problem* $(1),(2)$ *it is valid asymptotic representation* (60), *where the functions* W_i *are determined by the first iteration process,* V_j,η_j *are the boundary layer type functions near the boundaries* Γ_3,Γ_2 *that are determined by appropriate iteration processes,* z *is a remainder term and estimation* (62) *is valid for it.*

Remark. We can reject from conditions III imposed on $\varphi_1(y),\varphi_2(y)$. Then instead of conditions $(10)-(12)$ for $y=x$, appropriate conditions for $y=x+\varphi_1(a)-a$ should be taken.

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