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ON n-DIMENSIONAL ANALOGUE OF HIRSCHMAN'S INEQUALITY

Abstract

In the paper we obtain n- dimensional analogue of Hirschman inequality.

Let $\hat{f}$ is Fourier transform of the function $f(x)$:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{R^n} e^{i(x,\xi)} f(x) dx$$

and $\|\cdot\|$ is a norm in $L^2(R^n)$.

Theorem 1. *It is valid the integral inequality:*

$$I(f) + I(\hat{f}) \le K_H. \tag{1}$$

Here

$$I(f) = \int\limits_{R^n} \frac{|f(x)|^2}{\|f\|^2} \ln\left(\frac{|f(x)|^2}{\|f\|^2}\right) dx, K_H = -n\ln(\pi e).$$

Inequality (1) is exact in the following sense: the equality therein is attained for $f(x) = a\exp(-b\left|x - \overset{0}{x}\right|^2 + (c,x))$, *where a is any complex number differ from zero, b is any real positive number, c is any complex vector of dimension n,* $\forall \overset{0}{x} \in R^n$

It is assumed that the integrals $I(f)$ and $I(\hat{f})$ exist.

The proof of the theorem is based on the following.

Proposition 1 (On Titchmarsh-Hausdorff-Young inequality). Let $1 \le p \le 2$ and $f(x) \in L^2(R^n) \cap L^1(R^n)$. Then it is valid the following Titchmarsh-Hausdorff-Young inequality with Beckner-Babenko best constant:

$$\left\|\hat{f}\right\|_{p/(p-1)} \le K_B \|f\|_p \tag{2}$$

where

$$K_B(p) = \left[\left(\frac{p}{2\pi}\right)^{1/p} \Big/ \left(\frac{p}{2\pi(p-1)}\right)^{(p-1)/p}\right]^{n/2}. \tag{3}$$

Remark 1. Inequality (2) was proved by Young ([1]) and Hausdorff ([2]) for series with best constant $K_B = 1$; for Fourier integrals for $n = 1$ with in exact constant $K_B = 1$-by Titchmarsh ([3]); Babenko ([4]) derived inequality (2) for $p = 2k/(2k-1)$ $k = 2,3,4,....$ and for $n = 1$ with exact constant (3); Beckner ([5]) proved inequality (2) with exact constant (3) for any $n \in N$ and for all $1 < p < 2$;

Lieb ([6]) proved that for $1 < p < 2$ and $\forall n \in N$ the equality in (2) is attained if and only if

$$f(x) = a\exp[-(x, Mx) + (c, x)], \tag{4}$$

where a is a non-zero arbitrary complex number; M is an arbitrary non-zero, real, symmetric, positive-definite matrix; c is any complex vector of dimension n.

Proof of the theorem.

Let

$$\Phi_1(p) = \left\|\hat{f}\right\|_{p/(p-1)}, \quad \Phi_2(p) = \|f\|_p$$

and introduce the function $\Phi(p) = \Phi_1(p) - K_B(p)\Phi_2(p)$, $1 \le p \le 2$. The function $\Phi(p)$ on the segment [1,2] is negative and has a derivative for $\forall p \in (1, 2)$.

Its left derivative at the point $p = 2 - 0$ is non-negative:

$$\Phi'(2-0) \ge 0. \tag{5}$$

For $p = 2$ $\Phi_1(2) = \Phi_2(2)$, $K_B = 1$. Calculate the derivative

$$\Phi'(p) = \Phi_1'(p) - K_B'(p)\Phi_2(p) - K_B(p)\Phi_2'(p).$$

We have:

$$\Phi_2'(p) = \Phi_2(p)\left[-\frac{1}{p^2}\ln\int\limits_{R^n}|f(x)|^p\,dx + \frac{1}{p}\frac{\int\limits_{R^n}|f(x)|^p\ln|f(x)|\,dx}{\int\limits_{R^n}|f(x)|^p\,dx}\right].$$

Hence

$$\Phi_2'(2-0) = \|f\|\left[-\frac{1}{4}\ln\|f\|^2 + \frac{1}{2}\frac{\int\limits_{R^n}|f(x)|^2\ln|f(x)|\,dx}{\|f\|^2}\right]. \tag{6}$$

Further

$$\Phi_1'(p) = \Phi_1(p)\left[\frac{1}{p^2}\ln\int\limits_{R^n}\left|\hat{f}(\xi)\right|^{p/(p-1)}d\xi - \frac{1}{p(p-1)}\frac{\int\limits_{R^n}\left|\hat{f}(\xi)\right|^{p/(p-1)}\ln|f(\xi)|\,d\xi}{\int\limits_{R^n}\left|\hat{f}(\xi)\right|^{p/(p-1)}}\right].$$

Consequently:

$$\Phi_1'(2-0) = \left\|\hat{f}\right\|\left[\frac{1}{4}\ln\left\|\hat{f}\right\|^2 - \frac{1}{2}\frac{\int\limits_{R^n}\left|\hat{f}\right|^2\ln\left|\hat{f}(\xi)\right|d\xi}{\left\|\hat{f}\right\|^2}\right]. \tag{7}$$

For $K_B'(p)$ we have:

$$\frac{dK_B(p)}{dp} = K_B(p)\frac{n}{p^2}\ln\frac{2\pi e\sqrt{p-1}}{p}.$$

Consequently:

$$K'_B(2-0) = \frac{n}{4}\ln(\pi e) \tag{8}$$

Substituting relations (6)-(8) into (5) we have:

$$I(f) + I(\hat{f}) \le -n\ln(\pi e).$$

Q.E.D.

Now we prove the exactness of the constant K_H. Without loss of generality we shall assume $\overset{0}{x} \equiv 0,\ c \equiv 0$.

Using the following definite integrals

$$\int_{R^n} e^{-\gamma|x|^2} dx = \left(\frac{\pi}{\gamma}\right)^{n/2},$$

$$\int_{R^n} e^{-\gamma|x|^2} |x|^2 dx = \frac{n}{2\gamma}\left(\frac{\pi}{\gamma}\right)^{n/2},$$

where $\forall\gamma > 0$, we calculate $I(f)$, $I(\hat{f})$ for $f(x) \equiv \Psi_G(|x|) = a\exp\left(-b|x|^2\right)$,

$$\hat{f}(\xi) \equiv \hat{\Psi}_G = \frac{a\exp\left(-|\xi|^2/4b\right)}{\left(\sqrt{2b}\right)^n}.$$

We have:

$$I(\Psi_G) = -\left[\frac{n}{2} + \frac{n}{2}\ln\left(\frac{\pi}{2b}\right)\right], \tag{9}$$

$$I\left(\hat{\Psi}_G\right) = -\left[\frac{n}{2} + \frac{n}{2}\ln(2\pi b)\right]. \tag{10}$$

From (9) and (10) we get:

$$I(\Psi_G) + I\left(\hat{\Psi}_G\right) = -n\ln(\pi e).$$

That is the fact to be verified.

Remark 2. It is easy to verify that the functions of the form (4) transform inequality (1) to the equality.

Remark 3. If we determine the Fourier transform of the function $f(x)$ as

$$\hat{f}(\xi) = \int_{R^n} e^{2\pi i\xi x} f(x)dx,$$

then the constant K_n is determined in the following way:

$$\tilde{K}_H = -n(1-\ln 2),$$

since in this case the constant K_B in inequality (1) is determined as

$$\tilde{K}_B = \left[p^{1/p}(p')^{-1/p'}\right]^{n/2},$$

where $p' = p/(p-1)$.

The constant $\tilde{K}_B$ for $n = 1$ was found by Hirschman ([6]) in 1957 before calculation its exact value, assuming that $\|f\| = 1$ and equality in (1) is attained in the case when f is Gaussian.

This inequality is applied in the quantum physics. The uncertainties between impulse and coordinate measurements are relates and expressed by this inequality.

Below we give application of Hirschman's inequality. To this end we give some preliminary information on some inequalities (see [7]).

Statement 1. For any $f \in H^1(R^n)$, the following Gross-Sobolev logarithmic inequality is valid:

$$I(f) \leq \frac{n}{2} \ln \left(\frac{2}{\pi e n} \frac{\|\nabla f\|^2}{\|f\|^2} \right). \tag{11}$$

Inequality (11) is exact, it becomes an equality if and only if $f = ae^{-b|x-\bar{x}|^2}$, for $a > 0,\ b > 0$, and any $\bar{x} \in R^n$.

Statement 2. Suppose that $f \in H^1(R^n)$ and $rf \in L_2(R^n)$ for $r = |x|$. Then, the following Pauli-Heisenberg-Weyl inequality is valid:

$$\|f\| \leq \sqrt{\frac{2}{n}} \|\nabla f\|^{1/2} \|rf\|^{1/2} = \sqrt{\frac{2}{n}} \left\| |\xi| \hat{f} \right\|^{1/2} \|rf\|^{1/2}. \tag{12}$$

Inequality (12) is exact, it becomes an equality if and only if $f = ae^{-b|x-\bar{x}|^2}$, for $a > 0,\ b > 0$, and any $\bar{x} \in R^n$.

Statement 3. Suppose that $f \in H^1(R^n)$ and $rf \in L_2(R^n)$. Then, the following entropy inequality is valid:

$$I(f) \geq -\frac{n}{2} \ln \left(\frac{2\pi e}{n} \frac{\|rf\|^2}{\|f\|^2} \right). \tag{13}$$

Inequality (13) is exact it becomes an equality if and only if f is a Gaussian function, i.e., $f = ae^{-b|x-\bar{x}|^2}$, for any $a > 0,\ b > 0$, and any $\bar{x} \in R^n$.

Theorem 2. *The following deductions are valid: (i) The entropy and Hirschman inequalities imply the Pauli-Heienberg-Weyl inequality. (ii) The entropy and Hirschman inequalities imply the Gross-Sobolev inequality.*

Proof. (i) By virtue of the Plancherel –Parseval theorem (13) implies

$$I(\hat{f}) \geq \frac{-n}{2} \ln \left(\frac{2\pi e}{n} \frac{\|\nabla f\|^2}{\|f\|^2} \right).$$

Adding this inequality to (13), we obtain

$$I(f) + I(\hat{f}) \geq \frac{-n}{2} \ln \left[\left(\frac{2}{n} \right)^2 \frac{\|\nabla f\|^2}{\|f\|^2} \frac{\|rf\|^2}{\|f\|^2} \right] - n \ln(\pi e)$$

or

$$\frac{-n}{2}\ln\left[\left(\frac{2}{n}\right)^2\frac{\|\nabla f\|^2}{\|f\|^2}\frac{\|rf\|^2}{\|f\|^2}\right]\le I(f)+I(\hat{f})+n\ln(\pi e)\le 0.$$

This implies (12), as required. (ii) By virtue of the Plancherel-Parseval equality, (13) implies

$$I(\hat{f})\ge\frac{-n}{2}\ln\left(\frac{2\pi e}{n}\frac{\|\nabla f\|^2}{\|f\|^2}\right) \tag{14}$$

It follow from (13) and (14) that

$$\frac{-n}{2}\ln\left(\frac{2\pi e}{n}\frac{\|\nabla f\|^2}{\|f\|^2}\right)+I(f)\le I(f)+I(\hat{f})\le -n\ln(e\pi).$$

Therefore,

$$I(f)\le\frac{n}{2}\ln\left(\frac{2\pi e}{n}\frac{\|\nabla f\|^2}{\|f\|^2}\right)-\frac{n}{2}\ln(e^2\pi^2)=\frac{n}{2}\ln\left(\frac{2}{\pi e n}\frac{\|\nabla f\|^2}{\|f\|^2}\right),$$

as required.

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